Question 1

Suppose:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ldots\left(1-\frac{1}{2026^2}\right)=\frac{a}{2b}$$

Find the values of a and b, where $a, b \in \mathbb{Z}$.

Solution:

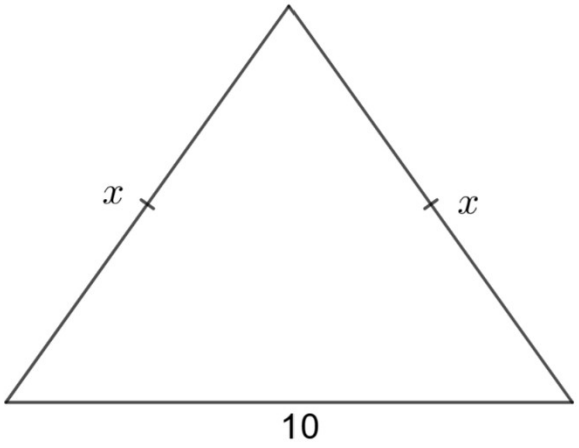
$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\ldots\left(1-\frac{1}{2026}\right)\left(1+\frac{1}{2026}\right)$$

$$\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right) \quad \ldots \quad \left(\frac{2025}{2024}\right)\left(\frac{2024}{2025}\right)\left(\frac{2026}{2025}\right)\left(\frac{2025}{2026}\right)\left(\frac{2027}{2026}\right)=\frac{1}{2}\left(\frac{2027}{2026}\right)$$

$$a = 2027; b = 2026$$

Question 2

An isosceles triangle has a fixed base of length 10 cm, with other sides of length x cm. The triangle is growing as time changes. The rate of change of x is 4 cm per second.



Find the rate of change of the area of the triangle when $x = 13$cm.

$$\frac{dA}{dt}|_{x=13} = ?$$

$\frac{dx}{dt} = 4;$

$A = \frac{1}{2}(10)\left(\sqrt{x^2-25}\right) = 5(x^2-25)^{\frac{1}{2}}$

$$\frac{dA}{dx} = \frac{5}{2}(2x)\frac{1}{\sqrt{x^2-25}} = \frac{5x}{\sqrt{x^2-25}}$$

$$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = \frac{5x}{\sqrt{x^2-25}}(4) = \frac{20x}{\sqrt{x^2-25}}$$

$$\frac{dA}{dt}|_{x=1} = \frac{20(13)}{\sqrt{13^2-25}} = \frac{65}{3}\ cm^2/s$$

Question 1

Solve for all $x \in \mathbb{R}$, such that:

$$\frac{1}{x^2-10x-29}+\frac{1}{x^2-10x-45}=\frac{2}{x^2-10x-69}$$

Solution:

Re-write:

$$\frac{1}{(x-5)^2-54}+\frac{1}{(x-5)^2-70}=\frac{2}{(x-5)^2-94}$$

Let $y=(x-5)^2$

$$\frac{1}{y-54}+\frac{1}{y-70}=\frac{2}{y-94}$$

$$\Rightarrow (y-70)(y-94)+(y-54)(y-94)=2(y-54)(y-70)$$

$$\Rightarrow -64y=-4096$$

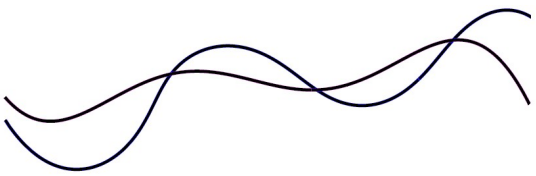
$$\Rightarrow y=64$$

$$\Rightarrow (x-5)=\pm 8$$

$$\Rightarrow x=13,-3$$

Question 2

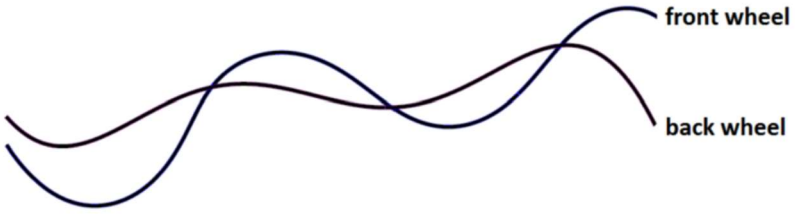
The diagram shows the tyre tracks made by the front and back wheel of a bicycle. Was the bicycle:



1. Going from right to left.
2. Going from left to right,
3. It doesn't make any difference – it's the same either way.

Solution:

The front wheel of the bicycle can turn left and right, but the back wheel of a bicycle is fixed in its frame. This means that the more "stable" track is likely to be the back-wheel track and the wobblier one the front wheel track. We deduce so far:

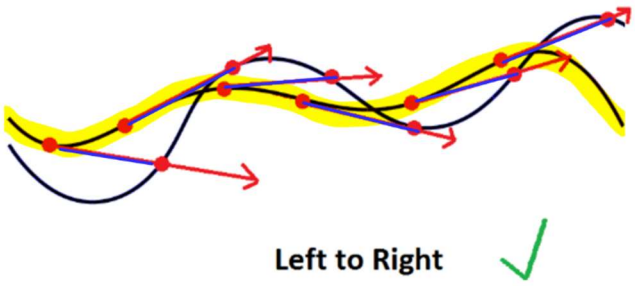


At any instantaneous moment in time, the back wheel always points towards the point of contact on the ground of the front wheel, no matter which way the front wheel is pointing.

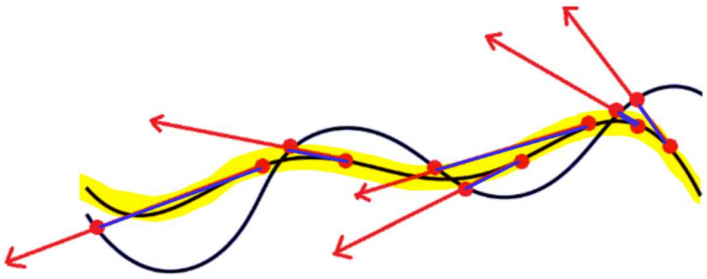
The distance to that point of contact is fixed: it is the distance between the axles of the wheels.

Hence, the ***tangent line*** at each point on the **back-wheel track intercepts the front wheel track at a fixed distance**.

Construct tangent lines from the back wheel track (highlighted) to intersect the front wheel track. Examine the distances from the point of contact of the tangent, to the front wheel track. These distances should be equal. Hence, left to right is the correct direction.



Left to Right ✓



Right to Left ✗

Round 3 **Babtha 3**

Question 1

Solve the following equation for $x \in \mathbb{R}$:

$$3^x - \left(\sqrt{3}\right)^{x+4} + 20 = 0$$

Give your answer(s) in their simplest possible form of: $2\log_a b$ where $a, b \in \mathbb{N}$.

Solution:

Let $y = 3^{\frac{x}{2}}$

$$\Rightarrow 3^x - \left(\sqrt{3}\right)^{x+4} + 20 = y^2 - 9y + 20 = 0$$

$$\Rightarrow y = 4, 5$$

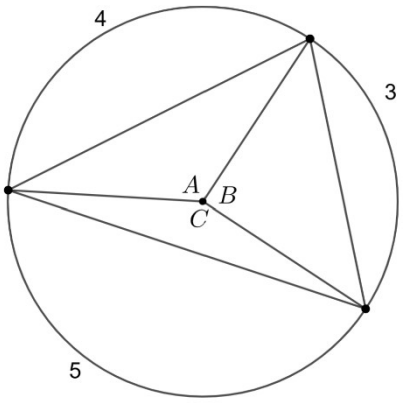
$$\Rightarrow 3^{\frac{x}{2}} = 4; \quad 3^{\frac{x}{2}} = 5$$

$$\Rightarrow \frac{x}{2} = \log_3 4\,; \quad \frac{x}{2} = \log_3 5$$

$$\Rightarrow x = 2\log_3 4\,; \quad x = 2\log_3 5$$

Question 2

A triangle is inscribed in a circle as shown. The vertices of the triangle divide the circle into three **arcs of lengths** 3, 4 and 5.



