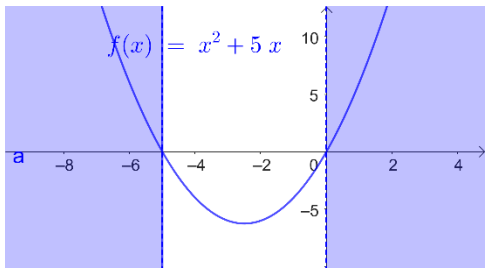
Round 1 **Babtha 1**

Question 1 Solve the following inequality where $x \in \mathbb{R}, x \neq 0$ $\frac{4x+5}{x} \geq 3$

Solution:



$$\frac{x^2(4x+5)}{x} \geq 3x^2$$

$$4x^2 + 5x - 3x^2 \geq 0$$

$$x(x+5) \geq 0$$

2 disjoint regions that cannot occur at the same time.

Solution: $\{x \leq -5\} \cup \{x > 0\}$

Question 2 Find the equation of the circle with the following properties:

- Its centre is in the 1st quadrant
- The origin, $(0,0)$, is on the circle
- The circle cuts the y-axis at points that are 4 units apart
- The circle cuts the x-axis at points that are 8 units apart

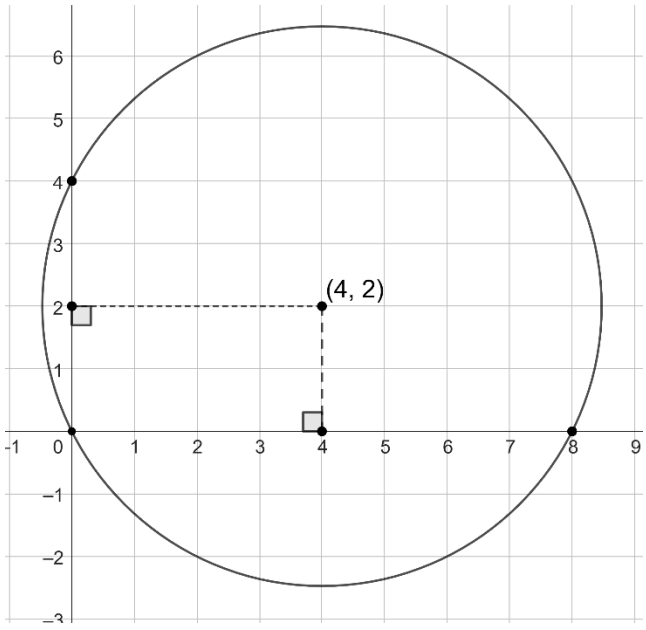
Solution:

From the diagram, the centre is clearly (4, 2).

Radius: $\sqrt{4^2 + 2^2} = \sqrt{20}$

Equation of circle:

$(x-4)^2 + (y-2)^2 = 20$



Round 2 **Babtha 2**

Question 1

Find the value of the term which is independent of x in the expansion of

$$\left(x^2 - \frac{2}{x}\right)^9$$

Solution:

General term of $\left(x^2 - \frac{2}{x}\right)^9$ is $\binom{9}{r}(x^2)^{9-r}\left(\frac{-2}{x}\right)^r$. The term independent of x has power on the x equal to zero. Hence:

$$x^{18-2r}\left(\frac{1}{x^r}\right) = x^{18-2r}x^{-r} = x^{18-3r}$$

$$<=> 18 - 3r = 0$$

$$=> 6 = r$$

Coefficient:

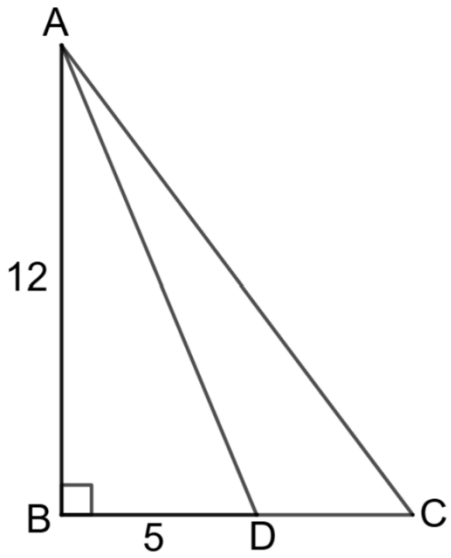
$$\binom{9}{6}(-2)^6 = 5376$$

Question 2

The triangle ABC is right angled at B, with $|AB| = 12$.
Point D, on BC, is such that $|BD| = 5$.

The area of ΔADC is 80% of the area of ΔABD.

Determine the perimeter of triangle ADC.



Solution:

From Pythagoras, $|AD| = 13$.

Let $|DC| = x$.

Area of ΔADC is 80% of the area of ΔABD

$$=> \frac{1}{2}(x)(12) = (.8)\frac{1}{2}(5)(12)$$

$$=> x = 4$$

From Pythagoras, $|AC| = 15$.

$$\therefore P = 4 + 15 + 13 = 32$$

Round 3 **Babtha 3**

Question 1

Express z in the form $2(a) + ai$, where $a \in \mathbb{Z}$, given that

$$z = \frac{30}{3-i} - 3$$

Solution:

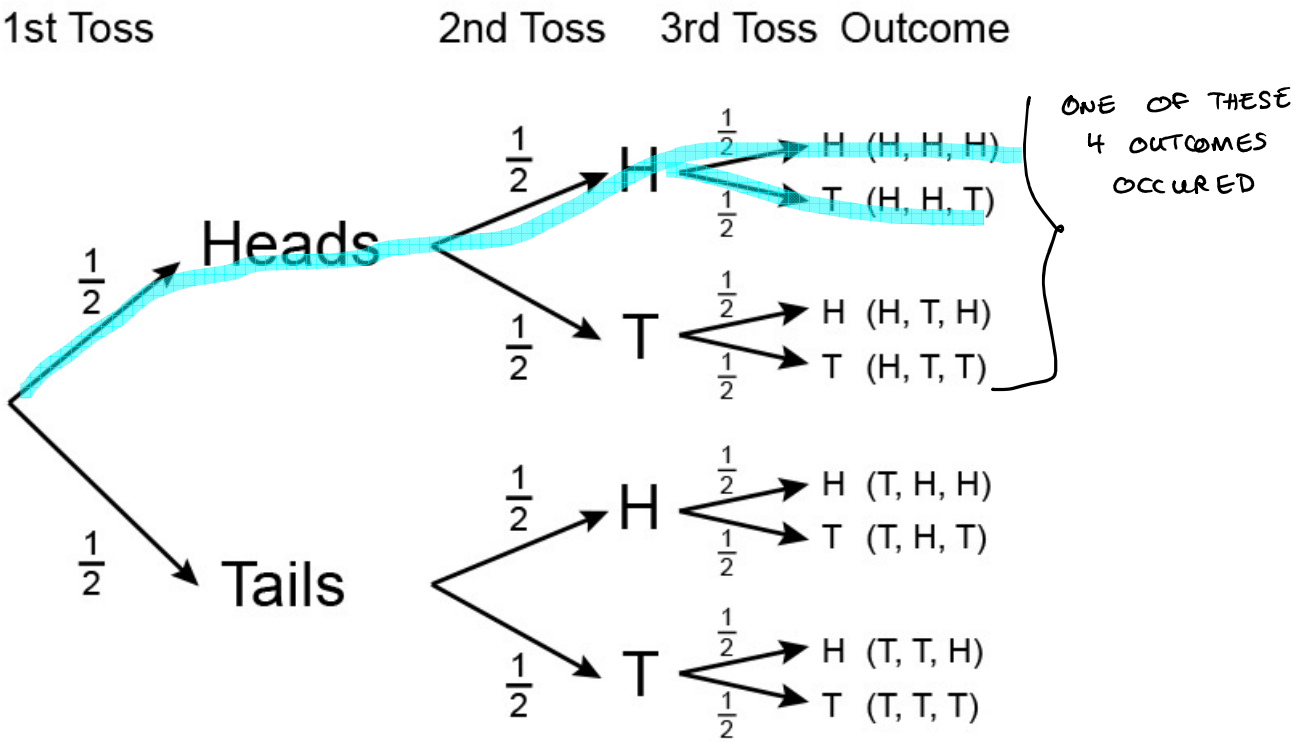
$$z = \frac{30(3+i)}{(3-i)(3+i)} - 3 = \frac{90+30i}{10} - 3 = 9 + 3i - 3 = 6 + 3i = 2(3) + 3i$$

Question 2

A fair coin is tossed three times. What is the probability that the second toss was heads ***given that*** the first toss was heads?

Solution:

All outcomes equally likely. 2 possible ways out of 4 as highlighted => $\frac{1}{2}$.



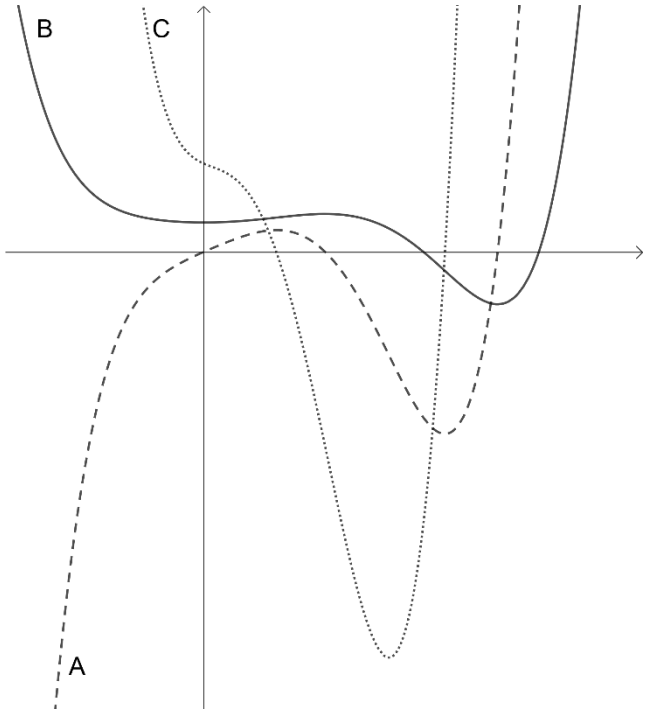
Or

$$P(1st\ H\ |2nd\ H) = \frac{P(1st\ H\ \cap 2nd\ H)}{P(1st\ H)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Round 4 **Babtha 4**

Question 1

Part of the graph of a function $f(x)$, along with the graphs of both its first and second derivatives $f'(x)$ and $f''(x)$ are shown below.



The graphs are labelled A, B and C. Put these labels in order of $f(x), f'(x)$ and $f''(x)$.

Solution:

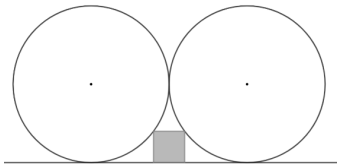
Link a root to a turning point, link + values to increasing and – values to decreasing. Then use logical deductions.

B, A, C

Link turning points and increasing /decreasing behaviour of parent function to derivative function values.

Question 2

Two tangential circles with equal length radius, r, and area 100π, are shown in the diagram.



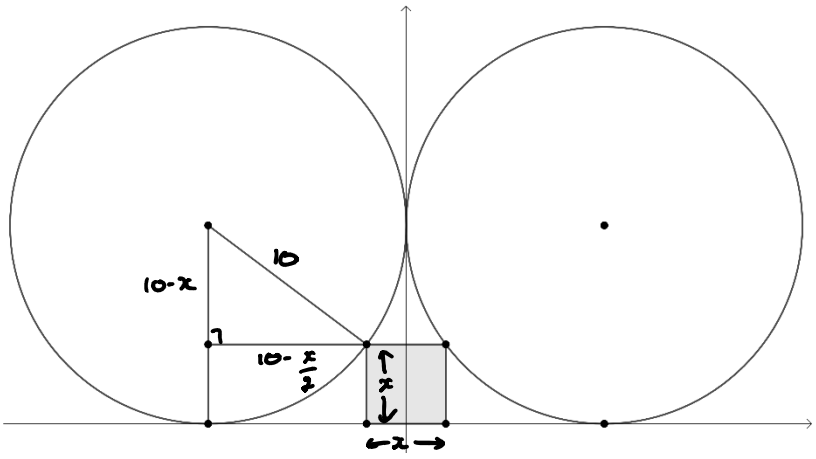
A square is drawn between the circles as shown, touching each circle at one point only.

Find the area of the square.

Solution:

$100\pi = 10^2\pi$. Radius =10

Use Pythagoras on the right triangle shown.



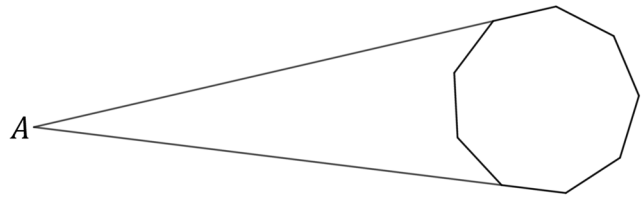
$$=> x = 4$$

Area of square 16.

Round 5 **Babtha 5**

Question 1

The diagram shows a regular nonagon (nine-sided polygon) with two of the sides extended to meet at the point A. What is the size, in degrees, of the acute angle at A?

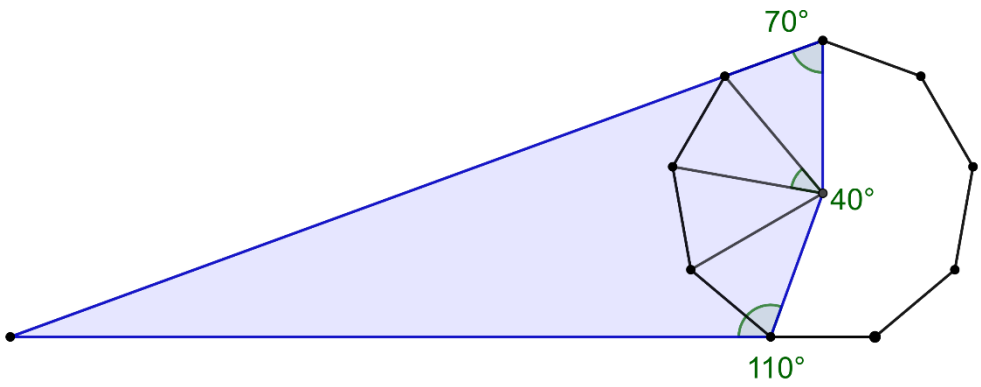


Solution: (Many methods available.)

Each angle at the centre of the regular nonagon is $\frac{360}{9} = 40°$.

Each of the 9 triangles are isosceles with base angle $70°$.

Consider the quadrilateral shown. All interior angles add to $360°$.



Hence the angle at A is $360 - 4(40) - 70 - 110 = 20°$.

Question 2

Find the values of $x \in \mathbb{R}$ such that $|x|^2 + 4|x| - 5 = 0$.

Solution:

Let $y = |x|$. $y^2 + 4y - 5 = (y - 1)(y + 5) = 0$

Hence $y = -5,\ 1$.

Hence $|x| = -5$ (reject) or $|x| = 1$ => $x = \pm 1$.

Round 6 **Babtha 6**

Question 1

Let $f(x) = \sin^{2026} x + \cos^{2026} x$

$f'(x)$ can be written in the form:

$$\frac{2026}{k} \sin kx \left(\left(\sin^b x\right)^k - \left(\cos^b x\right)^k\right)$$

Find the values of $b, k \in \mathbb{N}\backslash\{1\}$, such that b is as large as possible and k is as small as possible.

Solution:

$$f'(x) = 2026\,(\sin^{2025} x)(\cos x) - 2026\,(\cos^{2025} x)(\sin x)$$

$$= 2026(\sin x \cos x)\,[(\sin^{2024} x) - (\cos^{2024} x)]$$

$$= \frac{2026}{2}(2 \sin x \cos x)[(\sin^{1012} x)^2 - (\cos^{1012} x)^2]$$

$$= \frac{2026}{2}(\sin 2x)[(\sin^{1012} x)^2 - (\cos^{1012} x)^2]$$

$$\therefore b = 1012, \qquad k = 2$$

Question 2

Ciarán has been making two types of puzzles with wooden blocks: a smaller puzzle that contains exactly ***six*** blocks and a larger puzzle that contains exactly ***seven*** blocks.

Exactly 167 blocks in total were used to create a certain number of smaller and larger puzzles.

Determine ***all*** possibilities for how many of each type of puzzle Ciarán has made.

Solution:

Letting s and l equal the number of small and large puzzles respectively we get $6s + 7l = 167$.

Since $6s$ is even, $7l$ must be odd as they add to an odd number and hence l must be odd.

$\frac{167}{7} = 23.87..$ which means that l must be an odd number less than or equal to 23.

Testing all possibilities yields the following solutions:

Smaller	Larger
22	5
15	11
8	17
1	23

Round 7 **Babtha 7**

Question 1

Find the value of $n \in \mathbb{R}$ for which:

$$3 = -\log_n\left(\log_n\left(\sqrt{\sqrt{\sqrt{n}}}\right)\right)$$

Solution:

$$-3 = \log_n\left(\log_n\left(n^{\frac{1}{8}}\right)\right)$$

$$n^{-3} = \log_n n^{\frac{1}{8}} = \frac{1}{8} = 2^{-3}$$

$$n^{-3} = 2^{-3}$$

$$\therefore n = 2$$

Question 2

Ms White says that their friend, Mr Green, walks from his house into town at the rate of 5 km/h but walks back at only 3 km/h. The total journey – there and back – takes exactly 7 hours. What is the distance from Mr Green's house to the town, in km?

Solution:

Let distance $= d$, time outbound $= T_1$, return time T_2.

$$T_1 + T_2 = 7$$

$$T_1 = \frac{d}{5}, T_2 = \frac{d}{3}$$

$$\frac{d}{5} + \frac{d}{3} = 7$$

$$d = 13.125\ km$$

Question 3

The points $P(2,3)$ and $Q(4,3)$ are two vertices of the triangle PQR. The angle bisector at vertex Q is $x - 3y + 5 = 0$. The equation of the line through QR can be written in the form $ax - 4y + c = 0$. Find the value of $a^2 + c^2$.

Solution:

Tan of the angle between PQ and bisector is equal to tan of the angle between bisector and the line $ax - 4y + c = 0$. Let the slope of this line equal m.

$$=> \tan\theta = \frac{1}{3} = \frac{m - \frac{1}{3}}{1 + \frac{m}{3}}$$

$$m = \frac{3}{4} => a = 3$$

$Q(4, 3)$ is on the line $3x - 4y + c = 0$

$$\therefore 3(4 - 4(3) + c = 0$$

$$c = 0$$

$$a^2 + c^2 = 3^2 + 0^2 = 9.$$

Question 4

Let $3^a + 3^b = p$ and $3^a - 3^b = q$.

Express 3^{a+b} in terms of p and q.

Give your answer in its simplest form.

Solution:

Need an expression with 3^{a+b} so try squaring.

$$p^2 = \left(3^a + 3^b\right)^2 = 3^{2a} + 3^{2b} + 2(3^{a+b})$$

$$q^2 = \left(3^a - 3^b\right)^2 = 3^{2a} + 3^{2b} - 2(3^{a+b})$$

$$\therefore p^2 - q^2 = 4\left(3^{a+b}\right)$$

$$\therefore \frac{p^2 - q^2}{4} = 3^{a+b}$$

Round 8 **Babtha 8**

Question 1

Solve for $x \in \mathbb{R}$, giving your answer in its simplest form:

$$3^x = x^9$$

Solution:

$$3^x = x^9$$

$$(3^x)^{\frac{1}{9x}} = (x^9)^{\frac{1}{9x}}$$

$$3^{\frac{1}{9}} = x^{\frac{1}{x}}$$

$$3^{\frac{1}{9}\cdot\frac{3}{3}} = x^{\frac{1}{x}}$$

$$3^{\frac{3}{27}} = x^{\frac{1}{x}}$$

$$(3^3)^{\frac{1}{27}} = x^{\frac{1}{x}}$$

$$(27)^{\frac{1}{27}} = x^{\frac{1}{x}}$$

$$\therefore x = 27$$

Question 2

Find all positive integers n, for $1 \leq n \leq 10$, such that:

$$n^{2025} + n^{2026}$$

is divisible by 5.

Solution:

$$n^{2025} + n^{2026} = (n^{2025})(1 + n)$$

One of the factors must then be divisible by 5.

$n + 1$ is divisible by 5 only if $n + 1 = 10$ *or* 5.

$$\therefore n = 4, 9$$

n^{2025} is divisible by 5 only if n is divisible be by 5.

$$\therefore n = 5, 10$$

Complete answer:

$$\therefore n = 4, 5, 9, 10$$

Question 3

Evaluate the following, where the angles are in degree measure, giving your answer in its simplest form:

$$(\tan 1°)(\tan 2°)(\tan 3°)(\tan 4°) \ldots . (\tan 89°)$$

Solution:

$$(\tan 1°) \ldots (\tan 89°) = \frac{\sin 1}{\cos 1}\frac{\sin 2}{\cos 2}\frac{\sin 3}{\cos 3} \ldots \frac{\sin 88}{\cos 88}\frac{\sin 89}{\cos 89}$$

Recall that $\sin A = \frac{1}{\cos(90-A)}$ and $\cos A = \frac{1}{\sin(90-A)}$. E.g. $\frac{\sin 1}{\cos 1}.\frac{\sin 89}{\cos 89} = 1$

Hence

$$(\tan 1°)(\tan 2°)(\tan 3°)(\tan 4°) \ldots . (\tan 89°) = 1$$

Question 4

Consider the letters of the word COFFEE.

How many ***distinct*** arrangements of any size less than or equal to 6 letters can be made using each letter at most once. I.e. The total number of distinct 1 letter, 2 letter, 3 letter, 4 letter, 5 letter and 6 letter arrangements.

E.g. FOC, OFC, COF, FFE, FEF, EFF are all examples of possible 3 letter distinct arrangements.

Solution: $\mathbf{180 + 180 + 102 + 42 + 14 + 4 = 522}$

6 letter arrangements total: $\mathbf{180}$

All **6 letters**, divide out repeats: $\frac{6!}{2!2!} = 180$

5 letter arrangements total: $\mathbf{180}$

Same count for COFFE and COEEF $2 \times \frac{5!}{2!} = 120$ Same count for OFFEE and CFFEE $2 \times \frac{5!}{2!2!} = 60$

4 letter arrangements total: 102

Same count for COFF, COEE, CFEE, CFFE, OFFE, OEEF $6 \times \frac{4!}{2!} = 72$; COFE $4! = 24$; FFEE $\frac{4!}{2!2!} = 6$

3 letter arrangements total: $\mathbf{42}$

Same count for COF, COE, CEF, OFE $4 \times 3! = 24$

Same count for CFF, CEE, FEE, FFE, OFF, OEE $6 \times \frac{3!}{2!} = 18$

2 letter arrangements total: $\mathbf{14}$

Same count for CO, CF, CE, OF, OE, FE $6 \times 2! = 12$; Same count for FF, EE $2 \times \frac{2!}{2!} = 2$

1 letter arrangements total: $\mathbf{4}$ C, O, F, E

Tie Break

1 What is the common name of the number 10 raised to the power of 100?
Solution: Googol.

2 A list of 5 natural numbers has mean 5, mode 5, median 5 and range 5. How many such lists are possible?
Solution:2. They are: 3,4,5,5,8 and 2,5,5,6,7

3 Evaluate $\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right)$ giving your answer in its most simplified form.
Solution: $\lim_{n\to 9}\frac{(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{n\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{3+3} = \frac{1}{6}$

4 A "micro-century" is one millionth of a century. How long, to the nearest minute, is a micro-century?
Solution: $\frac{100\times 365\times 24\times 60}{1{,}000{,}000} = 52.56 \sim 53$

5 Find the image of the point $(3,4)$ under axial symmetry in the line $2x+4y-7=0$
Solution: Line through $(3,4) \perp$ to $2x+4y-7=0$: $y-4=2(x-3)$
Point of intersection of both lines: $\left(\frac{3}{2},1\right)$. Then $(3,4)\to\left(\frac{3}{2},1\right)\to(0,-2)$

6 Solve $4\sin^2 x + 2\cos x - 3 = 0$ where $0 \le x \le \pi$ Give your answers in their most simplified form.
Solution: $4(1-\cos^2 x)+2\cos x-3=0$

$$4\cos^2 x - 2\cos x - 1 = 0$$

$$\cos x = \frac{1\pm\sqrt{5}}{4}, x = \frac{3\pi}{5}, \frac{\pi}{5}\ as\ 0 \le x \le \pi$$

7 If $2a+3, 2b+3$ are the roots of the equation $x^2-5x-2=0$, construct an equation with roots $a+2, b+2$. Give your answer in its simplest form of $px^2+qx+r=0$ with $p,q,r \in \mathbb{Z}$.
Solution: $2a+2b+6=5; (2a+3)(2b+3)=-2$
Giving: $2a+2b=-1; \quad ab=-2$
Sum/Product needed $a+b+4=\frac{7}{2}; ab+2a+2b+4=1;$

$$x^2 - x\left(\frac{7}{2}\right)+1=0$$

$$2x^2-7x+2=0$$

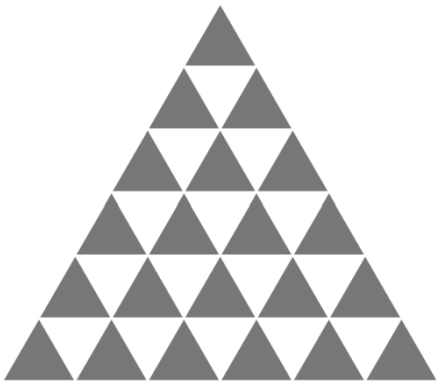
8 Consider the 4 integers: $2, 14, p, 2p-2$. If the standard deviation $\sigma=\sqrt{20}$, find the value of $p \in \mathbb{N}$.
Solution: $\mu = \frac{14+3p}{4}$ Using standard deviation formula and some algebra arrive at:

$$11p^2 - 116p + 300 = 0$$

$$p = 6, p = \frac{50}{11}\ reject\ as\ p \in \mathbb{N}$$

9 A bowl contains raspberries and blackberries. 13 more blackberries are added to the bowl, resulting in making $\frac{4}{5}$ of the bowl ***now*** blackberries. 25 raspberries are *then* added to the bowl resulting in $\frac{3}{5}$ ***now*** raspberries. What fraction of the original berries in the bowl were raspberries?
Solution: Use multiples and ratios or algebra.
$\frac{b+13}{r+b+13}=\frac{4}{5}; \frac{r+25}{r+b+38}=\frac{3}{5}$; Solve $r=5, b=7 \ \therefore \frac{5}{12}$

10 How many regular hexagons are in the following shape: **Solution**: Many ways to count. 11.



ANSWERS

R1	Q1	$\{x \leq -5\} \cup \{x > 0\}$ OR $x \leq -5, x > 0$ Do NOT accept $0 > x \geq -5$ (AND indicated which is incorrect)
	Q2	$(x-4)^2 + (y-2)^2 = 20$ OR $x^2 + y^2 - 8x - 4y = 0$
R2	Q1	5376
	Q2	32
R3	Q1	$2(3) + 3i$
	Q2	$\frac{1}{2}$
R4	Q1	B, A, C
	Q2	16
R5	Q1	$20°$
	Q2	± 1
R6	Q1	$k = 2, b = 1012$
	Q2	(S, L): $(22,5)$; $(15, 11)$; $(8,17)$; $(1,23)$
R7	Q1	2
	Q2	$13.125\ km$
	Q3	9
	Q4	$\frac{p^2 - q^2}{4}$
R8	Q1	27
	Q2	$n = 4, 5, 9, 10$
	Q3	1
	Q4	522
Tiebreak	1	A googol.
	2	2

	3	$\frac{1}{6}$
	4	53
	5	$(0, -2)$
	6	$\frac{\pi}{5}, \frac{3\pi}{5}$
	7	$2x^2 - 7x + 2 = 0$
	8	6
	9	$\frac{5}{12}$
	10	11