Round 1 **Babtha 1**

Question 1

Find the ***exact*** value of the following infinite product: $\sqrt[3]{2025}\left(\sqrt[3]{\sqrt[3]{2025}}\right)\left(\sqrt[3]{\sqrt{\sqrt[3]{\sqrt[3]{2025}}}}\right)\left(\sqrt[3]{\sqrt{\sqrt[3]{\sqrt[3]{\sqrt[3]{2025}}}}}\right)\ldots$

$$\sqrt[3]{2025}\left(\sqrt[3]{\sqrt[3]{2025}}\right)\left(\sqrt[3]{\sqrt{\sqrt[3]{\sqrt[3]{2025}}}}\right)\left(\sqrt[3]{\sqrt{\sqrt[3]{\sqrt[3]{\sqrt[3]{2025}}}}}\right)\ldots = (2025)^{\frac{1}{3}}(2025)^{\frac{1}{9}}(2025)^{\frac{1}{27}}\ldots.$$

$$= 2025^{\frac{\frac{1}{3}}{1-\frac{1}{3}}} = 2025^{\frac{1}{2}} = \sqrt{2025} = 45$$

Where $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = S_\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

Question 2

When $x^3 + px^2 + qx + 1$ is divided by $x - 1$, the remainder is 9.

When $x^3 + px^2 + qx + 1$ is divided by $x + 2$, the remainder is -9.

Find the values of $p, q \in \mathbb{Z}$.

When $x^3 + px^2 + qx + 1$ is divided by $x - 1$, the remainder is 9.

Therefore:

$x - 1$ is a factor of $x^3 + px^2 + qx - 8$ and $x = 1$ is a root.

$x + 2$ is a factor of $x^3 + px^2 + qx + 10$ and $x = -2$ is a root.

=> $1^3 + p(1)^2 + q(1) - 8 = 0$ and => $(-2)^3 + p(-2)^2 + q(-2) + 10 = 0$

=> $p + q = 7$ and => $2p - q = -1$

Solving we get $p = 2, q = 5$

Round 2 **Babtha 2**

Question 1

In the decimal expansion of $\frac{1}{13}$, what is the 2025th digit?

$\frac{1}{13} = 0.076923076923 \ldots$

Starting after the decimal point: The infinite decimal repeats after every 6 digits.

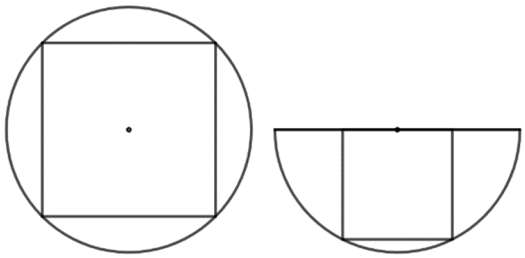
$\frac{2025}{6} = 337 + 3$ Since the remainder is 3, the third digit in the expansion is the required digit which is 6.

Or

Including the zero preceding the decimal point, the required digit which is 7.

Question 2

A circle and semicircle with equal radii are shown. Squares, of different sizes, are inscribed in each shape.



The area of the small square in the semicircle is x.

The area of the large square in the full circle is y.

Find the ratio $x:y$.

Let the length of the radius be r.

Large square area = y, so the side length is $\sqrt{y}$. From Pythagoras:

$$r^2 = \frac{y}{4} + \frac{y}{4} = \frac{y}{2}$$

$$y = 2r^2$$

Small square area =x, so the side length is $\sqrt{x}$.

From Pythagoras:

$$r^2 = x + \frac{x}{4} = \frac{5x}{4}$$

$$x = \frac{4r^2}{5}$$

$$x:y = \frac{4r^2}{5} : 2r^2 = 2:5$$

Round 3 **Babtha 3**

Question 1

Four-digit numbers are made using the numbers 1, 2, 3, 4 such that each number is used only once in the four-digit number e.g. 4213 or 3124. What is the sum of such four-digit numbers?

Start by fixing one digit in the thousands position. Firstly, count the number of numbers that can be made by fixing that digit in that position:

$4 \ast\ast\ast => (1)(3)(2)(1) = 3! =$ The number of different numbers that can be made with 4 in that position

$3 \ast\ast\ast => (1)(3)(2)(1) = 3! =$ The number of different numbers that can be made with 4 in that position

$2 \ast\ast\ast => (1)(3)(2)(1) = 3! =$ The number of different numbers that can be made with 4 in that position

$1 \ast\ast\ast => (1)(3)(2)(1) = 3! =$ The number of different numbers that can be made with 4 in that position

We next add $1 + 2 + 3 + 4$ and afterwards multiply by 10^3 to made it a digit in the thousands position.

Putting that together we get:

$$3!\,(1 + 2 + 3 + 4)(10^3)$$

Similarly we get the following for the hundredths, tens and unit positions:

$$3!\,(1 + 2 + 3 + 4)(10)^2$$

$$3!\,(1 + 2 + 3 + 4)(10)^1$$

$$3!\,(1 + 2 + 3 + 4)(1)$$

Adding all we get $66{,}660$.

Question 2

$1, \omega, \omega^2$ are the three roots of $z^3 - 1 = 0$, with $\omega, \omega^2 \neq 1$;

$\omega, \omega^2 \in \mathbb{C}$. Evaluate:

$$(1(\omega)(\omega^2))^{2025}$$

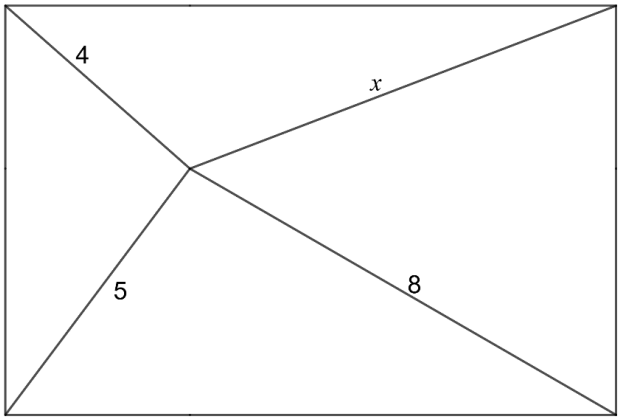
Since ω is a root, then $\omega^3 - 1 = 0 => \ \omega^3 = 1$

$$(1(\omega)(\omega^2))^{2025} = (\omega^3)^{2025} = (1)^{2025} = 1$$

Round 4 **Babtha 4**

Question 1

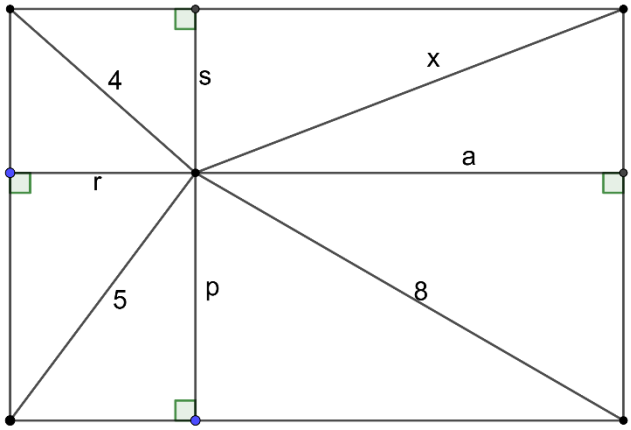
The figure shown is a rectangle. Find the value of x, giving your answer in surd form.



The shortest solution is by using the "Flag Theorem":

$$x^2 + 5^2 = 4^2 + 8^2$$

$$=> x = \sqrt{4^2 + 8^2 - 5^2} = \sqrt{55}$$



To see where this comes from, divide the shape as shown, possible since it is a rectangle.

We get:

$$5^2 = p^2 + r^2$$

$$4^2 = r^2 + s^2 => s^2 = 4^2 - r^2$$

$$8^2 = a^2 + p^2 => a^2 = 8^2 - p^2$$

$$x^2 = a^2 + s^2 = 8^2 - p^2 + 4^2 - r^2$$

$$x^2 = 8^2 + 4^2 - (p^2 + r^2) = 8^2 + 4^2 - 5^2$$

$$x = \sqrt{8^2 + 4^2 - 5^2}$$

Question 2

Maryam tells only the truth on Mondays, Tuesdays, Wednesdays and Thursdays. She tells only lies on all the other days.

William tells only the truth on Fridays, Saturdays, Sundays and Mondays. He tells only lies on all the other days. On one day last week, they both said, "Yesterday I told lies."

