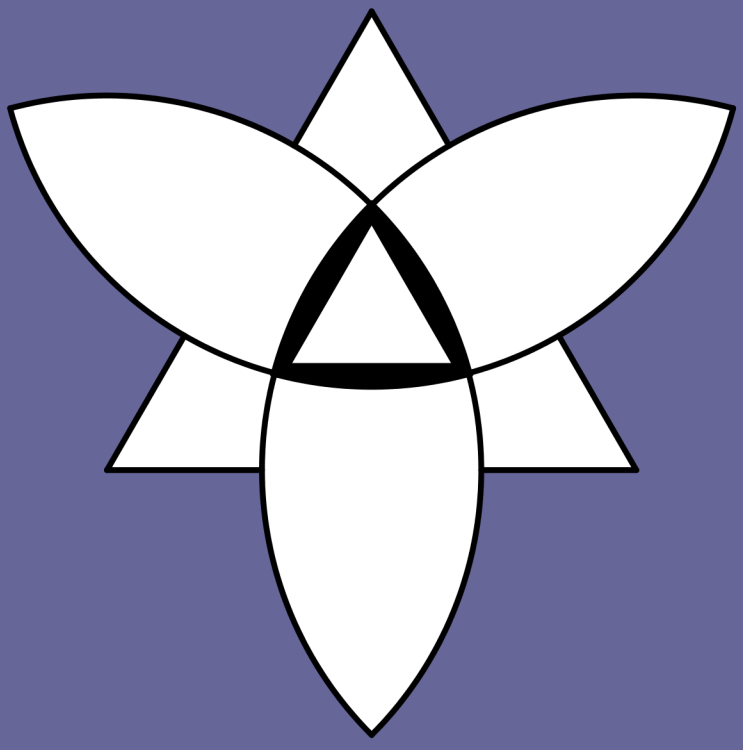
Cumann Oidí Matamaitice na hÉireann

Newsletter

Irish Mathematics Teachers' Association



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Acknowledgement
The IMTA would like to thank **Folens Publishers** for sponsoring the printing and distribution of this *Newsletter*.

Welcome to the 2018 edition of the Newsletter. The aim is to include articles on mathematics itself, mathematics education/pedagogy, the history of mathematics and — to counter the view that "nobody uses mathematics except maths teachers" — applications of mathematics.

Under mathematics education/pedagogy the Maths Development Team - PDST gives a report on Maths Counts 2018 which includes a valuable lesson plan on teaching a problem solving topic by the keynote speaker Professor Tad Watanabe. Cornelia Connolly and Tony Hall write about the importance of mathematical knowledge for the mathematics teacher and how their BA in mathematics and education develops that. Brendan O'Sullivan reports on MathsFest 2017 which was held in Cork while the editor writes about calculating with negative integers.

Under the history of mathematics Ciarán Mac an Bhaird writes about the great mathematician, Euler and how he went about his work. In it we not only learn the history but also the mathematics involved so this article falls under mathematics and mathematics education also.

Under mathematics Neil Hallinan extends our knowledge of the rotations of a cube via a very tricky puzzle involving one. For this it is very helpful to have a cube to hand to try out the rotations. If you don't have one then construct the net of a cube and put it together, labelling the faces top, bottom, left, right, front and back. Peter's Problem 2017 and solution are provided by Dominic Guinan and Neil Hallinan.

Under applications of mathematics we have contributions from two Young Scientist projects. Yaduvir Harangi develops an extension of Escott's infinite product method for calculating square roots. Adam Kelly and Sam Enright write about their method for visualising higher dimensional objects. This project involved not only mathematics but computer programming as well. The emphasis in the article is on the mathematics. It is heartening to see the level of mathematical understanding attained by these three young students; it is well above higher level Leaving Certificate standard.

Two men who both had the remarkable distinction of making contributions to the IMTA right up to their nineties are remembered here by members of the Dublin Branch.

Thanks to the authors who gave their time and effort to compose their articles. Thanks also to Elizabeth Ivory and Sue Reilly and to their team at **Folens** for their very efficient work.

Information about the IMTA is available at `www.imta.ie` and also on Facebook at `https://www.facebook.com/Irish-Maths-Teachers-Association-355409477856681/`. MathsFest 2018 takes place at the Castletroy Park Hotel, Limerick on $19^{\text{th}}/20^{\text{th}}$ October 2018. See `http://www.mathsfest.ie`.

Note that the views expressed in this Newsletter are those of the individual authors and do not necessarily reflect the position of the IMTA. While every care has been taken to ensure that the information in this publication is up-to-date and correct, no responsibility will be taken by the IMTA for any errors that might occur.

If you would like to have an article published in the Newsletter or have an idea for an article then email the editor at imta.newsletter@gmail.com. Michael O'Loughlin (editor)

Maths Counts: Engaging Post-Primary Mathematics Teachers in *Lesson Study*

A conference entitled, "**Maths Counts: Engaging Post-Primary Mathematics Teachers in *Lesson Study***" was hosted on 11th and 12th May in the O'Brien Centre for Science, UCD, Dublin. The conference was open to all teachers of Mathematics, prospective teachers of Mathematics and third-level personnel. This was the fifth annual Maths Counts conference organised by the Maths Development Team - PDST; approximately 400 post-primary teachers attended the festival of teaching and learning Mathematics through problem solving.

The international guest was Tad Watanabe a Professor of Mathematics Education and Assistant Department Chair in the Department of Mathematics at Kennesaw State University, Kennesaw, Georgia. As a native of Japan, Tad was always interested in Mathematics education practices in Japan such as *Lesson Study* and teaching through problem solving. Since 2000, he has attended numerous research lessons in Japan, both at primary and secondary level and has written several articles on *Lesson Study* as well as Japanese elementary school mathematics curriculum materials. He has worked with a number of *Lesson Study* groups internationally and in the United States. He is also one of the leaders of the *Lesson Study* Immersion Program organised by Tokyo Gakugei University every summer. Professor Watanabe attended the first Maths Counts Conference in 2013 where he wowed teachers with his approach to teaching geometry using open-ended problems. This year, Professor Watanabe prepared a series of 3 one-hour demonstration lessons entitled "*Fractions and their Curious Behaviour*". Please see Appendix 1 for the full lesson proposal that Professor Watanabe drafted specifically for Maths Counts.

The first demonstration lesson was taught to a group of fifth year Higher Level students in their own school — Ardee Community School. Professor Tad Watanabe continued the second demonstration lesson live at the Conference on Friday night to the same group of students, watched by an audience of 400 teachers. He also delivered a keynote address: "*Lesson Study in Ireland: Where do we go from Here*?". The third demonstration lesson was taught live to the same group of students on Saturday morning.

Following this, 3 research lessons each of 1 hour duration which were developed over the Lesson Study cycle were taught live by our own teachers in parallel sessions. The teachers involved in the research teams participated in the observation of the lesson and gathered data to inform the post-lesson discussion and reflection. The students from each of the research team's own school had never been taught or had any exposure to the problem-solving lesson prior to the Conference. Professor Tad Watanabe acted as a "*knowledgeable other*" at one of these lessons. The 3 lessons were taught in front of conference attendees. The live problem-solving lessons were as follows:

1. *Plane Clothes:* Teachers Pat O'Leary, Eimear White, Celine Buckley and Declan Cronin from Coláiste na Toirbhirte, Bandon, Co. Cork designed a lesson to encourage students to be flexible in their thinking, specifically in the area of geometry. They hoped that the "area" problem they created would help students recognise the benefits of applying algebra, in the form of coordinate geometry, to geometric problems and that students would also identify the benefits of using surds when dealing with irrational numbers. This lesson was taught to a group of second year students.
2. *Heavenly Slopes:* Teachers Emer Hickey and David Walton from Coláiste Éamann Rís,

Callan, Co. Kilkenny and Aoife Cronin from Holy Family Secondary School, Newbridge, Co. Kildare collaborated across their schools to design a lesson which aimed to help students develop a deep understanding of the slopes of parallel and perpendicular lines. They hoped that the problem they had chosen would challenge students to apply their prior knowledge to forge new links between geometry and coordinate geometry and that in the process of doing so they would discover the relationship between the slopes of parallel lines and perpendicular lines. This lesson was taught to a group of third year students.

3. *Untangle the Angles:* Teachers Brigid Ring, Aishling Donohue and Lauren Matthews from Ardee Community School, Co. Louth, designed a lesson which focused on encouraging students to use prior knowledge, in the form of axioms, theorems and constructions, to identify a number of new geometric relationships. They hoped that this lesson would provide students with an introduction to the type of reasoning needed when dealing with geometry problems and that this will form a solid foundation for being able to prove various relationships at a later stage. This lesson was taught to a group of first year students.

Following the live lessons, a further 24 lessons from the 72 lessons developed over the course of the Lesson Study cycle were showcased as interactive workshops where attendees at the Conference got the chance to hear, first-hand, teachers describe their experience of designing their problem-solving lesson, how the teaching of the lesson went, what they learned about students understanding of mathematics and how they feel about teaching mathematics through problem solving. The 24 workshops were divided into three time slots with 8 workshops happening simultaneously during each slot. In addition to the interactive workshops and by way of celebrating the huge amount of work carried out by over 450 teachers from 150 participating schools in *Lesson Study*, poster presentations (72 in total) summarising the key points of each lesson proposal and the research findings of each group were displayed throughout the Conference in the foyer of the O'Brien Building. The conference concluded with an address by Shane Flanagan, Team Leader for Mathematics with the Junior Cycle for Teachers Support Service. Shane presented on Mathematics in the context of Junior Cycle reform and the Framework for Junior Cycle (2015).

All materials from the conference can be accessed online at `www.projectmaths.ie`

APPENDIX 1:

Mathematics Lesson Proposal

Maths Counts 2018

May 11^{th}, 2018 Ardee Community School

May 11^{th} & 12^{th}, 2018 University College, Dublin

Teacher: Tad Watanabe

1. Name of Unit: Fractions and their curious behaviour

2. Goals of the unit

   - Students will understand that algebraic expressions are useful explanation tools.
   - Students will extend their ability to add/subtract algebraic fractions.
   - Students will continue to develop mathematical proficiency.

3. About the unit

Addition and subtraction of algebraic fractions challenge many students throughout the world, and apparently, Irish students are not an exception. This topic is included in both Junior and Senior Cycle syllabi even though only advanced students in Junior Cycle study this topic. There are probably many reasons why this topic is so challenging. For one, fractions challenge many primary students. Moreover, algebra in general is also challenging to many students. The Irish syllabi state, "Two aspects of algebra that underlie all others are algebra as a systematic way of expressing generality and abstraction, including algebra as generalised arithmetic, and algebra as syntactically guided transformations of symbols." Thus, if students' understanding of fraction arithmetic is fragile, it is not surprising that they have difficulty "generalising" the arithmetic process into "syntactically guided transformations of symbols."
This 3-lesson unit is designed with the following ideas in mind:

- Students will understand symbol manipulation procedures better if they realise the usefulness of algebra as their own reasoning tool.
- Students' learning will be enhanced when the need for algebraic transformations arises naturally in a problem.

The second bullet is consistent with what the Irish syllabi say about problem solving:

> In the mathematics classroom problem solving should not be met in isolation, but should permeate all aspects of the teaching and learning experience. Problems may concern purely mathematical matters or some applied context.

In Lessons 1 and 2, students will first formalise the procedure for addition and subtraction of fractions, $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$.
We will then explore a very common mistake students often make, $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$
Looking at several examples, students will verify that the fraction, $\frac{a+c}{b+d}$, actually lies in between the original two fractions. Students will then try to generalise this observation using algebraic expressions:
Given $\frac{a}{b} < \frac{c}{d}$ $(b \neq 0,\ d \neq 0)$, show $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Students will then be told that one way mathematicians compare two quantities represented by algebraic expressions is to consider their difference. If we can determine the difference is positive or negative, we can determine which quantity is the greater of the two. Students will then try to prove the two inequalities by trying to determine the differences between $\frac{a}{b}$ and $\frac{a+c}{b+d}$ and also $\frac{c}{d}$ and $\frac{a+c}{b+d}$ by calculating their differences.

When students calculate the differences of these quantities, they are left with algebraic fractions. At this point, they have to realise that in order to determine whether or not the differences are positive or negative, they must use the given information, $\frac{a}{b} < \frac{c}{d}$. Because $\frac{a}{b} < \frac{c}{d}$ they know that $\frac{(ad-bc)}{bd} < 0$. Because b and d are positive integers, the only way

this algebraic fraction is negative is when $ad - bc$ is negative. By using this fact, students will then be able to conclude that the two inequalities are indeed true.

In Lesson 3, students are asked which fraction is greater, $\frac{10000000001}{9898989898}$ or $\frac{10000000002}{9898989899}$. When these fractions are converted to decimal numbers using a TI-84 calculator, both will return 1.010204082. However, it seems unlikely that these two fractions are equal. Upon closer observation, students will notice that the second fraction can be obtained by adding 1 to both the numerator and the denominator of the first fraction. We will then explore if there is any pattern when 1 is added to both the numerator and the denominator of a fraction, or compare $\frac{a}{b}$ and $\frac{(a+1)}{(b+1)}$ where a and b are positive integers. As we did in lessons 1 and 2, students will examine the difference between these two fractions. When they calculate the difference, once again, they are left with an algebraic fraction. However, unlike the first problem, there was no condition given. Thus, students realise that the relationship depends on the relationship between a and b, or whether the fraction, $\frac{a}{b}$, is a proper fraction or an improper fraction.

In both of these tasks, students need to find the differences between algebraic fractions. Moreover, by using algebraic expressions and algebraic transformations, students can justify that the relationships they observed hold true in general. Therefore, they can experience the usefulness of algebraic expressions as an explanation tool. It is hoped that through these experiences of formalising the addition and subtraction procedures and making use of subtraction of algebraic fractions, students will deepen their understanding of the particular algebraic transformation.

Finally, developing students' mathematical proficiency is an important goal in the Irish mathematics syllabi for both Junior and Senior Cycles. Concerning the 5 strands of mathematical proficiency, the authors of *Adding It Up* (National Research Council, 2001) noted:

> These strands are not independent; they represent different aspects of a complex whole. ... The most important observation we make here, one stressed throughout this report, is that **the five strands are interwoven and interdependent in the development of proficiency in mathematics** ... helping children acquire mathematical proficiency calls for instructional programs that address all its strands. (emphasis original, p. 116)

Thus, the lessons in this unit will not focus on a specific proficiency. Rather, through these lessons, all 5 strands will be addressed.

4. About mathematics

Every teacher who taught addition and subtraction of fractions or algebraic fractions has seen students mistakenly add the numerators together and the denominators together. Both fractions and decimal numbers are useful representations for expressing quantities involving amounts less than 1. Decimal numbers extend the base-10 numeration system of whole numbers, and as a result, the same computational algorithms may be adopted. Fractions, on the other hand, allow us to create units more flexibly by equally partitioning 1 into however

many pieces as necessary. This flexibility allows us to express quantities such as $\frac{1}{3}$ more concisely than what we can do with decimal numbers. However, it also means that different ways of calculation are needed with fractions.

With respect to addition and subtraction, students must understand that the denominator specifies the fraction unit in a given number while the numerator counts the number of that unit. Thus, when adding or subtracting, we can simply add or subtract the numerators if they refer to the same unit, or the denominator. Thus, in order to add or subtract fractions, or algebraic fractions, we need a common unit. Sometimes, in primary schools, the use of the least common denominator is over-emphasised — and not enough attention given to the idea that the product of the denominators will always be a possible common denominator. Understanding that idea more explicitly might help some of the students in post-primary schools as they deal with algebraic fractions.

Interestingly, this process of adding the numerators together and the denominators together has some mathematical significance. The resulting fraction is called the *mediant* of the original fractions. Moreover, if we consider (b, a) associated with the fraction, $\frac{a}{b}$, the operation to find the mediant of $\frac{a}{b}$ and $\frac{c}{d}$ corresponds to vector addition. There is an interesting article about the tree of fractions (called the Stern-Brocot tree) consisting of a series of mediants starting with fractions $\frac{0}{1}$ and $\frac{1}{1}$ by David Austin (`http://www.ams.org/publicoutreach/feature-column/fcarc-stern-brocot`). Austin describes some of the interesting properties of this tree of fractions, and how a famous 19^{th} century clock maker utilised this tree in clock making.

The second task, investigating the relationship between $\dfrac{a}{b}$ and $\dfrac{(a+1)}{(b+1)}$ can be easily connected to the first task as $\dfrac{(a+1)}{(b+1)}$ is the mediant of $\dfrac{a}{b}$ and $\dfrac{1}{1}$. Thus, $\dfrac{(a+1)}{(b+1)}$ will be between $\dfrac{a}{b}$ and $\dfrac{1}{1}$, or 1. Therefore, if $\dfrac{a}{b}$ is a proper fraction, the successive mediants of $\dfrac{a}{b}$ and $\dfrac{1}{1}$ will get larger and larger towards 1, while if $\dfrac{a}{b}$ is an improper fraction, they will get smaller and smaller towards 1. I am anticipating (hoping?) that no student will see this relationship at the beginning of the lesson. However, at the end of the third lesson, I may ask students to observe if they notice any relationship between the two tasks they worked on.

As I planned this unit, I debated the order of these tasks. As I just noted, the second task can be trivial if students see the relationship of the task to the operation to find the mediant. In addition, the calculation of the difference in the second task is simpler than the calculations in the first task. However, the set up for the first task is a more natural context to talk about generalisation of fraction addition and subtraction. Moreover, after students find the difference in the second task, they will have to actually consider two cases, $a < b$ or $a > b$, and generate distinct conclusions. I felt this reasoning was more mathematically challenging than what is required in the first task. The optimal order of these tasks is something we need to examine further based on how students respond to these lessons.

5. Unit Plan

Lesson #	Learning Task
1	Explore fraction addition. Write the formula for fraction addition/subtraction using variables. Problematise the observation, given two non-equivalent fractions, the third fraction obtained by adding the numerators and the denominators together will be in between the original two, using variables. Express this observation using algebraic expressions. Think about how to explain why this observation is true.
2	Prove: Given two non-equivalent fractions $\frac{a}{b} < \frac{c}{d}$, show that $$\frac{a}{b} < \frac{(a+c)}{(b+d)} < \frac{c}{d}$$ Discuss how to subtract algebraic fractions.
3	Examine if there is any pattern when 1 is added to both the numerator and the denominator of a fraction. Express this enquiry using algebraic expressions. Students will "read/interpret" the algebraic fraction to reach a conclusion.

Flow of Lesson — Days 1 & 2

Steps	**Learning task, teacher questions, anticipated responses**	**Instructional points of consideration, assessment**
1	Introduction T: Do you like fractions? T: What's so different about fractions? T: How do you calculate $\frac{1}{4}+\frac{5}{6}$? $\frac{1}{3}+\frac{3}{5}$? T: Why do we have to find a common denominator?	Include an example where the least common denominator is not the product of the denominators.
2	Generalise fraction addition and subtraction. T: How can we show how to add and subtract fractions using algebra?	Can students use variables to express general fractions? Emphasise the idea that we need a common denominator, not necessarily the least common, in order to add or subtract fractions.

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
3	Observe patterns in the two fractions and the mediant. T: What's the common mistake students often make when adding fractions? S: $\frac{1}{4}+\frac{5}{6}=\frac{(1+5)}{(4+6)}=\frac{6}{10}=\frac{3}{5}$ S: $\frac{1}{3}+\frac{3}{5}=\frac{(1+3)}{(3+5)}=\frac{4}{8}=\frac{1}{2}$ T: We know the correct answers, but even without knowing the correct answer, is there a way we can tell that these answers cannot be the correct answers? S: $\frac{3}{5}$ is less than $\frac{5}{6}$. S: $\frac{1}{2}$ is less than $\frac{3}{5}$. T: I noticed that $\frac{3}{5}$ is between $\frac{1}{4}$ and $\frac{5}{6}$, and $\frac{1}{2}$ is between $\frac{1}{3}$ and $\frac{3}{5}$. Do you think when we add the numerators together and the denominators together, the new fraction is always in between the original two fractions? S: I think so.	Do students recognise that the sum should be greater than both addenda? Have students try out additional examples on their own to confirm this pattern.
4	State the observation using algebra. T: Can we express your observation using algebra? S: We can call the smaller fraction $\frac{a}{b}$. S: We can call the bigger fraction $\frac{c}{d}$. S: $\frac{a}{b}<\frac{c}{d}$. S: $\frac{a}{b}<\frac{(a+c)}{(b+d)}<\frac{c}{d}$.	Can students use algebraic expressions to express the relationship observed above? For the purpose of this lesson, a, b, c, and d need to be positive.

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
5	Think about how to show if this is true. T: How can we show $\frac{a}{b} < \frac{(a+c)}{(b+d)}$?	Give students time to think about ways to compare. However, if the idea of finding the difference does not come up, the teacher will provide that idea.
	T: When mathematicians want to know which expression is greater than the other, they find the difference of the two expressions. T: If $A - B > 0$, which is greater, A or B? S: A	
	T: If $A - B < 0$, which is greater, A or B? S: B	Focus on the first inequality.
	T: Do you think we can use this idea to show $\frac{a}{b} < \frac{(a+c)}{(b+d)}$? S: Yes.	Possible Exit Point for Day 1 T: OK, we will try to show this inequality tomorrow. Have students write a brief reflection — what they learned, what they are not sure about, what they want to learn next, etc.
6	Independent work Students will try to find the difference between $\frac{a}{b}$ and $\frac{(a+c)}{(b+d)}$. S: $\frac{a}{b} - \frac{a+c}{b+d} = \frac{a(b+d)-b(a+c)}{b(b+d)} = \frac{ad-bc}{b(b+d)}$. S: $\frac{a+c}{b+d} - \frac{a}{b} = \frac{(a+c)b-a(b+d)}{b(b+d)} = \frac{bc-ad}{b(b+d)}$.	As necessary, discuss the procedure of subtracting algebraic fractions — either individually, in a small group, or as a whole class. Possible prompts: Does the formula we created earlier help? How do we add/subtract fractions? What can we use as a common denominator?

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
7	Sharing and discussing [Have a couple of students show their work on the white board.] T: Did they calculate the difference correctly?	Verify the differences are calculated correctly.
	T: So, how can we tell if the difference is positive or negative?	Possible Exit Point for Day 1 T: Let's think about how we can determine if the difference is positive or negative tomorrow.
	T: What do we know about $\frac{a}{b}$ and $\frac{c}{d}$?	Have students write a brief reflection — what they learned, what they are not sure about, what they want to learn next, etc.
	S: $\frac{a}{b} < \frac{c}{d}$. S: $\frac{a}{b} - \frac{c}{d} < 0$. $\frac{ad-bc}{bd} < 0$. S: $\frac{c}{d} - \frac{a}{b} > 0$. $\frac{bc-ad}{bd} > 0$. T: What do we know about these four variables? S: They are positive.	
	S: So, $ad - bc$ must be negative because bd is positive.	T: Let's see if we can use this relationship to find out whether the difference we found was positive or negative tomorrow.
	S: So, $bc - ad$ must be positive because bd is positive.	
	T: Now, what can we say about the difference we found?	Have students write a brief reflection — what they learned, what they are not sure about, what they want to learn next, etc.
8	Complete the problem T: So, can we say the difference we found was positive or negative? S: $\frac{a}{b} - \frac{a+c}{b+d} = \frac{a(b+d)-b(a+c)}{b(b+d)} = \frac{ad-bc}{b(b+d)} < 0$ because $ad - bc < 0$ and $b(b + d) > 0$.	
	S: $\frac{a+c}{b+d} - \frac{a}{b} = \frac{(a+c)b-a(b+d)}{b(b+d)} = \frac{bc-ad}{b(b+d)} > 0$	Make sure students understand the reasoning process.

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
	because $bc - ad > 0$ and $b(b+d) > 0$. T: So, what can we say about $\frac{a}{b}$ and $\frac{a+c}{b+d}$? S: $\frac{a}{b} < \frac{a+c}{b+d}$. T: Can you show the relationship between $\frac{c}{d}$ and $\frac{a+c}{b+d}$ on your own? Students independently find the difference and reach the conclusion. S: $\frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d)-d(a+c)}{b(b+d)} = \frac{bc-ad}{b(b+d)} > 0$ because $bc - ad > 0$ and $b(b+d) > 0$. S: $\frac{a+c}{b+d} - \frac{c}{d} = \frac{(a+c)d-c(b+d)}{b(b+d)} = \frac{ad-bc}{b(b+d)} < 0$ because $ad - bc < 0$ and $b(b+d) > 0$. S: $\frac{a+c}{b+d} < \frac{c}{d}$.	Observe how well students are finding the difference of the algebraic fractions. Have students share their calculations on the board and discuss the conclusions.
9	Summary T: Mathematicians call $\frac{a+c}{b+d}$ the *mediant* of the fractions $\frac{a}{b}$ and $\frac{c}{d}$. What can we say about the mediant?	Have students write a brief reflection — what they learned, what they are not sure about, what they want to learn next, etc.

Flow of Lesson — Day 3

Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
1	Grasp the problem T: Which is greater, $\frac{10000000001}{9898989898}$ or $\frac{10000000002}{9898989899}$? T: Let's see what we get if we change these fractions to decimal numbers. T: We get 1.010204082 for both. So, these fractions are equal. What do you think?	Do students realise it is highly unlikely that these two fractions are equivalent?

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
	S: I don't think they are equal. S: I think the calculator cannot show the small difference in these two fractions. T: So, what do you notice about these fractions? S: You get the second one by adding 1 to both the numerator and the denominator. T: So, what do you think will happen when we add 1 to the numerator and the denominator of a fraction? Let's try it out. S: It gets bigger. I started with $\frac{3}{4}$ and $\frac{4}{5}$ is greater than $\frac{3}{4}$. S: It gets bigger. I started with $\frac{1}{3}$ and $\frac{2}{4}$ $\left(\text{or } \frac{1}{2}\right)$ is greater than $\frac{1}{3}$. T: How can we use algebra to express a fraction and a new fraction you can get by adding 1 to the numerator and the denominator? S: $\frac{a}{b}$ and $\frac{a+1}{b+1}$. T: Do you think you can compare these fractions — decide which is greater? S: Yes	I suspect most students will start with a proper fraction. However, if anyone starts with an improper fraction, there will be two competing observations — another reason for algebraically examining the situation. Can students express the situation using algebraic expressions?
	Independent problem solving S: $\frac{a}{b} - \frac{a+1}{b+1} = \frac{a(b+1)-b(a+1)}{b(b+1)} = \frac{a-b}{b(b+1)}$. S: $\frac{a+1}{b+1} - \frac{a}{b} = \frac{b(a+1)-a(b+1)}{b(b+1)} = \frac{b-a}{b(b+1)}$.	Discuss how we subtract algebraic fractions as necessary — individually, in small groups, or as a whole class.
3	Sharing and discussing Have a couple of students share their work on the board. T: So, what can we say about $\frac{a}{b}$ and $\frac{a+1}{b+1}$?	Make sure that everyone agrees that calculations are done correctly.

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
	S: We can't determine if the difference is positive or negative. T: Why not? S: Because we don't know if $a-b$ (or $b-a$) is positive or negative. S: It depends on a and b. T: How so? If $a>b$ then $\frac{a}{b}-\frac{a+1}{b+1}=\frac{a(b+1)-b(a+1)}{b(b+1)}=\frac{a-b}{b(b+1)}>0.$ So $\frac{a}{b}>\frac{a+1}{b+1}$. Or adding 1 to the numerator and the denominator makes the new fraction smaller. If $a<b$ then $\frac{a}{b}-\frac{a+1}{b+1}=\frac{a(b+1)-b(a+1)}{b(b+1)}=\frac{a-b}{b(b+1)}<0.$ So $\frac{a}{b}<\frac{a+1}{b+1}$. Or adding 1 to the numerator and the denominator makes the new fraction greater. T: Is it OK that the answers are different based on these two cases? S: Yes.	Do students realise that whether $\frac{a-b}{b(b+1)}$ $\left(\text{or } \frac{b-a}{b(b+1)}\right)$ is negative or positive depends on the relationship between a and b?
4	Summarise T: So, what does $a>b$ mean in the context of this problem? S: $\frac{a}{b}$ is an improper fraction. T: How about $a<b$? S: $\frac{a}{b}$ is a proper fraction. T: So, what can we say about $\frac{a}{b}$ and $\frac{a+1}{b+1}$? S: $\frac{a+1}{b+1}$ will be greater than $\frac{a}{b}$ if $\frac{a}{b}$ is a proper fraction. S: $\frac{a+1}{b+1}$ will be less than $\frac{a}{b}$ if $\frac{a}{b}$ is an improper fraction. T: By the way, do you remember we discussed the mediant of 2 fractions?	Can students describe the two cases as proper fractions vs. improper fractions?

Continued on next page

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Steps	Learning task, teacher questions, anticipated responses	Instructional points of consideration, assessment
	T: Is $\frac{a+1}{b+1}$ the mediant of 2 fractions?	Do students see $\frac{a+1}{b+1}$ as the mediant of $\frac{a}{b}$ and $\frac{1}{1}$?
	S: Yes. $\frac{a}{b}$ and $\frac{1}{1}$.	
	T: Can we use what we discovered yesterday to answer today's question?	Can they apply the conclusion for Day 2 to this problem?
	S: Because $\frac{a+1}{b+1}$ is the mediant of $\frac{a}{b}$ and $\frac{1}{1}$, it will be in between them, or it will get closer to 1.	Have students write a brief reflection — what they learned, what they are not sure about, what they want to learn next, etc.

Maths. Development Team — PDST

STEPS Engineers Ireland

Peter's Problem 2017

The Irish Mathematics Teachers' Association

A school wishes to build a stand-alone library on a green-field plot of land beside the school.

The school authorities agree to adapt the plan for the building as outlined on the Department of Education and Skills website — **Room Layouts for Post Primary Schools:** Reference RT – 023.1 (Library and Ancillary stores $100\,\text{m}^2$) available at `https://www.education.ie/en/School-Design/Design-Guidance/pbu_23_1_room_layout_library_and_ancillary_stores_pdf.pdf`.

The building is stand-alone and so there is no corridor to be built.

All external walls are to be 2-leaf cavity walls using standard $440 \times 215 \times 100$ mm solid blocks with a 90 mm gap between them for insulation.

1. **Verify that there are approximately 10 blocks per square metre of the wall, allowing 10 mm of mortar between the blocks.**

All internal walls are to be built as single leaf using the same standard block.

There is a precast lintel installed above the openings left in the walls for the window and doors. Each lintel is 215 mm high. The lintel extends 600 mm into the wall on either side of the opening left in the wall for the window and frame (i.e. Area left vacant before installation of the window frame) and 300 mm into the walls on either side of the door openings.

A pre-cast window sill is to be inserted in the front leaf of the wall only. The sill is of height 215 mm and extends 200 mm either side of the window opening.

All internal doors have width as specified in the general notes with an allowance of 25 mm on either side for the width of the frame. The external door frame unit width will be 2.6 times the width of the opening left for the internal door. The lintels above the doors are inserted in the 12^{th} course of blocks.

The area of the window opening is 40% of the area of the wall in which it is set as viewed from the room.

2. **Calculate suitable dimensions for the window.**

3. **Make a scale drawing of the elevation D – D, to include the position of the window.**

The building is to be constructed on a concrete base with no blocks in the foundation.

4. **Calculate the number of concrete blocks required to complete the building, using the dimensions given in the DES file drawing.**

Allow for 5% wastage with the blocks.

5. **Using the standard bale found on-line in Ireland, calculate the number of bales of blocks required to complete the building.**

Dominic Guinan and Neil Hallinan

Peter's Problem 2017 — Suggested Solution

Q. 1 Verify that there are approximately 10 blocks per square metre of the wall, allowing 10 mm of mortar between the blocks.

Block dimensions (in mm): $440 \times 215 \times 100$. With mortar (10 mm) : $450 \times 225 \times 100$.
The face area covered per block will be $450 \times 225 = 101250\,mm^2$ or $0.10125\,m^2$.
The number of blocks required to cover $1\,m^2$ is $1 \div 0.10125 = 9.87654321$ blocks.

Conclusion:
9.8765 blocks per square metre is 10 blocks per square metre to the nearest block.
Or, 10 blocks will cover $1.0125\,m^2$, slightly more than one square metre.
This verifies that there are approximately 10 blocks per square metre.

Q. 2 Calculate suitable dimensions for the window.

Visible wall: view D–D between the Book Store and the end wall.
Length of visible wall (in metres) = $14.2 - (3 + 0.1 + 2.16 + 0.1) = 14.2 - 5.36 = 8.84\,m$.
Height of wall = $3.15\,m$ which allows for 14 block courses $(3.15 \div 0.225 = 14)$.
Area of visible wall = $8.84 \times 3.15 = 27.846\,m^2$.
Window area = 40% of $27.846 = 11.1384\,m^2$.

Possibilities:
To calculate possible lengths and heights for the window we observe that we require a product of height $\times$ length which gives the result 11.1384.
However, for building purposes it may be more practical to consider heights which match the courses of blocks. This is the approach taken here.

Allowing for a minimum lintel space of 1 course and a minimum radiator space of 2 courses we could have up to 11 courses $(2475\,mm)$ available for the window height.

Furthermore, a window height of less than $1350\,mm$ (6 courses) would produce a window length $(11.1384 \div 1.125\ (5\text{ courses}) = 9.9008\,m)$ greater than the $8.84\,m$ visible wall.

Thus, there are six practical possibilities of window design with heights ranging from 6 to 11 courses of blocks. Here we provide the calculations for a design which has a height of $1575\,mm$ (7 courses). This design allows for a radiator space of 5 courses $(1125\,mm)$ and a course above the lintel. It also allows for the lintel overhang of $600\,mm$ at both ends.

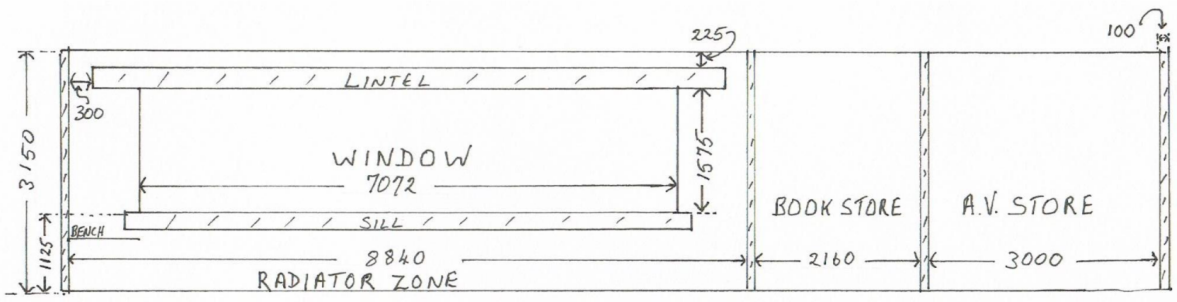
Window Dimensions:

$$\begin{aligned}\text{Height:} &= 1.575\,m.\\ \text{Length:} &= 11.1384 \div 1.575 = 7.072\,m\\ \text{Window lintel length:} &= 7.072 + 2 \times 0.6 = 8.272\,m.\\ \text{Window sill length:} &= 7.072 + 2 \times 0.2 = 7.472\,m.\end{aligned}$$

Q. 3 Make a scale drawing of the elevation D–D, to include the position of the window.

Considerations:

The drawing will reflect the choice of window dimensions.



Note: All measurements are in mm.

Q. 4 Calculate the number of concrete blocks required to complete the building, using the dimensions given in the DES file drawing.

There are numerous valid methods of doing this calculation. One method is presented here.

Method: The gross wall area is calculated on a piece-by-piece basis without making allowances for doors or window. The total area of the doors, window, lintels and sill is calculated and removed from the gross total area.

Observations on Calculations: The height of the building is $3.15\,m$. We must allow for overlapping end-joins, insulation width and both leaves of blocks when finding lengths.

Calculation of Gross Area of Walls (in m^2)			
View	Inner Wall (H × L)	Outer Wall (H × L)	Total area (m^2)
A–A	3.15×7 $= 22.05$	$3.15 \times (7 + 2 \times 0.1 + 2 \times 0.09)$ $= 3.15 \times 7.38$ $= 23.247$	45.297
B–B	$3.15 \times (14.2 + 2 \times 0.1)$ $= 3.15 \times 14.4$ $= 45.36$	$3.15 \times (14.4 + 2 \times 0.09 + 2 \times 0.1)$ $= 3.15 \times 14.78$ $= 46.557$	91.917
C–C	As for A–A: 22.05	As for A–A: 23.247	45.297
D–D	As for B–B: 45.36	As for B–B: 46.557	91.917
Total for A–A, B–B, C–C, D–D:			**274.428**
	AV Store (H×L)	Book Store (H×L)	Total area (m^2)
Side Wall	3.15×3 $= 9.45$	3.15×3 $= 9.45$	18.9
	AV Store with Book Store		
Front Wall interior	$3.15 \times (3 + 2.16 + 2 \times 0.1)$ $= 3.15 \times 5.36$ $= 16.884$		16.884
Total for AV Store and Book Store:			**35.784**
Grand Total Gross Area:		**274.428 + 35.784 =**	**310.212**

Observations on Areas to Remove:
Door height: $11 \times 0.225 = 2.475\,m$ (Lintel is in 12^{th} course of blocks).
Door width: Internal door $850 + 2 \times 25 = 900\,mm = 0.9\,m$ (allowing for the frame width).
Internal door lintels length: $900 + 2 \times 300 = 1500\,mm = 1.5\,m$.
External door: $2.6 \times 900 = 2340\,mm = 2.34\,m$.
External door lintel length: $2340 + 2 \times 300 = 2940\,mm = 2.94\,m$.

Parts to remove	H×L	**Areas to Remove (m^2)**
Two Internal door openings	2.475×0.9	$2.2275 \times 2 = 4.455$
Two Internal door lintels	0.225×1.5	$0.3375 \times 2 = 0.675$
External door openings (inner and outer walls)	2.475×2.34	$5.7915 \times 2 = 11.583$
External door lintel (inner and outer walls)	0.225×2.94	$0.6615 \times 2 = 1.323$
Window area × 2	1.575×7.072	$11.1384 \times 2 = 22.2768$
Window lintel × 2	0.225×8.272	$1.8612 \times 2 = 3.7224$
Window sill	0.225×7.472	1.6812
Total area to remove		**45.7164**

Area remaining for blocks: $310.212 - 45.7164 = \mathbf{264.4956\,m^2}$.

Number of blocks required @ 10 blocks per m^2 and rounded up: 2645

Added wastage @ 5% = 132.25 **which requires 133 blocks.**

Total number of blocks: $2645 + 133 = 2778$.

Q. 5 Using the standard bale found on-line in Ireland, calculate the number of bales of blocks required to complete the building. Any suitable online reference may be used.
The number of blocks per bale is 44 for our calculations.
$2778 \div 44 = 63.13636$ bales.
This rounds up to 64 bales required.

Neil Hallinan and Dominic Guinan

Extensions of Escott's Infinite Product Method for Calculating Square Roots

Abstract:
Algorithms for calculating square roots have a long history. Many go back all the way to Mesopotamia and ancient Greece. Some involve iterative methods; others make use of infinite series, infinite continued fractions and infinite products.

How does your calculator find square roots? Very probably by using the Newton-Raphson algorithm. This algorithm is very popular because it has quadratic convergence, that is, the number of correct digits doubles with each calculation iteration.

Escott's method [1] is not so well known but it is an excellent algorithm which has cubic convergence. It is somewhat unusual in that it involves an infinite product.

Escott's infinite product arises from the manipulation of two algebraic identities. In this project I ask three questions:

(i) Are there other algebraic identities of this type that generate infinite products that can be used to calculate square roots?

(ii) Is there a systematic method of producing such identities?

(iii) If so, do the resulting infinite products have even faster convergence than Escott's infinite product?

I show that the answer to all three questions is 'yes'.

I have discovered new algebraic identities that lead to infinite products with quintic and septic convergence (the number of correct digits increases by a factor of five and by a factor of seven, respectively, when one extra term is included in the product).

I made many attempts to adapt Escott's infinite product method to calculate cube roots, fourth roots and so on, but without success. But, arising from these unsuccessful attempts, I have discovered a type of infinite product expansion that can calculate cube roots, fourth roots and so on, with great accuracy. It appears that this type of infinite product is new.

Escott's Infinite Product

Escott started with two identities:

$$(x-1)^2(x+2) = x^3 - 3x + 2 \tag{1}$$
$$(x+1)^2(x-2) = x^3 - 3x - 2 \tag{2}$$

He let

$$x_{i+1} = x_i^3 - 3x_i \tag{3}$$

and re-wrote equations (1) and (2) as:

$$\frac{\dfrac{x_i+2}{x_i-2}}{\dfrac{x_{i+1}+2}{x_{i+1}-2}} = \left(\frac{x_i+1}{x_i-1}\right)^2 \tag{4}$$

Letting $i = 1, 2, 3, \ldots, n$ in turn he got:

$$\frac{\dfrac{x_1+2}{x_1-2}}{\dfrac{x_2+2}{x_2-2}} = \left(\frac{x_1+1}{x_1-1}\right)^2 \tag{5}$$

$$\frac{\dfrac{x_2+2}{x_2-2}}{\dfrac{x_3+2}{x_3-2}} = \left(\frac{x_2+1}{x_2-1}\right)^2 \tag{6}$$

$$\vdots$$

$$\frac{\dfrac{x_n+2}{x_n-2}}{\dfrac{x_{n+1}+2}{x_{n+1}-2}} = \left(\frac{x_n+1}{x_n-1}\right)^2 \tag{7}$$

When the left-hand sides are multiplied everything cancels except for the numerator in (5) and the denominator in (7). For example, if we multiply the left-hand sides of (5) and (6) we get:

$$\frac{\dfrac{x_1+2}{x_1-2}}{\dfrac{x_2+2}{x_2-2}} \times \frac{\dfrac{x_2+2}{x_2-2}}{\dfrac{x_3+2}{x_3-2}} = \frac{\dfrac{x_1+2}{x_1-2}}{\dfrac{x_3+2}{x_3-2}}$$

Multiplying all the right-hand sides gives:

$$\prod_{i=1}^{n}\left(\frac{x_i+1}{x_i-1}\right)^2 \text{ which equals } \left(\prod_{i=1}^{n}\left(\frac{x_i+1}{x_i-1}\right)\right)^2$$

The result is

$$\frac{\dfrac{x_1+2}{x_1-2}}{\dfrac{x_{n+1}+2}{x_{n+1}-2}} = \left(\prod_{i=1}^{n}\left(\frac{x_i+1}{x_i-1}\right)\right)^2$$

Hence

$$\sqrt{\frac{\dfrac{x_1+2}{x_1-2}}{\dfrac{x_{n+1}+2}{x_{n+1}-2}}} = \prod_{i=1}^{n}\left(\frac{x_i+1}{x_i-1}\right) \tag{8}$$

Since $x_{i+1} = x_i^3 - 3x_i \to \infty$ as $i \to \infty$ for $x_1 > 2$ it follows that

$$\sqrt{\frac{x_1+2}{x_1-2}} = \prod_{i=1}^{\infty}\left(\frac{x_i+1}{x_i-1}\right) = \prod_{i=1}^{\infty}\left(1+\frac{2}{x_i-1}\right) \tag{9}$$

since in equation (8) $\dfrac{x_{n+1}+2}{x_{n+1}-2} \to 1$ as $n \to \infty$

Escott's	Algorithm	
x(1) =	3	
Root	5	
Index	x(i)	Product
1	3	2.000000000000000
2	18	2.235294117647060
3	5778	2.236067977476610
4	1.93E+11	2.236067977499790
5	7.18E+33	2.236067977499790
6	3.7E+101	2.236067977499790

Table 1: Calculating $\sqrt{5}$

Table 1 is taken from an Excel spreadsheet that makes use of equation (9) to calculate $\sqrt{5}$ (with $x_1 = 3$).

Equation (9) contains the expression $\sqrt{\dfrac{x_1+2}{x_1-2}}$ so to calculate $\sqrt{5}$ we need to set $\dfrac{x_1+2}{x_1-2} = 5$ and solve for x_1. We find that $x_1 = 3$.

The x_i column in the table is obtained by starting with $x_1 = 3$ and using equation (3) to obtain the next value and so on down the column.

The product column is obtained from the right-hand side of equation (9). Start with $1 + \dfrac{2}{x_1 - 1}$ to get the value for index 1. Then calculate $1 + \dfrac{2}{x_2 - 1}$ and multiply it by the value just above it (the value for index 1). This gives the value for index 2. Continue on down the column like this.

Note the rapid increase in x_i, justifying the transition from equation (8) to equation (9).

In order to study the accuracy of Escott's algorithm I wrote a function in PARI/GP. The purpose of the function is to calculate the error arising from terminating Escott's infinite product after one term, two terms and so on. The user must specify 3 parameters:

(i) The number (r) whose square root is to be calculated.

(ii) The maximum number (t) of terms to be used in the product.

(iii) The precision (p) to be used while the calculations are being performed.

```
es3(r,t,p) =
{
 default(realprecision, p);
 ans = sqrt(r);
 x = 2 * (r + 1) / (r - 1);
 pr = (1 + 2.0 / (x - 1));
 er = pr - ans;
 printf("%.1E\n", er);
 for (i = 1, t - 1,
        x = x^3 - 3*x;
        trm = 1 + 2 / (x - 1);
        pr = pr * trm;
        er = pr - ans;
        printf("%.1E\n", er);
 );
 default(realprecision, 38);
}
```

$es3(5, 8, 10000)$ generated the following output:

No. of Terms	Error
1	-2.4×10^{-1}
2	-7.7×10^{-4}
3	-2.3×10^{-11}
4	-6.2×10^{-34}
5	-1.2×10^{-101}
6	-8.8×10^{-305}
7	-3.5×10^{-914}
8	-2.1×10^{-2742}

Table 2: Calculating the error

Table 2 shows us that Escott's infinite product is an excellent algorithm for calculating square roots — it has cubic convergence.

New Identities Leading to Faster Convergence

In order to obtain a new Escott-like infinite product I began with the expression:

$$(x^2 + ax + b)^2 (x - 1) \tag{10}$$

I expanded this expression in powers of x. In Escott's identities the even powers of x vanish. I chose a and b so that this happened $\left(a = \frac{1}{2},\ b = -\frac{1}{4}\right)$. This gave me:

$$\left(x^2 + \frac{1}{2}x - \frac{1}{4}\right)^2 (x - 1) = x^5 - \frac{5}{4}x^3 + \frac{5}{16}x - \frac{1}{16} \tag{11}$$

The second step is to obtain an identity of the form:

$$(x^2+px+q)^2(x+1) = x^5 - \frac{5}{4}x^3 + \frac{5}{16}x + \frac{1}{16} \tag{12}$$

Solving for p and q gives $p = -\frac{1}{2}$ and $q = -\frac{1}{4}$ and the identity:

$$\left(x^2 - \frac{1}{2}x - \frac{1}{4}\right)^2 (x+1) = x^5 - \frac{5}{4}x^3 + \frac{5}{16}x + \frac{1}{16} \tag{13}$$

It is more convenient to write the identities (11) and (13) as:

$$(4x^2+2x-1)^2(x-1) = 16x^5 - 20x^3 + 5x - 1 \tag{14}$$

and

$$(4x^2-2x-1)^2(x+1) = 16x^5 - 20x^3 + 5x + 1 \tag{15}$$

Replacing x by x_i and $16x^5 - 20x^3 + 5x$ by x_{i+1} we re-write (14) and (15) as follows:

$$\frac{\dfrac{x_i+1}{x_i-1}}{\dfrac{x_{i+1}+1}{x_{i+1}-1}} = \left(\frac{4x_i^2+2x_i-1}{4x_i^2-2x_i-1}\right)^2 \tag{16}$$

Letting $i = 1, 2, 3, \ldots, n$ in turn we get:

$$\frac{\dfrac{x_1+1}{x_1-1}}{\dfrac{x_2+1}{x_2-1}} = \left(\frac{4x_1^2+2x_1-1}{4x_1^2-2x_1-1}\right)^2 \tag{17}$$

$$\frac{\dfrac{x_2+1}{x_2-1}}{\dfrac{x_3+1}{x_3-1}} = \left(\frac{4x_2^2+2x_2-1}{4x_2^2-2x_2-1}\right)^2 \tag{18}$$

$$\vdots$$

$$\frac{\dfrac{x_n+1}{x_n-1}}{\dfrac{x_{n+1}+1}{x_{n+1}-1}} = \left(\frac{4x_n^2+2x_n-1}{4x_n^2-2x_n-1}\right)^2 \tag{19}$$

When we multiply the left-hand sides everything cancels except the numerator of (17) and the denominator of (19) so that we get:

$$\frac{\dfrac{x_1+1}{x_1-1}}{\dfrac{x_{n+1}+1}{x_{n+1}-1}}$$

When we multiply the right-hand sides we get:

$$\prod_{i=1}^{n}\left(\frac{4x_i^2+2x_i-1}{4x_i^2-2x_i-1}\right)^2$$

which equals

$$\left(\prod_{i=1}^{n}\left(\frac{4x_i^2+2x_i-1}{4x_i^2-2x_i-1}\right)\right)^2$$

Taking the square root of both sides we finally get:

$$\sqrt{\frac{\dfrac{x_1+1}{x_1-1}}{\dfrac{x_{n+1}+1}{x_{n+1}-1}}}=\prod_{i=1}^{n}\left(\frac{4x_i^2+2x_i-1}{4x_i^2-2x_i-1}\right) \tag{20}$$

Since $x_{i+1}=16x_i^5-20x_i^3+5x_i\to\infty$ as $i\to\infty$ for $x_1>1$ so that $\dfrac{x_{n+1}+1}{x_{n+1}-1}\to 1$ as $n\to\infty$ we get:

$$\begin{aligned}\sqrt{\frac{x_1+1}{x_1-1}}&=\prod_{i=1}^{\infty}\frac{4x_i^2+2x_i-1}{4x_i^2-2x_i-1}\\&=\prod_{i=1}^{\infty}\left(1+\frac{4x_i}{4x_i^2-2x_i-1}\right)\\&=\prod_{i=1}^{\infty}\left(1+\frac{1}{x_i-\dfrac{1}{2}-\dfrac{1}{4x_i}}\right)\end{aligned} \tag{21}$$

The middle expression in (21) is obtained by dividing the denominator into the numerator and the last expression by dividing the numerator and denominator by $4x_i$.

Table 3 is taken from an Excel worksheet that makes use of equation (21) to calculate $\sqrt{2}$. First we set $\dfrac{x_1+1}{x_1-1}=2$ to calculate x_1. We get $x_1=3$.

x(1) =	3	
Root	2	
Index	**x(i)**	**Product**
1	3	1.413793103448280
2	3363	1.414213562373090
3	6.88E+18	1.414213562373090
4	2.47E+95	1.414213562373090

Table 3: Calculating $\sqrt{2}$

The x_i column is obtained by using $x_{i+1} = 16x_i^5 - 20x_i^3 + 5x_i$.
The product column is obtained by using the last part of equation (21). For index 1 we calculate

$$1 + \frac{1}{x_1 - \frac{1}{2} - \frac{1}{4x_1}}$$

For index 2 we calculate the same expression but with x_1 replaced by x_2 and then multiply it by the value obtained for index 1. We carry on down the column calculating like this.

In Table 3 we see that x_i increases very rapidly. This justifies the transition from equation (20) to equation (21). The rate of increase of x_i is even faster than in Table 1.

Because of the limited precision available in Excel it was impossible to make a definitive statement about the rate of convergence of this algorithm using an Excel worksheet. I conjectured that the rate of convergence was probably quintic and I wrote a PARI/GP function, $quin(r, t, p)$ to test this conjecture. The function is very similar to the function $es3(r, t, p)$ shown earlier.

Table 4 shows the output generated by $quin(2, 8, 400000)$.

No. of Terms	Error
1	-4.2×10^{-4}
2	-2.1×10^{-19}
3	-5.7×10^{-96}
4	-9.6×10^{-479}
5	-1.3×10^{-2392}
6	-5.1×10^{-11962}
7	-5.6×10^{-59809}
8	-8.9×10^{-299044}

Table 4: Calculating the error

Table 4 confirms that the product in (21) has quintic convergence.

The identities

$$(8x^3 + 4x^2 - 4x - 1)^2 (x - 1) = 64x^7 - 112x^5 + 56x^3 - 7x - 1 \quad (22)$$
$$(8x^3 - 4x^2 - 4x + 1)^2 (x + 1) = 64x^7 - 112x^5 + 56x^3 - 7x + 1 \quad (23)$$

were obtained using a similar technique to that used to find identities (11) and (13).

Replacing x by x_i and $64x^7 - 112x^5 + 56x^3 - 7x$ by x_{i+1} we obtain the infinite product:

$$\sqrt{\frac{x_1+1}{x_1-1}} = \prod_{i=1}^{\infty} \frac{8x_i^3 + 4x_i^2 - 4x_i - 1}{8x_i^3 - 4x_i^2 - 4x_i + 1}$$
$$= \prod_{i=1}^{\infty} \left(1 + \frac{1}{x_i - \frac{1}{2} - \frac{1}{4x_i - \frac{1}{x_i}}} \right) \tag{24}$$

The rate of convergence of this algorithm is septic. A PARI/GP function, $sep(r, t, p)$ was written to confirm this.

Typing $sep(2, 5, 100000)$ generated the following output:

No. of Terms	Error
1	-1.2×10^{-5}
2	-8.7×10^{-38}
3	-7.4×10^{-263}
4	-2.3×10^{-1838}
5	-6.8×10^{-12867}

Table 5: Calculating the error

Conclusion

I describe a systematic method of producing identities that lead to infinite products that have even faster convergence than Escott's.

References

1. E.B. Escott: *A Rapid Method for Extracting a Square Root*, The American Mathematical Monthly, Vol 44, No. 10 (Dec., 1937), pp. 644–646.

Yaduvir Harhangi, student at Synge Street CBS, Dublin.

Rev. Brendan P Steen C.M., 1921–2015 — An Appreciation

Fr. Brendan P Steen C.M., who passed away on 30th November 2015 in his 95th year, was a member of the IMTA from its very beginnings in 1964. He served the Association faithfully and with great distinction for more than fifty years.

Born on 29th November 1921 in Ardee, Co. Louth, to Dr. Patrick J and Mrs. Eileen Steen, he followed a family tradition in attending St Vincent's College, Castleknock, where he excelled academically and at sports including tennis and rugby; the latter, as his friends know, became a lifelong passion for him. He entered the Vincentian community in September 1940 and was ordained on Saturday 22nd May 1948 in St Mary's Pro-Cathedral, Dublin, by Archbishop John Charles McQuaid. In addition to his theological studies during this time, he graduated from UCD in 1944 with an honours BSc degree in Mathematics / Mathematical Physics with first class honours in Mathematics.

From 1948 to 1965 he served on the staff of Castleknock College as a teacher of mathematics, and also took on roles of responsibility including those of Dean of Discipline and of Prefect of Studies, in which he was responsible for the teaching programme of the school. He was deeply involved in extra-curricular activities with students, and in later years he remembered with great pride coaching the rugby team to win the prestigious Leinster Schools Senior Cup in 1965, beating Blackrock College 12–8 in the final.

It was also in 1965 that he was appointed President of St. Paul's College, Raheny, in Dublin. This was an exciting time of great change and expansion in second level education. Fr. Brendan presided over the building of a new classroom wing, so that the college could double in size to meet the demand for places; a swimming pool was also constructed. The most important decision during his presidency, however, was to include St. Paul's in the newly introduced "free education scheme." In addition, he contributed to general educational policy development through his role in the Catholic Headmasters' Association, notably by being its representative on the Dublin Advisory Committee on Post-Primary Educational Facilities; this group consulted with the Department of Education on the number and location of new schools to cope with population growth on the north side of the city. He also served — as a Vincentian trustee — as Chair of the Board of Management of Coolmine Community School, Clonsilla, one of the first two community schools in Ireland.

The year 1974 saw his appointment as Head of Mathematics in St. Patrick's College, Drumcondra. The Bachelor of Education degree (replacing the longstanding two-year diploma as a qualification for primary teaching) had just been introduced. The model adopted in Drumcondra included the study of Education and one other subject to degree level, and Fr. Brendan had responsibility for drawing up the programme in Mathematics. As well as leading the Department and teaching Calculus and Analysis courses, he mentored students on school placement, was Resident Adviser in one of the student residences, and supported students in sporting activities, particularly basketball. Moreover, his pastoral work with students was outstanding.

During the 1970s also, he furthered his own knowledge and skill in the burgeoning area of IT in education. He attended a ground-breaking course for teachers in University College Galway led by Prof. A C Bajpai, an early pioneer of computer education in schools, and then took the Computers in Education Diploma course in Trinity College. After obtaining the Diploma in 1976, he went

on (with other high-achieving students from the early years of the course) to complete a Master's degree in Computer Practice in 1979. His thesis was entitled "College of Education: student record system," reflecting his interest in serving the institution in which he worked.

Fr. Brendan after his Master's Graduation ceremony,
Trinity College Dublin, July 1979.

On his retirement from St. Patrick's in 1987, he moved to the parish of St. Peter's, Phibsborough, where he ministered to parishioners until 1997. He also played a role with the parish primary schools there and was involved in preliminary work on their amalgamation. From 1997 until his death in 2015, he assisted in the work of the Vincentian Provincialate based in St. Paul's, Raheny.

In this article, we pay special tribute to Fr. Brendan's role in the IMTA. His contribution to the life of the Association was enormous. He attended its first meeting in 1964, and over the following decades was active at both national and branch level.

At national level, he became a member of what was then known as the Standing Committee (now Council) in 1967, and served for almost thirty years. In the session 1970–71 he was the first holder of the post of Vice-Chairman; the following year he was elected as Chairman — succeeding the original Chairman, Denis Buckley — and he held that office until 1977. His active leadership is reflected and acknowledged in the many activities reported in Newsletters during that period. Syllabus committees (later known as course committees, now as development groups) were very busy at the time, and through Fr. Brendan and other IMTA personnel, the Association was involved in providing feedback to the committees on draft syllabus proposals. Discussions at Standing Committee meetings also addressed the relationship between computer science and mathematics and the use of calculators in public examinations. With regard to teacher professional development, issues included the organisation of courses and related payments and expenses for lecturers, together with travel grants for teachers attending courses — topics on which support was sought from the ASTI and TUI and other subject associations.

When he vacated the chair in 1977, he was appointed Vice-Chairman with special responsibility for courses and liaison with the Department of Education. Thus, when the Department of Education set up a committee on "inservice training" in 1980 and requested submissions, it was Fr. Brendan who prepared a document for the IMTA, based on consultations with branches. After discussion at

Former chairs, representatives of deceased former chairs, and original members, with their guests, together with President Michael D. Higgins and Mrs. Sabina Higgins, at the IMTA Golden Jubilee Celebration, October 2014.

the 1980 Delegate Conference — at that time, a meeting held annually as the policy-making arm of the Association — he submitted a report to the Department; it appears in Newsletter No. 45, and contains proposals that are still relevant today. Fr. Brendan continued as Vice-Chairman up to and including the session 1994–95, after which he finally stepped down from Council.

His service to the Dublin Branch was perhaps even more remarkable. The Branch was founded in 1964, soon after the birth of the Association. Fr. Brendan was not actually the first treasurer, but took over in 1965 and held the position for fully fifty years before handing over the reins in 2015. As well as being an excellent steward of the funds, he contributed greatly to the Branch and its Committee through his knowledge of the history of the Association and his firm grasp of protocol, as well as his wide experience of mathematics teaching at second and third levels. On his retirement as treasurer, he was presented with a medal and a scroll acknowledging all that he had done for the Branch.

The Branch and the Association also celebrated major events in his life outside the IMTA. At the Delegate Conference in 1988, he was given a presentation marking his retirement from St. Patrick's College; in 1998, the Branch Committee invited him to a dinner in honour of the fiftieth anniversary of his ordination.

His immense contribution to the Association was also acknowledged, notably by his being awarded Honorary Membership. Furthermore, a presentation was made to him at the IMTA Golden Jubilee celebrations attended by His Excellency President Michael D. Higgins and Mrs. Sabina Higgins on 11th October 2014. The plaque that he was given (designed by another long-standing IMTA member and Council Officer, Michael Moynihan) was later offered as one of the symbols of his life at his funeral. Appropriately, it depicted the structure of an ancient computing machine recovered from a Roman shipwreck in the Mediterranean!

Fr. Brendan before his final Dublin Branch AGM, May 2015.

Fr. Brendan holding the plaque presented to him
at the IMTA Golden Jubilee Celebration, October 2014.

Looking back over Fr. Brendan's life, we recall a modest man who wore his talents, learning and achievements lightly. His many friends will attest to his kindness, loyalty and sense of humour. Fr. Brendan is very fondly and gratefully remembered by his friends in the IMTA, his colleagues and many generations of students, as well as by his Vincentian confrères and his extended family. It was a privilege to know him. *Ar dheis Dé go raibh a anam uasal*.

Olivia Bree, formerly of the Mathematics Department, St. Patrick's College, Drumcondra.
Elizabeth Oldham, School of Mathematics, Trinity College, Dublin.

Editor's note: Fr. Steen also found time to make contributions to the Newsletter. His articles appeared in the following issues:

No. 26 (Feb. 1976) No. 30 (Oct. 1976) No. 33 (Sep. 1977) No. 34 (Oct. 1977)
No. 45 (Mar. 1981) No. 60 (Jan. 1986.)

Counting . . . As Easy as 1, 2, 3?

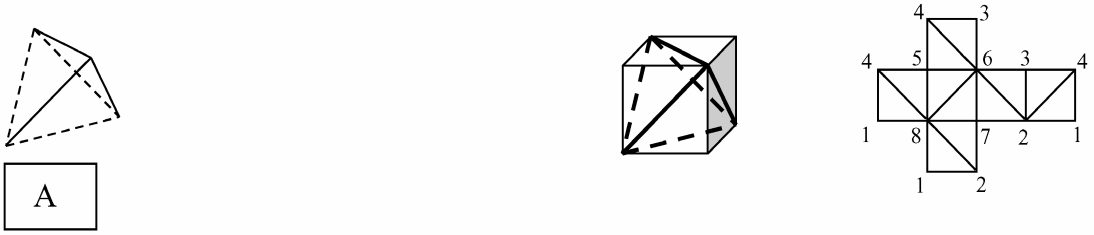
Counting is usually thought of as being easy. But, it can also be puzzling. Such a puzzle appeared in *The Sunday Times* Teaser section some time ago (No. 2835, 22 Jan 2017). It detailed a problem where a jeweller designed a cube with a single diagonal of rubies on each face. How many distinctly different such cubes could be designed?

A Solution

Each face of a cube has two diagonals. With 6 faces there are 12 diagonals in total. By joining all diagonals on the 6 faces we can see that two interlaced tetrahedrons may be formed. Label them A and B. Between both A and B all 12 diagonals are included.

Case 1: Starting with tetrahedron A:

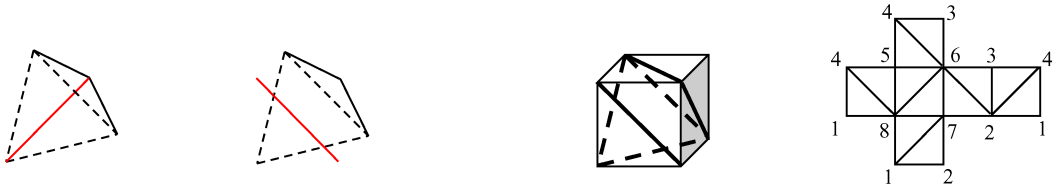
If all the jewelled diagonals form the pattern of A we have the formation of a distinct decoration of the cube. Then, if all the diagonals of A are switched to the opposite diagonal of each face we have the formation of tetrahedron B. However, this is not a distinct pattern of decoration as it may be formed by a simple rotation of the cube. The pattern formed by switching all the diagonals is the same as that formed by switching none.



Case 1: The tetrahedron shape. One pattern of rubies.

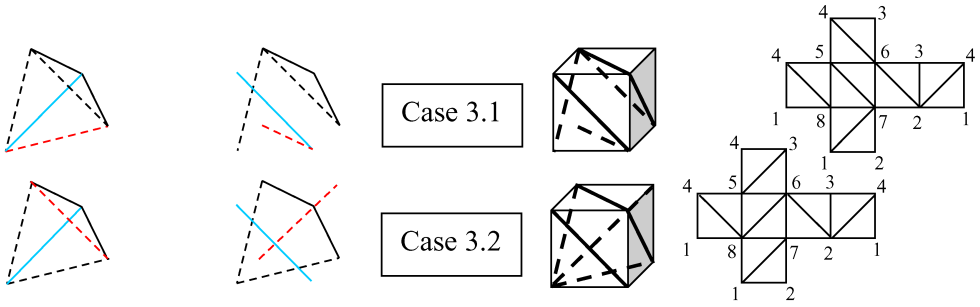
Case 2: Switch one diagonal from orientation A to orientation B (shown in red).

Any diagonal may be chosen. The resulting pattern has 5 connected diagonals and one on its own. (This is the same as switching 5 diagonals from B to A or from A to B). There is only one way in which this may give a distinct pattern.



Case 2: Switching one of the diagonals. One pattern of rubies.

Case 3: Switch two diagonals from A to B. This is the same as switching 4 diagonals from B to A or from A to B.



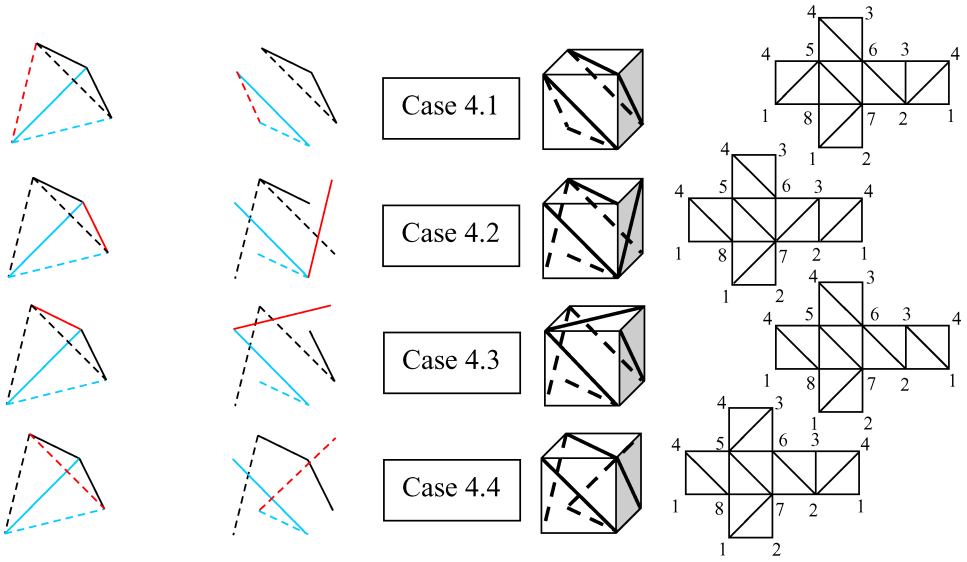
Case 3: Switching two of the diagonals. Two patterns of rubies.

It may be done in two ways. Having already switched one diagonal (shown in blue) the next diagonal to switch may either be connected (Case 3.1) to that one or not connected to it, i.e. it is on the opposite face (Case 3.2). In Case 3.1 only one distinct pattern emerges due to symmetry. In Case 3.2 only one pattern is possible.

Case 4: Switch 3 diagonals from A to B.

From Case 3.1: The third diagonal may be chosen so that:
4(i) one end is common with the other two (ie., all three stem from a common vertex on tetrahedron A) **or**
4(ii) it joins the free ends of the other two (ie., the three are linked to form a triangle face of the tetrahedron A) **or**
4(iii) and 4(iv) it joins to one end of either of the other two but not to another end on tetrahedron A (leaving two free ends) in which case it has similar patterns but with two orientations.
Thus, there are 4 ways in which this may be done.

From Case 3.2: The third diagonal must be chosen so that it joins one end of each of the other two in tetrahedron A leaving two free ends. These choices have already been dealt with in the development of Case 3.1 (see 4(iii) and (iv) above) and do not add any further patterns.

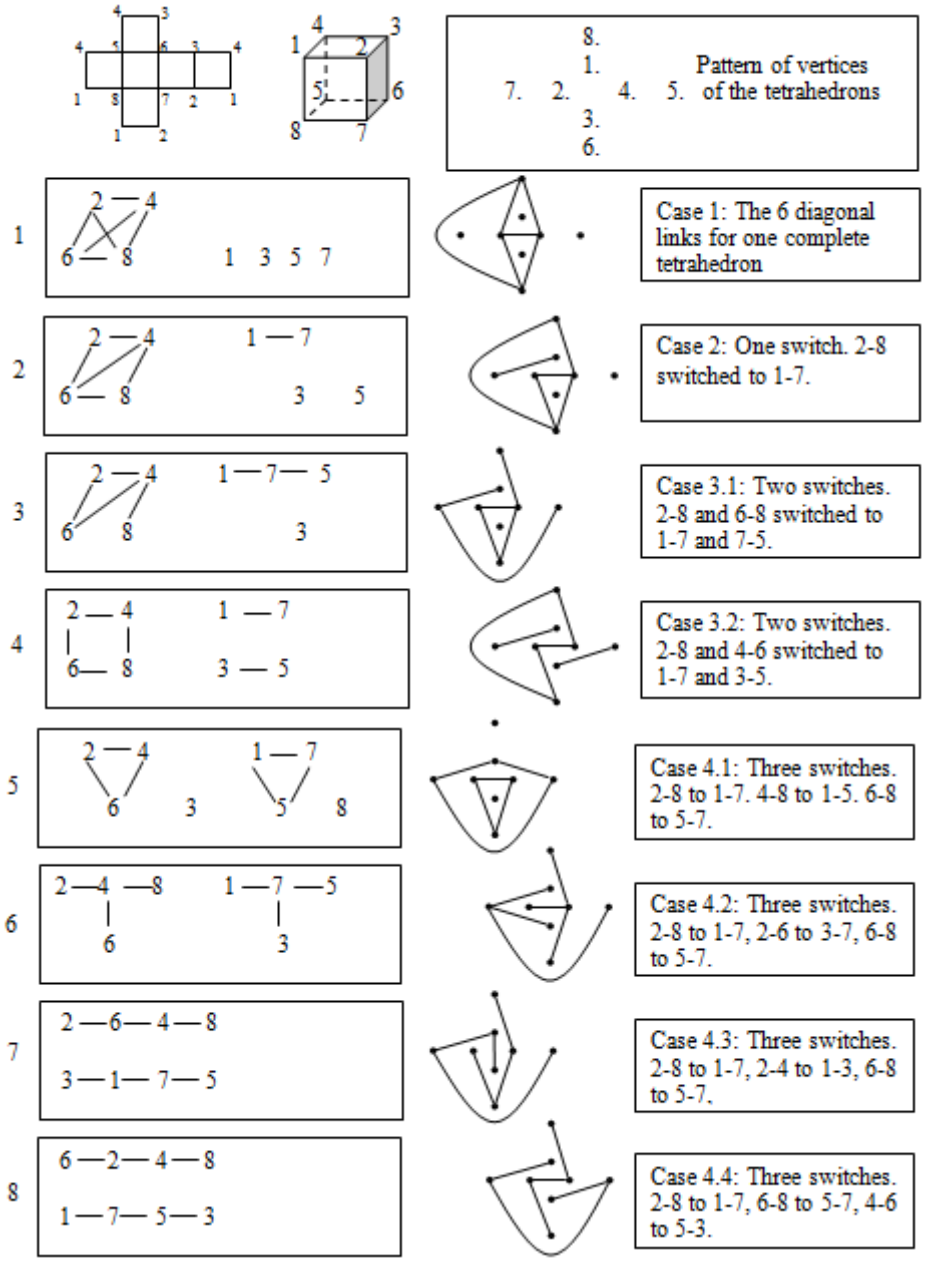


Case 4: Switching three of the diagonals. Four patterns of rubies.

Conclusion: There are no more switches that can be made. Therefore, the total number of patterns is $1 + 1 + 2 + 4 = 8$. The jeweller can make eight cubes with a different pattern of ruby diagonals on each one.

Alternative Methods using the Tetrahedrons

We may label the 8 vertices of the cube (1 to 8) and show them as vertices of the tetrahedrons as illustrated. The vertices may be labelled as shown and the linkages for each pattern identified.



A More Generalised Method

Another method of counting the patterns is to use the Pólya-Burnside Lemma or, simply, the Burnside Lemma. This method has been detailed by Nick Baxter [1] and forms the basis of what follows.

There are many ways to state this lemma. Baxter refers to C.L. Liu [2] and gives the lemma in a

relatively simple form as follows:
Theorem (Burnside):
The number of equivalence classes into which a set is divided by the equivalence relation induced by a permutation group G of S is given by

$$\frac{1}{|G|}\sum_{\pi\in G}\psi(\pi)$$

where:

- $\psi(\pi)$ is the number of elements that are invariant under the permutation π
- S is the set of all possible variations of an object, with no considerations for rotations and reflections
- $|G|$ is the number of permutations in the group G

An equivalence class is the set of objects that are the same, ignoring rotations and reflections.
An invariant element is any object that appears exactly the same before and after a permutation.

For the purposes of this puzzle a permutation is an action which takes the physical object (the cube) and places or re-places it in a way that does not change its structure as a cube.
There are 24 possible permutations of a cube — imagine picking up a cube and putting it back down in the same place — any of the 6 faces may be face down and there are 4 rotations while each face is down. The Burnside Lemma reduces all problems of symmetry to simply counting the number of invariant elements for each permutation.

The lemma may be restated in terms of a base object (such as a cube) and a puzzle object (a member of the set of all variations of the base object that satisfy the puzzle's constraints).

The Pólya-Burnside Lemma (restated) (in [1]):
The number of unique puzzle objects that are variations of a base object p is:

$$\frac{1}{n}\left(\psi(\pi_1)+\psi(\pi_2)+\ \ldots\ +\psi(\pi_n)\right)$$

where:

- $G=\{\pi_1,\pi_2,\ldots,\pi_n\}$ is the set of all physical permutations of p
- $\psi(\pi)$ is the number of invariant puzzle objects for the permutation π
- $n=|G|$

For our problem we have: The base object p is a normal cube.
Our question is: How many unique ways are there to cover a cube with a diagonal, say '/', on each face? We note that '/' has two orientations: '/' and '\', i.e. each face can have two diagonals (but not at the same time).

The sets of invariant objects may then be analysed.
G is the set of 24 cube permutations (ie. $n=24$).
The permutations may be grouped into families of similar operations. i.e. $G=\{I,Q,H,D,E\}$ and, as we shall see, $\#(I)=1$, $\#(Q)=6$, $\#(H)=3$, $\#(D)=8$, $\#(E)=6$, totalling 24.

***I* is the Identity**

a) Permutations: There are no rotations involved. Only 1 permutation is possible since the cube may be placed once.

b) Invariants: There are 2 ways to place a diagonal on each face and therefore $2^6 = 64$ ways of placing one diagonal on each of the 6 faces of the cube. Since every object is invariant under the identity permutation, there are thus 64 invariant objects for this permutation.

64 invariants for 1 permutation gives a count of 64 invariants for *I*

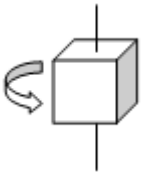
***Q* is the set of 90° face rotations**

a) Permutations: There are 6 permutations, as follows:

The cube has an axis of rotation which goes vertically through the centre of the faces on opposite sides (see diagram).

From a placing on a given face the cube may be re-placed on the same face but with a 90° turn (clockwise, say). From the same starting position it may also be re-placed on the same face but with a 90° anti-clockwise turn. Thus, one pair of opposite faces allows for 2 permutations.

The cube has 3 pairs of opposite faces — top/bottom; front/back; right/left. Thus, there are 3 axes of rotation through the pairs of opposite sides. Hence, there are $2 \times 3 = 6$ permutations in total.



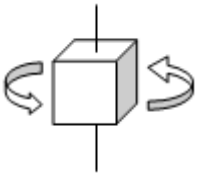
No invariants for each of 6 permutations gives a count of 0 invariants for *Q*

b) Invariants: Looking at one of the permutations and comparing with the original placement we can observe the following: the top and bottom faces stay in place for this rotation. But '/' (the diagonal on the top face which goes from lower left to upper right), does not have 90° rotational symmetry since it switches to '\' (a diagonal going from upper left to lower right) under a 90° rotation. Consequently, there are no invariant objects for Q and the count is zero.

***H* is the set of 180° face rotations**

a) Permutations: There are 3 permutations, as follows:

There are 3 pairs of opposite faces (as for set Q) (see diagram). In each case, re-placing the cube on the same face but with a 180° clockwise turn is the same as re-placing it with a 180° anti-clockwise turn. Thus, each pair of opposite faces allows for only 1 permutation and so there are only 3 permutations in total.



16 invariants for each of 3 permutations gives a count of 48 invariants for *H*

b) Invariants: Again, looking at one of these permutations and observing what happens we can see that the top and bottom faces can have either orientation of the diagonal. The orientation of the diagonal on the top and bottom faces will remain the same following a 180° turn, so there are 2 possibilities for each face (the two possible diagonals) and there are 2 independent faces which gives 2^2 invariant

objects altogether for these faces. Also, each side-face rotates to the opposite face and must have the same orientation as its opposite companion in order that no change may be apparent — i.e. remain invariant. There are two such opposite pairs (front/back and left/right) which are independent and there are 2 possibilities for each pair (the opposite diagonals have to match) and, thus, 2^2 invariant objects. Thus, in total, there are $2^2 \times 2^2 = 2^4$ invariant objects for each permutation in H and, therefore, $3 \times 16 = 48$ invariants altogether in H.

D is the set of 120° major diagonal rotations

a) Permutations: There are 8 permutations as follows: Two vertices are connected by this internal diagonal (see diagram). Three faces surround each vertex e.g. top/left/front for one vertex and bottom/right/back for the other. The top face rotates to the left side; the left side rotates to the front and the front rotates to the top by a rotation of 120° anti-clockwise. Similarly, the reverse re-placements happen for a clockwise rotation of 120°. This gives rise to 2 permutations. There are 4 possible internal diagonals for this set of permutations. Therefore, there are $2 \times 4 = 8$ permutations in the set.

4 invariants for each of 8 permutations gives a count of 32 invariants for *D*

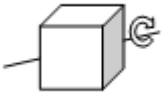
b) Invariants: Each of the three faces must have the same orientation of the face diagonal, '/', in order to remain invariant under a permutation. Thus, there are 2 possibilities for each set of three faces and, therefore, 2×2 or 2^2 invariant objects for each permutation in D. In total, then, there are $2^2 \times 8 = 32$ invariants in set D.

E is the set of 180° centre-edge rotations

a) Permutations: There are 12 edges on a cube which permit 6 pairs of axes through the centres of diagonally opposite edges (see diagram). Clockwise and anti-clockwise rotations do not change the permutations so the number of permutations remains at 6.

b) Invariants: To find invariants, we see that the top and bottom faces must have the same orientation of '/' since these faces swap places under the permutation. This gives 2 invariants — one for each orientation of the face diagonal. Also, the two faces adjacent to the edges that rotate must have the same orientation (front goes to left/left goes to front beside one edge and right goes to back/back goes to right beside the other edge). There are 2 invariants for each pair of faces thus giving a further 2^2 invariant objects. Overall, then, there are $2 \times 2^2 = 2^3 = 8$ invariant objects for each permutation in E.

In total, then, there are $8 \times 6 = 48$ invariants in set E.



8 invariants for each of 6 permutations gives a count of 48 invariants for *E*

The grand total

We may now count $\frac{1}{24}\left(\psi(I)+6\psi(Q)+3\psi(H)+8\psi(D)+6\psi(E)\right)$ which is

$$\begin{aligned}&\frac{1}{24}\left(2^6+6\times 0+3\times 2^4+8\times 2^2+6\times 2^3\right)\\&=\frac{1}{24}\left(64+0+48+32+48\right)\\&=\frac{1}{24}\times 192=8\end{aligned}$$

Conclusion: There are 8 different cubes that can be made with different patterns of face diagonals.

The method of counting using the Burnside Lemma may appear more difficult, at first. However, it allows many more such puzzles to be solved easily. For instance, in how many ways can the faces of a cube be coloured using only two colours? (The answer is 10.) Furthermore, the problem of colouring the faces of the cube may be extended to any number of colours, k, where the answer will be $\frac{1}{24}\left(k^6+6k^3+3k^4+8k^2+6k^3\right)$.

References:

1. Nick Baxter, *The Burnside Di-Lemma: Combinatorics and Puzzle Symmetry* in Cipra, B., Demaine, E.D., Demaine, M.L., Rodgers, T. (eds) Tribute to a Mathemagician, pp. 199–210; A K Peters Ltd, Wellesley, 2005.
   (Baxter notes that the lemma is known by the names Pólya-Burnside Lemma, Burnside Lemma and Cauchy-Frobenius Lemma).

2. C.L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, 1968.

See also: `www.baxterweb.com/puzzles` and `www.puzzle-place.com/wiki/Main_Page`

Note: Burnside (1897) attributed the lemma to Frobenius (1887), overlooking Cauchy (1845).

Neil Hallinan, Dublin Branch.

NUI Galway — Envisioning New Horizons for Mathematics and Computer Science Education

It is often attributed that the eminent American educationist and educational innovator, John Dewey once famously said: "If we teach today's students, as we taught yesterday's, we rob them of tomorrow".

The role of all new mathematics teachers entering the profession is not to maintain the *status quo* but rather to carry forward the tradition of excellent mathematics teaching in Ireland, and seek to promote continuous improvement in mathematics education in their classrooms and schools. The importance of creative, compassionate and committed teachers cannot be overstated; teaching is a crucially important profession, which contributes inestimably to the well-being of a country's culture, economy and society.

Continuous improvement in mathematics education is a worthy and attainable goal for all post-primary mathematics teachers. Related to this is the recognised shortage of available qualified teachers of STEM (science, technology, engineering and mathematics) disciplines, and never has there been a more important time than now to focus interest and build capacity in teacher education for STEM. In today's world, technology is becoming ever more ubiquitous and there is an educational imperative for new generations to move from being consumers *of* technology to creators *with* technology. In this context, there remains so much educational potential yet to be harnessed in STEM, and in particular the integration of mathematics education and computer science and computational thinking. It is heartening to read at time of writing the government's investment of €150 million over the next three years to support the rollout of the new Computer Science Leaving Certificate subject and digital assessment (including innovatively within the new PE subject at Senior Cycle) in Irish schools. However, alarmingly, if we don't have the requisite number of qualified teachers of STEM, we are negatively impacting capacity for careers in the field and doing a disservice to the future generations in the areas of national interest. While certainly not the only one, a salient measure of the quality of mathematics education is the proportion of excellent mathematics teachers in the system.

Mathematics teaching at post-primary level demands knowledge of mathematics beyond the level at which one may have to teach and it also requires different kinds of mathematical knowledge from that possessed by mathematicians. Ball et al. (2008), when developing the Mathematical Knowledge for Teaching (MKT) framework, argued that teachers' mathematical representations are dependent on their own conceptual understanding of the subject and the knowledge needed to teach mathematics goes beyond the Common Content Knowledge (CCK) needed for many other professions (Ball, Thames, & Phelps, 2008). The MKT framework contends that the teacher must be able to deconstruct content to build mathematical understanding (Specialised Content Knowledge or SCK), be aware of the ways in which mathematics concepts relate and are connected across the curriculum and other disciplines (Horizon Content Knowledge or HCK), understand how students might approach problems and the misconceptions which occur, (Knowledge of Content and Students or KCS), be familiar with the types of pedagogical strategies that might help students learn particular content (Knowledge of Content and Teaching or KCT), and have experiences with the content and curriculum (Knowledge of Content and Curriculum or KCC).

The MKT framework builds on the seminal work of Shulman in adding to pedagogical content

knowledge for mathematics teaching, however, the initial conceptualisation was focused on the knowledge of elementary teachers (Shulman, 1986). The pedagogical knowledge post-primary teachers require augments these models as mathematical content is more conceptually-challenging, requiring more focus on proof and attention to mathematical structure, and the Mathematical Understanding for Secondary Teaching (MUST) framework addresses this (Heid, Wilson, & Blume, 2015). The MUST framework includes mathematical activity such as justifying, proving and reasoning, when conjecturing and generalising, while also addressing the mathematical context of teaching, such as recognising how students think about particular mathematical ideas.

The National University of Ireland, Galway, had the vision in 2008 to build on their excellent tradition in mathematics and foresee the importance of mathematics teacher education, developing a Bachelor of Arts programme in Mathematics and Education. The four-year concurrent initial teacher education programme was designed to address the identified need in Irish education for excellent teachers of Mathematics and Applied Mathematics in schools. The BA Mathematics and Education has now for almost a decade made a very significant contribution — at source — to the stream of new mathematics teachers entering the profession in Ireland every year.

At the time of its introduction, the concurrent initial teacher education degree programme represented the first and only teacher education degree programme to focus in such a deep and specialised manner on the mathematical sciences. Historically, it was the first specialist programme of its kind since the establishment of the Department of Education in 1922. As well as secondary teachers recognised by the Teaching Council, graduates of the BA Mathematics and Education programme are highly qualified mathematicians, educated beyond the requirements of the Leaving Certificate Honours Level Mathematics/Applied Mathematics. Embedded within the education strand of the programme are mathematics methodology modules which focus on pedagogical approaches to teach the subject at post-primary level, aligning to the MKT and MUST frameworks.

In a report by Ní Ríordáin and Hannigan published in 2009, 48% of the 324 mathematics teachers in Ireland surveyed, did not have a mathematics teaching qualification (Ní Ríordáin and Hannigan, 2009). Further research suggested that teachers were "assigned by school administrators to teach subjects which do not match their training or education" (Ní Ríordáin and Hannigan, 2011). Coordinated and led by the EPI*STEM (formerly National Centre for Excellence in Mathematics and Science Teaching and Learning, NCE-MSTL), University of Limerick, with NUI Galway as the senior partner, a national programme was developed for the upskilling of out-of-field mathematics teachers to redress this (the Professional Diploma in Mathematics for Teaching, a national mathematics education, bilingual and blended learning continuous professional development (CPD) programme for teachers is funded by the Department of Education and Skills, and jointly accredited by the University of Limerick and NUI Galway). While the success of the PDMT is unparalleled, further continuous professional development opportunities are necessary to support the integration of the Senior Cycle key skills, reform of new curricula and the education and development of teachers. A high-potential area of focus here would be the integration of mathematics and computer science education, including algorithmic and computational thinking.

Our education system is certainly moving in the right direction, however there is no panacea and we need to keep John Dewey's words firmly in our minds, as we seek always to advance the state of the art in STEM, mathematics and computer science education, to ensure we educate ourselves and our students for todays's and tomorrow's, highly technological worlds.

References

Ball, D. L., Thames, M. H., & Phelps, G., (2008). