Round 1 **Babtha 1**

Question 1

2024 is a tetrahedral number and, as such, can be written in the form:

$$2024 = \frac{n(n+1)(n+2)}{6}$$

Solve for $n \in \mathbb{N}$.

Solution

Trial & Improvement

$n, n+1, n+2$ are consecutive integers.

$$n(n+1)(n+2) = 2024(6) = 12144$$

Try: $n = 30$ $(30)(31)(32) = 29760$ too big

Try: $n = 17$ $(17)(18)(19) = 5814$ too small

Try: $n = 22$ $(22)(23)(24) = 12144$ just right

$$\Rightarrow n = 22$$

Cubic

$$n^3 + 3n^2 + 2n - 12144 = 0$$

Prime decomposition of 12144: $2^4(3)(11)(23)$

Testing factors from this as roots, $n = 22$ is a solution.

$$\Rightarrow n = 22$$

Question 2

Given that $z = 3 + i$, where $i^2 = -1$, find the value of $k \in \mathbb{R}$ such that:

$$z + \frac{k}{z}$$

is real.

Solution

$$3 + i + \frac{k(3-i)}{(3+i)(3-i)} = 3 + \frac{3k}{10} + i - \frac{ik}{10}$$

Real $\Leftrightarrow 1 - \frac{k}{10} = 0$

$$\Rightarrow k = 10$$

Round 2 **Babtha 2**

Question 1

James is the penalty taker for his soccer team. The probability James scores his first penalty is $\frac{7}{10}$. If subsequent penalties in a match are awarded, the probability of James scoring depends on his previous success or failure. If James scored the previous penalty, the probability of scoring again is $\frac{3}{4}$. If he missed, the probability is $\frac{1}{2}$.

On a particular day three penalties are awarded.

What is the probability James scores exactly two penalties? Give your answer as a fraction written in its simplest form.

Solution

James scores exactly 2 out of 3. Letting S denote score and N denote not score we have three possible situations with specific orders: SSN, SNS, NSS

$$=> \left(\frac{7}{10}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{7}{10}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{10}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{53}{160}$$

Question 2

Let $f(x) = -\frac{5}{2}x + 1$. Solve for $k \in \mathbb{Q}$, for which:

$$f\big(x + f(x)\big) = kf(x)$$

Solution

$$x + f(x) = x - \frac{5x}{2} + 1 = -\frac{3x}{2} + 1$$

$$=> f\big(x + f(x)\big) = f\left(-\frac{3x}{2} + 1\right)$$

$$= -\frac{5}{2}\left(-\frac{3x}{2} + 1\right) + 1$$

$$= -\frac{5}{2}\left(-\frac{3x}{2}\right) - \frac{5}{2} + 1$$

$$= -\frac{5}{2}\left(-\frac{3x}{2}\right) - \frac{3}{2}$$

$$= -\frac{3}{2}\left(-\frac{5x}{2} + 1\right)$$

$$= -\frac{3}{2}f(x)$$

$$=> k = -\frac{3}{2}$$

Round 3 **Babtha 3**

Question 1

A box contains discs of equal size, coloured red, white or black. There are 11 red discs, 7 white discs and x black discs. Three discs are selected together at random from the box, without replacement.

If the probability that all three discs are all different in colour is equal to the probability that they are all black, find the value of x.

Solution

P(All different colours) = P(All discs are black)

$$3!\frac{11(7)(x)}{(18+x)(17+x)(16+x)} = \frac{x(x-1)(x-2)}{(18+x)(17+x)(16+x)}$$

$$462 = x^2 - 3x + 2$$

$$x^2 - 3x - 460 = 0$$

$$(x-23)(x+20) = 0$$

$$x = 23 \; or \; \cancel{x = -20}$$

Or:

$$\frac{\binom{11}{1}\binom{7}{1}\binom{x}{1}}{\binom{18+x}{3}} = \frac{\binom{x}{3}}{\binom{18+x}{3}}$$

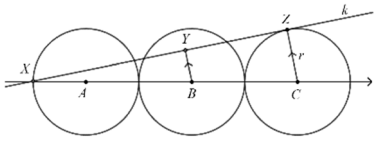
$$77x = \frac{x!}{3!\,(x-3)!}$$

$$x^2 - 3x - 460 = 0$$

$$x = 23 \; or \; \cancel{x = -20}$$

Question 2

3 circles of equal radius r, have centres A, B, and C respectively, on the x-axis. They touch externally as shown in the diagram. The line k is a tangent to the third circle at the point Z. The line k intersects the first circle on the x-axis at the point X. BY is drawn parallel to CZ.



Express $|BY|$ in terms of r. Your answer must be given in its simplest form.

Solution

BY || CZ and hence ΔXBY and ΔXCZ are similar.

Counting radii, $XC = 5r$ and $XB = 3r$.

Hence $\frac{XB}{XC} = \frac{3r}{5r} = \frac{3}{5}$.

$$=> \frac{BY}{CZ} = \frac{3}{5} = \frac{BY}{r}$$

$$=> BY = \frac{3r}{5}$$

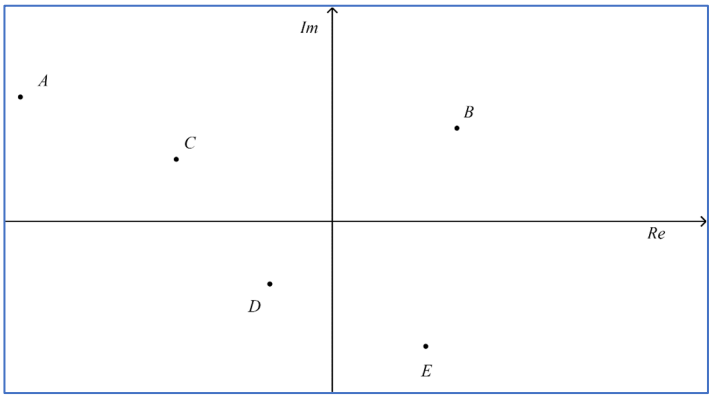
Round 4 **Babtha 4**

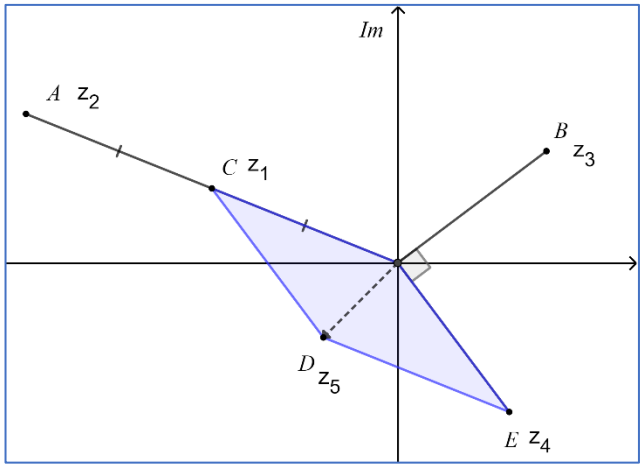
Question 1

5 complex numbers are shown with the same scale on each axis. They satisfy the following conditions:

$z_2 = 2z_1$; $z_5 = z_1 + z_4$; $z_4 = -iz_3$

Identify each point A, B, C, D, E correctly as to their z_n equivalent. **Solution**





Question 2

A battery manufacturer claims a mean battery life of 2190 minutes and standard deviation 76 minutes for a particular battery. They conducted a random sample of 200 batteries, found the mean battery life of the sample and calculated a p-value of 0.037.

The sample mean battery life was lower than the previously established mean. Calculate the sample mean battery life from their sample. Give your answer in minutes, correct to 4 significant figures.

Solution

$$n = 200, \mu = 2190, \sigma = 76, \bar{x} = ?$$

$p - val = 0.037 = 2\big(1 - P(Z < |z|)\big)$

$$0.037 = 2\big(1 - P(Z < |z|)\big)$$

$$0.0185 = 1 - P(Z < |z|)$$

$$P(Z < |z|) = 1 - 0.0185 = .9815$$

Closest z-score(s) to ~ 0.9815 are either 2.09 or 2.08.

Taking $|z| = 2.09$

$$|z| = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Two choices:

$+2.09 = \frac{\bar{x}-2190}{\frac{76}{\sqrt{200}}}$ or $-2.09 = \frac{\bar{x}-2190}{\frac{76}{\sqrt{200}}}$

We need the $\bar{x}$ to be lower than the original μ so: $-2.09 = \frac{\bar{x}-2190}{\frac{76}{\sqrt{200}}}$

$$\bar{x} = 2178.7683 \sim 2179$$

Using a z-score of 2.08 also gives $\bar{x} = 2178.822 \sim 2179$.

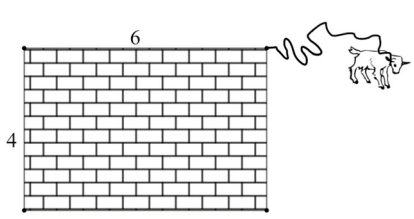
Round 5 **Babtha 5**

Question 1

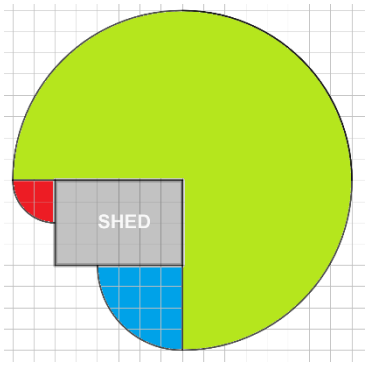
Gertie, the mathematical goat, is tied by a rope to the corner of a 4x6 metre solid brick shed that is surrounded entirely by grass. The length of the rope is 8 metres. The goat can move freely in all directions that the rope and shed do not impede.

Calculate the area of grass that the goat can eat.

Give your answer in terms of π.



Solution



Total Area

$$\frac{3}{4}\pi(8)^2 + \frac{1}{4}\pi(4)^2 + \frac{1}{4}\pi(2)^2 = 53\pi\ m^2$$

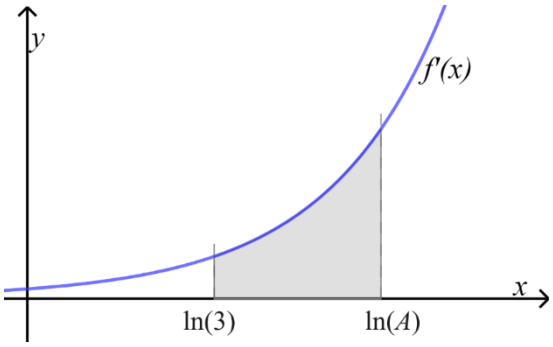
Question 2

Let $f(x) = \frac{2}{3}(1 + e^x)^{\frac{3}{2}} + C$, where C is a constant.

The diagram shows part of the graph of the function $f'(x)$. The shaded region in the diagram is between the graph and the x-axis, from $x = \ln(3)$ to $x = \ln(A)$.

This shaded region has an area of $\frac{38}{3}$ square units.

Find the value of A.



Solution

$$\frac{2}{3}\left(1 + e^{\ln A}\right)^{\frac{3}{2}} - \frac{2}{3}(1 + e^3)^{\frac{3}{2}} = \frac{38}{3}$$

$$(1 + A)^{\frac{3}{2}} - (1 + 3)^{\frac{3}{2}} = 19$$

$$(1 + A)^{\frac{3}{2}} = 27$$

$$1 + A = 9$$

$$A = 8$$

Round 6 **Babtha 6**

Question 1

Find the values of k for which the four numbers below have standard deviation $\sigma = \sqrt{20}$

$$1, \quad k, \quad 3k-2, \quad 9$$

Solution

$$\bar{x} = \frac{1+k+3k-2+9}{4} = 2+k$$

$$\sqrt{\frac{(1-2-k)^2+(k-2-k)^2+(3k-2-2-k)^2+(9-2-k)^2}{4}} = \sqrt{20}$$

$$\Rightarrow 3k^2-14k-5=0$$

$$\Rightarrow k = -\frac{1}{3}, k = 5$$

Question 2

Let $f(x) = \frac{e^{3x}+e^{-3x}}{3}$. Find the value of p for which:

$$f''(x) = pf(x)$$

where $p \in \mathbb{Z}$, and $f''(x)$ is the second derivative of $f(x)$.

Solution

$f(x) = \frac{1}{3}e^{3x} + \frac{1}{3}e^{-3x}$

$f'(x) = e^{3x} - e^{-3x}$

$$f''(x) = 3e^{3x} + 3e^{-3x}$$

$$= 9\left(\frac{1}{3}e^{3x} + \frac{1}{3}e^{-3x}\right)$$

$$= 9f(x)$$

$p = 9$

Round 7 **Babtha 7**

Question 1

Sand is pouring out of a tube at 1 cubic meter per second. It forms a pile which has the shape of a right cone. The height of the cone is equal to the radius. How fast is the sandpile rising when it is 2 meters high? Give your answer in terms of π.

Solution

$\frac{dV}{dt} = 1 \qquad h = r$

$V = \frac{1}{3}\pi(h)^2 h = \frac{1}{3}\pi h^3 \qquad => \frac{dV}{dh} = \pi h^2 \qquad => \frac{dh}{dV} = \frac{1}{\pi h^2}$

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \left(\frac{1}{\pi h^2}\right)(1)$$

$$\frac{dh}{dt}|_{h=2} = \frac{1}{\pi(2)^2} = \frac{1}{4\pi}\ m/s$$

Question 2

$u_1, u_2, u_3, u_4, u_5, u_6, \ldots\ldots$ is a sequence where $u_1 = 4$ and $u_{n+1} = (-1)^n u_n + 20$. Find the value of u_{2024}.

Solution

$$u_2 = (-1)^1(4) + 20 = 16$$

$$u_3 = (-1)^2(16) + 20 = 36$$

$$u_4 = (-1)^3(36) + 20 = -16$$

$$u_5 = (-1)^4(-16) + 20 = 4$$

$$=> u_6 = u_2 = 16$$

$$4, 16, 36, -16, 4, 16, \ldots$$

Repeats after 4.

$$\frac{2024}{4} = 506$$

2024 is a multiple of 4, hence $u_{2024} = u_4 = -16$.

Question 3

A is an obtuse angle such that:

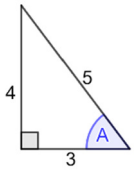
$$\sin\left(A+\frac{\pi}{3}\right)+\sin\left(A-\frac{\pi}{3}\right)=\frac{4}{5}$$

Find the value of $\tan A$.

Solution

$$\sin\left(A+\frac{\pi}{3}\right)+\sin\left(A-\frac{\pi}{3}\right)=2\sin A\cos\frac{\pi}{3}=\frac{4}{5}$$

$$\sin A=\frac{4}{5}$$



If A were acute, then $\tan A=\frac{4}{3}$.

For A obtuse $\tan(\pi-A)=-\tan A=-\frac{4}{3}$.

Question 4

A maths teacher has 7 pencils to give to four students and must give out all 7 pencils. The maths teacher can give none, or a mix of equal amounts and unequal amounts to the four students. In how many ways can the teacher do this?

Solution

Let the students be A, B, C, D. Distribute the pencils as below and count the total permutations among students possible.

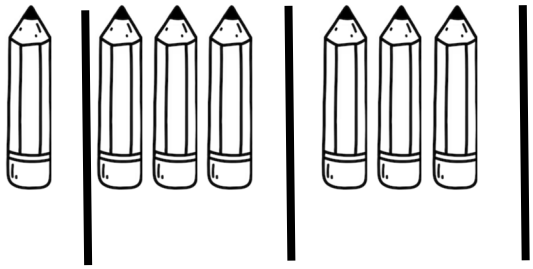
A	B	C	D	Permutations	Totals
0	0	0	7	4!/3!	4
0	0	6	1	4!/2!	12
0	0	5	2	4!/2!	12
0	0	4	3	4!/2!	12
0	1	1	5	4!/2!	12
0	1	2	4	4!	24
0	1	3	3	4!/2!	12
0	2	2	3	4!/2!	12
1	1	1	4	4!/3!	4
1	1	2	3	4!/2!	12
1	2	2	2	4!/3!	4
					120

Final Total

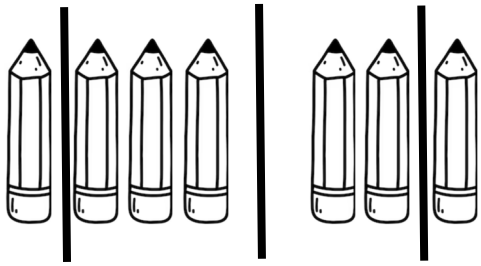
Alternative Solution

Visually, if we line up the 7 pencils, we need only 3 cuts/bars to divide the pencils into 4 groups e.g.

A gets 1, B gets 3, C gets 3 and D gets 0.



Or A gets 1, B gets 3, C gets 2 and D gets 1.



And so forth.

Consider this now as a counting question:

In how many ways can you choose 3 positions for the bars among all possible positions?

We take a pencil and each bar to be a 'position'. There are 3 + 7 = 10 positions available, hence $\binom{10}{3} = 120$

Round 8 **Babtha 8**

Question 1

Let $y = \frac{1}{x} + \cos 2x$. Find the value of $n \in \mathbb{Z}$, such that:

$$\frac{d^5y}{dx^5} = -\frac{n!}{x^6} - 2^n \sin 2x.$$

Solution

Shortcut

Notice that for $\cos 2x$ and subsequent derivatives, each will be multiplied by an extra 2. Thus for the fifth derivative, this must be $2^5$and hence $n = 5$.

$$y = \frac{1}{x} + \cos 2x = x^{-1} + \cos 2x$$

$$\frac{dy}{dx} = -1x^{-2} - 2\sin 2x$$

$$\frac{d^2y}{dx^2} = +2x^{-3} - 2^2 \cos 2x$$

$$\frac{d^3y}{dx^3} = +2(3)x^{-4} + 2^3 \sin 2x$$

$$\frac{d^4y}{dx^4} = +2(3)(4)x^{-5} + 2^4 \cos 2x$$

$$\frac{d^5y}{dx^5} = +2(3)(4)(5)x^{-6} - 2^5 \cos 2x = -\frac{5!}{x^6} - 2^5 \sin 2x.$$

Question 2

The National Lottery Daily Million is a draw game in which you can pick any six numbers from 1 to 39. At each draw six random numbers are drawn. If you match all 6 numbers, you win €1 million. It costs €1 to enter. You play the game once. What is your overall expected profit/loss, taking the entry fee into account? Give your answer correct to 4 decimal places.

Solution

Expected Value from Lotto

x	$P(x)$	$x.P(x)$
€0	$1 - \frac{1}{\binom{39}{6}}$	0
€1,000,000	$\frac{1}{\binom{39}{6}}$	$\frac{€1{,}000{,}000}{\binom{39}{6}}$

Expected Profit/Loss

$$\frac{€1{,}000{,}000}{\binom{39}{6}} - €1 = -€0.693498 \sim -€0.6935$$

Question 3

Let $f(x) = x^2 - 7x + 12$. Find the range of values of

$x \in \mathbb{R}, x \neq 2, x \neq 3$, for which

$$\frac{f(x)}{f(x+1)} \geq 3$$

Solution

$f(x) = x^2 - 7x + 12 = (x-4)(x-3)$

$f(x+1) = (x+1)^2 - 7(x+1) + 12 = (x-2)(x-3)$

$$\frac{f(x)}{f(x+1)} \geq 3$$

$$\frac{(x-4)(x-3)}{(x-2)(x-3)} \geq 3$$

$$(x-4)(x-2) \geq 3(x-2)^2$$

$$x^2 - 3x + 2 \leq 0$$

$$(x-1)(x-2) \leq 0$$

Roots $x = 1, 2$ but given $x \neq 2$.

Hence $1 \leq x < 2$

Question 4

What is the remainder when 2024^{2024} is divided by 9?

Solution

2024 has reminder 8 when divided by 9. Thus the question is equivalent to:

What is the remainder when 8^{2024} is divided by 9?

$8^2 \div 9$ has remainder 1.

$8^3 \div 9$ has remainder 8.

$8^4 \div 9$ has remainder 1.

$8^5 \div 9$ has remainder 8.

$8^6 \div 9$ has remainder 1.

Pattern:

$8^{odd} \div 9$ has remainder 8.

$8^{even} \div 9$ has remainder 1.

$\Rightarrow 8^{2024} \div 9$ has remainder 1.

Answers

R1	Q1	22
	Q2	10
R2	Q1	$\frac{53}{160}$
	Q2	$-\frac{3}{2}$
R3	Q1	23
	Q2	$\frac{3r}{5}$
R4	Q1	$A = z_2, \quad B = z_3, C = z_1, \quad D = z_5, \quad E = z_4$
	Q2	2179
R5	Q1	$53\pi\ m^2$
	Q2	8
R6	Q1	$5;\ -\frac{1}{3}$
	Q2	9
R7	Q1	$\frac{1}{4\pi}\ m/s$
	Q2	-16
	Q3	$-\frac{4}{3}$
	Q4	120
R8	Q1	5
	Q2	$-$ €0.6935 or €0.6935 loss
	Q3	$1 \leq x < 2$
	Q4	1