Find the area of the triangle in the form: $\frac{a}{\pi^2}(b + \sqrt{b})$ where $a, b \in \mathbb{N}$.

Solution:

$3 + 4 + 5 = 12 = 2\pi r \qquad \therefore r = \frac{6}{\pi}$

$$l = r\theta \ \therefore \ 3 = \frac{6}{\pi}A; \quad 4 = \frac{6}{\pi}B; \quad 5 = \frac{6}{\pi}C$$

$$Area = \frac{1}{2}r^2 \sin A + \frac{1}{2}r^2 \sin B + \frac{1}{2}r^2 \sin C$$

$$Area = \frac{(6/\pi)^2}{2}\left(\sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6}\right)$$

$$Area = \frac{18}{\pi^2}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{9}{\pi^2}(3 + \sqrt{3})$$

Round 4 **Babtha 4**

Question 1

A random sample of 250 schoolbags were weighed and the data was analysed.

Using only the data from the sample, a **95%** confidence interval (**LC HL method**) for the sample mean weight of the bags was found to be: [4.961kg, 4.999kg].

A school claimed that the population mean weight of schoolbags was 5kg. A hypothesis test was carried out at the **1%** level of significance.

1. Calculate the p-value, **as a percentage**, correct to **2 decimal places**.
2. Determine whether the null hypothesis was rejected or not at the **1%** level of significance.

p-value as %:_____

$\mathrm{H_o}$ Conclusion: Reject/Do Not Reject:_____

$\bar{x} = \frac{4.961+4.999}{2} = 4.98; \qquad \frac{1.96}{\sqrt{250}} = 4.98 - 4.961 = 0.019; \qquad \frac{\sigma}{\sqrt{250}} = \frac{(0.019)}{1.96}$

$$z = \frac{4.98 - 5}{\frac{0.019}{1.96}} = -2.063 \sim -2.06$$

$$p - value = 2(1 - P(Z \leq 2.06) = 2(1 - 0.9803) = 0.0394 => 3.94\%$$

Conclusion:

@ 1% level of significance, $3.94\% > 1\%$, therefore not significant at the 1% level of significance.

=> Do not reject $\mathrm{H_o}$ at the 1% level of significance.

Question 2

For certain **real** values of p, q, r and s, the equation

$$z^4 + pz^3 + qz^2 + rz + s = 0$$

has four **non-real** roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$.

Find the value of $q \in \mathbb{R}$.

Solution:

The conjugate roots theorem applies, so the roots occur in conjugate pairs. However, the sum of $13 + i$ is not the sum of a pair of conjugate roots as they would add to have $0i$.

Consider two pairs of conjugate roots: $a + bi; a - bi$ and $x + yi; x - yi$.

Look at the opposite sums $(a + x) + (b + y)i$ and $(a + x) - (b + y)i$. They are conjugates.

Look at the products: $(a + bi)(x + yi) = ax - by + (ay + bx)i$.

And the product: $(a - bi)(x - yi) = ax - by - (ay + bx)i$. They are also conjugates.

We build 2 quadratics with this knowledge:

$$(z^2 - z(3 + 4i) + 13 - i)(z^2 - z(3 - 4i) + 13 + i)$$

Focusing on the z^2 coefficient we get:

$$13 - i + 13 + i + (3 + 4i)(3 - 4i) = 51$$

Round 5 **Babtha 5**

Question 1

Let:

$$\frac{1}{x} - \frac{1}{y} = 2026$$

Find the value of:

$$\frac{x + 2027xy - y}{2y - 2026xy - 2x}$$

Solution:

$$\frac{1}{x} - \frac{1}{y} = 2026 \Rightarrow y - x = 2026xy$$

$$\frac{x + 2027xy - y}{2y - 2026xy - 2x} = \frac{x - y + xy + 2026xy}{2y - 2x - (y - x)} = \frac{x - y = xy + y - x}{y - x} = \frac{xy}{2026xy} = \frac{1}{2026}$$

Question 2

The sum of 18 consecutive positive integers is a perfect square. What is the smallest possible value of this sum?

Solution:

Let the sum of the 18 consecutive numbers be:

$$a + a + 1 + a + 2 + \cdots + a + 17 = 18a + (1 + 2 + 3 + \cdots + 17) = 18a + \frac{17(18)}{2} = 18a + 153$$

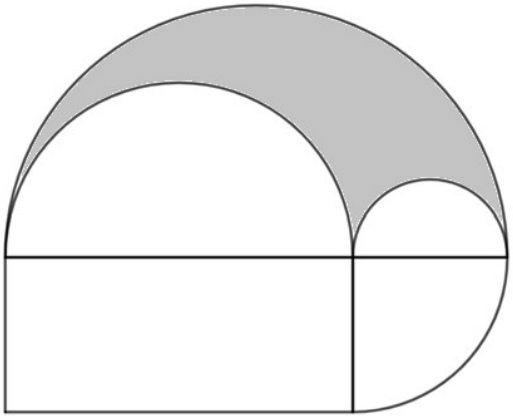
We now need the next perfect square, x^2, after 153 that satisfies $18a + 153 = x^2$.

With a little trial and error, we arrive at 225.

Round 6 **Babtha 6**

Question 1

The diagram shows a rectangle, with **area** of the **rectangle equal to 4** units squared, a quarter circle and three semicircles of various radii.



Find the ***exact*** area of the shaded region in the diagram.

Solution:

The area of the shaded region is the area of the large semi-circle less the areas of the small and medium semi-circles.

Let x and y be the radii of the small and medium semi-circles respectively.

The radius of the large semi-circle is then $\frac{2y+2x}{2} = x + y$

The area of the rectangle is $4 = 2x(2y)$ which gives $1 = xy$.

The area of the shaded region in units squared is then:

$$\frac{1}{2}\pi(x + y)^2 - \frac{1}{2}\pi x^2 - \frac{1}{2}\pi y^2 = \pi(xy) = \pi(1) = \pi.$$

Question 2

Simplify: $b^{\log_b(2a)! - 2\log_b(a)!}$

Give your answer in the form $\binom{n}{r}$, where n, r are both given in terms of a.

$$b^{\log_b(2a)! - 2\log_b a!} = b^{\log_b(2a)! - \log_b(a!)^2} = b^{\log_b \frac{(2a)!}{(a!)^2}} = \frac{(2a)!}{(a!)^2} = \frac{(2a)!}{(a)!(a)!} = \frac{(2a)!}{(a)!(2a-a)!} = \binom{2a}{a}$$

Question 1

For $x \in \mathbb{R}$, let:

$$f(x) = \begin{cases} (x-2)^2 + 1, & x \leq 3 \\ 2x - 4, & x > 3 \end{cases}$$

Solve for all values of x for which $f(x) > 2026$.

Solution:

For $x > 3$, solving $2x - 4 = 2026$ we get $x = 1015$.

Hence $f(x) > 2026$ for $x > 1015$.

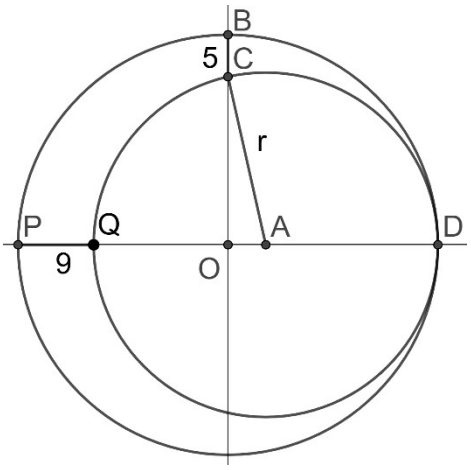
For $x \leq 3$, solving $(x-2)^2 + 1 = 2026$ to get $x = 47$ reject, $x = -43$.

Hence $f(x) > 2026$ for $x < -43$.

$\{x < -43\} \cup \{x > 1015\}$

Question 2

The diagram shows a larger circle, with centre at the origin $O(0,0)$ and radius R, and a smaller circle with centre A and radius r. The circles touch at the point D. The length of the line segment BC is 5 and of PQ is 9.



Find the length of the sum of the radii i.e. $R + r$.

Solution:

$$2R = 2r + 9$$

$$R - r = \frac{9}{2}$$

In the right triangle OAC we have lengths: $R - 5, r - (R - 9)$ and hypotenuse: r. But $r - (R - 9) = r - R + 9 = \frac{9}{2}$

$$(R-5)^2 + \left(\frac{9}{2}\right)^2 = r^2$$

Substituting $r = R - \frac{9}{2}$ and then solving we get $R = 25, r = \frac{41}{2}$ and $R + r = 45.5$

Question 3

Let z be a complex number satisfying the equation:

$2z^2 - 5z + 4 = 0$. Find the value of $\frac{z^3}{z^6+8}$.

Solution:

$2z^2 - 5z + 4 = 0 \Leftrightarrow z^2 = \frac{5z-4}{2}$ and $z^2 + 2 = \frac{5}{2}z$

$$\frac{z^3}{z^6+8} = \frac{z(z^2)}{(z^2+2)(z^4-2z^2+4)} = \frac{z\left(\frac{5z-4}{2}\right)}{\left(\frac{5}{2}\right)(z)[z^2(z^2-2)+4]} = \frac{5z-4}{5\left[\left(\frac{5z-4}{2}\right)\left(\frac{5z-8}{2}\right)+4\right]}$$

$$= \frac{5z-4}{5[\frac{25z^2-60z+32}{4}+4]} = \frac{4(5z-4)}{5[25z^2-60z+48]} = \frac{4(5z-4)}{5[25\left(\frac{5z-4}{2}\right)-60z+48]}$$

$$= \frac{4(5z-2)}{5\left[\frac{125z-100-120z+96}{2}\right]} = \frac{4(5z-4)}{5\left[\frac{5z-4}{2}\right]} = \frac{4}{\frac{5}{2}} = \frac{8}{5}$$

Question 4

The Binomial Expansion (given below) is defined for powers other than positive integer powers i.e. n does not have to be a positive integer.

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

Hence, or otherwise, find the first three terms, in their simplest form, in ascending powers of x, of the expansion:

$$\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$$

Solution:

Recall that $\binom{n}{0} = 1, \binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$.

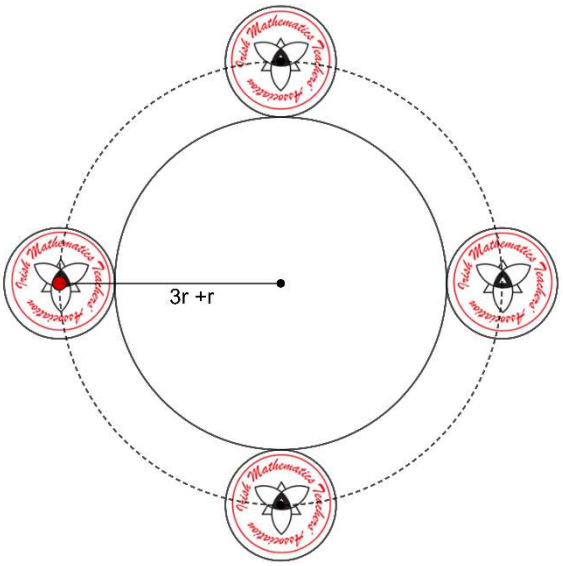
Hence, the first three terms in ascending powers of x are:

$$\binom{-\frac{1}{2}}{0}(1)^{-\frac{1}{2}}\left(-\frac{x}{4}\right)^0 = (1)(1)(1) = 1$$

$$\binom{-\frac{1}{2}}{1}(1)^{-\frac{1}{2}-1}\left(-\frac{x}{4}\right)^1 = \left(-\frac{1}{2}\right)(1)\left(-\frac{x}{4}\right) = \frac{x}{8}$$

$$\binom{-\frac{1}{2}}{2}(1)^{-\frac{1}{2}-2}\left(-\frac{x}{4}\right)^2 = \left(\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2}\right)(1)\left(\frac{x^2}{16}\right) = \left(\frac{3}{8}\right)\left(\frac{x^2}{16}\right) = \frac{3x^2}{128}$$

Question 1



The radius of circle B is three times the radius of circle A. Starting from the position shown in the figure, circle A rolls around circle B. When circle A returns to its starting point, how many rotations will it have completed?

Solution:

Let r be the radius of A and $3r$ be the radius of B. Then the centre of circle A traces a path of radius $r + 3r = 4r$ around the circle and hence 4 rotations are completed.

Question 2

A coin is selected at random from a bag containing exactly two coins. One of the coins is a fair, unbiased coin with a head and a tail and the other coin has two heads. The selected coin is tossed three times in succession and lands heads up each time - HHH.

Given that HHH was tossed in succession, what is the probability that the coin that was thrown was the coin with two heads?

Solution:

Let B be the event that the Biased coin is chosen and F be the event that the fair coin is chosen.

$$P(B|HHH) = \frac{P(B \cap HHH)}{P(B \cap HHH) \cup P(F \cap HHH)} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{16}} = \frac{8}{9}$$

Question 3

The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is 1 ,16, 31, ...). This process is continued until a number is reached which has already been marked, then it stops. Find the ***number*** of unmarked numbers at this point.

Solution:

1st sequence: 1, 16, 31 …. $T_n = 15n - 14$

Solving for $15n - 14 = 1000$ we get 67.6.

This means the largest number less than 1000 in this sequence is $15(67) - 14 = 991$.

This means that the next number is back to $15 - 9 = 6$.

2nd sequence: 6, 21, 36 …. $T_n = 15n - 9$

Solving for $15n - 9 = 1000$ we get 67.26666 …

This means the largest number less than 1000 in this sequence is $15(67) - 9 = 996$.

This means that the next number is back to $15 - 4 = 11$.

3rd sequence: 11, 26, 41…. $T_n = 15n - 4$

Solving for $15n - 4 = 1000$ we get 66.93333 …

This means the largest number less than 1000 in this sequence is $15(66) - 4 = 986$.

This means that the next number is back to $15 - 14 = 1$. This number is already on a sequence.

The numbers on our lists add to: $67 + 67 + 66 = 200$. Therefore, there are 800 unmarked numbers.

Question 4

While finding the sine of a certain angle, an absent-minded student failed to notice that their calculator was not in the correct angular mode. They were lucky to get the right answer.

The two smallest positive real values of x for which the sine of x in degrees is equal to the sine of x in radians are: $\frac{a\pi}{b+\pi}$ and $\frac{c\pi}{d-\pi}$, where $a, b, c, d \in \mathbb{N}$.

Find $a + b + c + d$.

Solution:

$$\sin x = \sin\left(\frac{180x}{\pi}\right)$$

Note: $x = \frac{180x}{\pi}$ yields nothing. Hence:

$$\Rightarrow x + 360 = \frac{180x}{\pi} \quad \text{or} \quad x = 180 - \frac{180x}{\pi}$$

Solving both we get:

$$\Rightarrow x = \frac{360\pi}{180} \quad \text{or} \quad x = \frac{18}{180+\pi}$$

$$\therefore a + b + c + d = 180 + 180 + 360 + 180 = 900$$

R1	Q1	$a = 2027; b = 2026$
	Q2	$\frac{65}{3} cm^2/s$ (Units not required).
R2	Q1	$-3, 13$
	Q2	Option 2: Left to right.
R3	Q1	$2\log_3 4; 2\log_3 5$
	Q2	$\frac{9}{\pi^2}\left(3+\sqrt{3}\right)$ (units not required).
R4	Q1	$p - value = 3.94\%$; Do Not Reject $\mathrm{H_o}$.
	Q2	51
R5	Q1	$\frac{1}{2026}$
	Q2	225
R6	Q1	π (Units not required.)
	Q2	$\binom{2a}{a}$
R7	Q1	$\{x < -43\} \cup \{x > 1015\}$
	Q2	45.5
	Q3	$\frac{8}{5}$
	Q4	$1, \frac{x}{8}, \frac{3x^2}{128}$
R8	Q1	4
	Q2	$\frac{8}{9}$
	Q3	800
	Q4	900