On which day of the week was that?

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Maryam	T	T	T	T	L	L	L
William	T	L	L	L	T	T	T

Friday is the only day that will meet the criteria for both on the same day.

Round 5 **Babtha 5**

Question 1

Find the values of $a, b \in \mathbb{N}, a < b$, such that:

$$\log_3 6! = a + a^a \log_3 a + \log_3 b$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 9 \times 16 \times 5$$

$$\log_3 6! = \log_3 9 \times 16 \times 5 = \log_3 9 + \log_3 16 + \log_3 5$$

$$= 2 + \log_3 2^4 + \log_3 5 = 2 + 2^2 \log_3 2 + \log_3 5 = a + a^a \log_3 a + \log_3 b$$

$$a = 2, b = 5$$

Or

$$6! = 3 \times 240$$

$$\log_3 6! = \log_3 3 \times 240 = \log_3 3 + \log_3 240 = 1 + 1^1 \log_3 1 + \log_3 240$$

$$a = 1, b = 240$$

Question 2

Find the range of values of $k \in \mathbb{R}$ for which the following functions ***do not*** intersect:

$$f(x) = 3x^2 + kx - 2$$

$$g(x) = x^2 + 2x - 10$$

If they intersect, then $3x^2 + kx - 2 = x^2 + 2x - 10$ has a real solution.

If they do not intersect, then $3x^2 + kx - 2 = x^2 + 2x - 10$ has no real solutions.

Hence

$$2x^2 + (k-2)x + 8 = 0$$

has no real roots and its discriminant is less than zero:

$$=> (k-2)^2 - 4(2)(8) < 0$$

$$=> k^2 - 4k - 60 < 0$$

$$-6 < k < 10$$

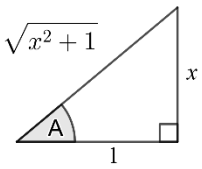
Round 6 **Babtha 6**

Question 1

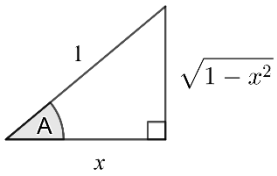
Find the value of $x^2 \in \mathbb{R}$ such that $\cos(\tan^{-1} x) = x$. Give your answer in the form:

$$x^2 = \frac{a + \sqrt{b}}{c}, \qquad a, b, c \in \mathbb{Z}$$

Let $\tan^{-1} x = A \quad => x = \tan A$. From this we get the right-angled triangle shown. Using Pythagoras can get an expression for the hypotenuse.



With $\tan^{-1} x = A$, we can also write $\cos(\tan^{-1} x) = \cos A = x$ and similarly we get the following right-angled triangle and side length expressions.



Equating an expression for $\cos A$ from both triangles we get:

$$\frac{x}{1} = \frac{1}{\sqrt{x^2+1}}$$

$$=> x^2(x^2+1) = 1$$

$$=> (x^2)^2 + x^2 - 1 = 0$$

$$x^2 = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$=> x^2 = \frac{-1 \pm \sqrt{5}}{2}$$

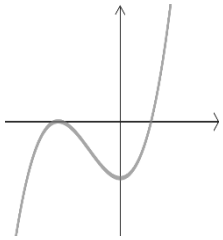
Question 2

Let $f(x) = x^3 + kx^2 - \frac{1}{2}$.

Find the value of $k \in \mathbb{Q}$, for which $f(x) = 0$ has three real roots, two of which are equal.

$$f'(x) = 3x^2 + 2kx$$

Min/max $x = 0, x = -\frac{2k}{3}$



$f(0) = -\frac{1}{2}$ must be the local minimum along with shape of cubic with positive leading coefficient we deduce that at the local maximum, $f\left(-\frac{2k}{3}\right) = 0$

Hence: $\left(-\frac{2k}{3}\right)^3 + k\left(-\frac{2k}{3}\right)^2 - \frac{1}{2} = 0$

$=> k = \frac{3}{2}$

Round 7 **Babtha 7**

Question 1

Solve the simultaneous equations for $x > 0, y > 0 \in \mathbb{R}$, giving your solutions as integers and or surd form if needed:

$$x^{\frac{1}{2}} + y^{\frac{2}{3}} = 6$$

$$x^{-\frac{1}{2}} + y^{-\frac{2}{3}} = \frac{3}{4}$$

Let $x^{1/2} = a, y^{\frac{2}{3}} = b$ =>

$$a + b = 6$$

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{4}$$

=> $4b + 4a = 3ab$

Since $b = 6 - a$ we get:

$$4(6 - a) + 4a = 3a(6 - a)$$

$$=> a^2 - 6a + 8 = 0$$

$$a = 2, b = 4$$

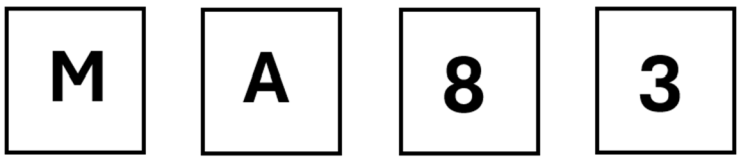
$$=> \sqrt{x} = 2 => x = 4; \qquad y^{\frac{2}{3}} = 4 => y = 8$$

$$a = 4, b = 2$$

$$=> \sqrt{x} = 4 => x = 16; \qquad y^{\frac{2}{3}} = 2 => y = 2\sqrt{2}$$

Question 2

Each of the cards shown has a letter on one side and a number on the other.



Which cards **must** you turn over to test the following rule:

"Any card with an A on one side has a 3 on the other side. "

The rule to be tested is that if there is an A on one side, then the other side is a 3.

We must therefore check the A card. We must also check the 8 card as if the other side was an A, the rule would be false.

We are not testing "if 3 => A" so we do not need to check the 3 card.

Question 3

The points $A(-4,3)$ and $B(-3,-4)$ are points on the circle k: $x^2+y^2=25$.

The perpendicular bisector of AB cuts the circle k at points P in the first quadrant and Q in the third quadrant.

Calculate $|PQ|$.

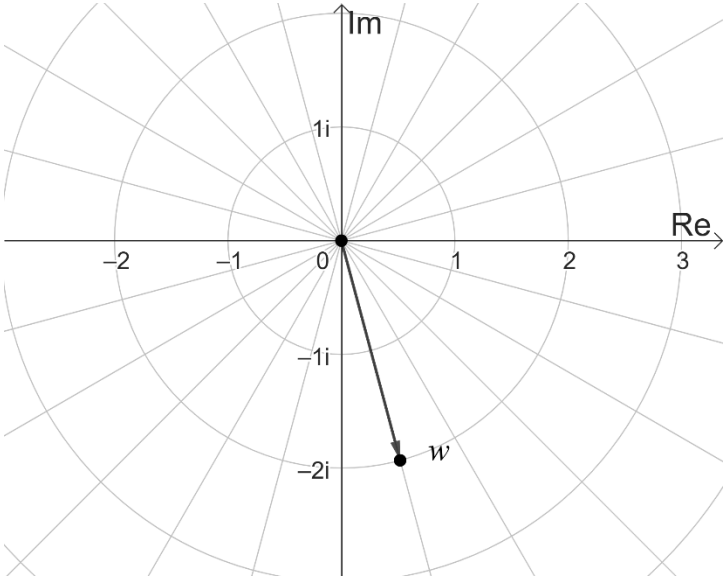
The perpendicular bisector of a chord must go through the centre and hence will cut end points of a diameter on the circle at P and Q. Since the radius is 5, the diameter is 10.

Question 4

One of the three roots, w, of a complex equation:

$z^n = a+bi$, where $a, b \in \mathbb{R}$

is plotted and shown on the diagram, which **is to scale**.



Find the ***original equation in surd form***.

Reading from the scaled diagram one root is given by:

$$w = 2(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12})$$

Since there were only 3 roots, then the original equation had $z^3 = a+bi$.

$$=> w^3 = a+bi$$

$$=> \left(2(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12})\right)^3 = 8\left(\cos\frac{3(19\pi)}{12} + i\sin\frac{19\pi}{4}\right)$$

$$= -4\sqrt{2} + 4\sqrt{2}i = a+bi$$

Hence the original equation was:

$$z^3 = -4\sqrt{2} + 4\sqrt{2}i$$

Round 8 **Babtha 8**

Question 1

The roots of the equation $x^2 + (4p - 5)x + p = 0$ are consecutive integers. Find all possible values of p.

If the roots are consecutive integers, they differ by 1. Hence $\sqrt{b^2 - 4ac} = 1$ which gives:

$$\sqrt{(4p-5)^2 - 4p} = 1$$

$$4p^2 - 11p + 6 = 0$$

$$=> \ p = 2, p = \frac{3}{4}$$

Testing both:

$p = 2$: $x^2 + 3x + 2 = (x + 1)(x + 2)$ valid.

$p = \frac{3}{4}$: $x^2 - 2x + \frac{3}{4} = \left(x - \frac{3}{2}\right)\left(x - \frac{1}{2}\right)$ roots are not integers => invalid.

Question 2

How many different 4-digit codes are there if you cannot have the digit 1 immediately followed by the digit 3 (with repeated digits allowed). E.g. 0055 allowed but 8139 not allowed.

The total number of possible codes without restriction is $10^4 = 10{,}000$. It remains to count the restrictions to remove and subtract from the total. Let * represent any digit from 0 to 9. We count as follows:

$13 **$: There are $(1)(1)(10)(10) = 100$

$* 13 *$: There are $(10)(1)(1)(10) = 100$

$** 13$: There are $(10)(10)(1)(1) = 100$

However, both $13 **$ and $** 13$ count the code 1313 which is an over count by 1.

The total is then: $10{,}000 - 100 - 100 - 100 + 1 = 9701$.

Question 3

A six-digit number is formed by taking a three-digit number and repeating it. E.g. 376 generates 376376 and 211 generates 211211. List any prime numbers that divide ***all*** such possible six-digit numbers.

Consider $376376 = 1000(376) + 376$.

Each six-digit number formed from a three-digit number x, can be written as $1000x + x = 1001x$.

Thus, the prime factors of all such six-digit numbers must be the prime factors of 1001 which are $7, 11, 13$.

Question 4

Find the exact value of:

$$\frac{1}{\sqrt{2025+\sqrt{2025^2-1}}}$$

Give your answer in the form: $\sqrt{b}-\sqrt{a}$, where $b>a;\ a,b\ \in \mathbb{N}$

$$\frac{1}{\sqrt{2025+\sqrt{2025^2-1}}}=\frac{1}{\sqrt{2025+\sqrt{(2026)(2024)}}}=\frac{1}{\sqrt{2025+\sqrt{2(1013)2(1012)}}}$$

$$=\frac{1}{\sqrt{2025+\sqrt{4}\sqrt{(1013)(1012)}}}=\frac{1}{\sqrt{2025+2\sqrt{(1013)(1012)}}}$$

$$=\frac{1}{\sqrt{1013+1012+2\sqrt{(1013)(1012)}}}=\frac{1}{\sqrt{\left(\sqrt{1013}\right)^2+\left(\sqrt{1012}\right)^2+2\sqrt{(1013)}\sqrt{1012}}}$$

$$=\frac{1}{\sqrt{\left(\sqrt{1013}+\sqrt{1012}\right)^2}}=\frac{1}{\sqrt{1013}+\sqrt{1012}}$$

$$=\frac{1}{\sqrt{1013}+\sqrt{1012}}\times\frac{\sqrt{1013}-\sqrt{1012}}{\sqrt{1013}-\sqrt{1012}}=\frac{\sqrt{1013}-\sqrt{1012}}{1013-1012}=\sqrt{1013}-\sqrt{1012}$$

Answers

R1	Q1	45
	Q2	$p = 2, q = 5$
R2	Q1	6 Or 7
	Q2	$2:5$
R3	Q1	66,660
	Q2	1
R4	Q1	$\sqrt{55}$
	Q2	Friday
R5	Q1	$a = 2, b = 5;$ Or $a = 1, b = 240;$
	Q2	$-6 < k < 10$
R6	Q1	$\frac{-1 \pm \sqrt{5}}{2}$
	Q2	$\frac{3}{2} = 1.5$
R7	Q1	$(16, 2\sqrt{2}); (4,8)$
	Q2	A, 8
	Q3	10
	Q4	$z^3 = -4\sqrt{2} + 4\sqrt{2}i$
R8	Q1	2
	Q2	9701
	Q3	$7, 11, 13$
	Q4	$\sqrt{1013} - \sqrt{1012}$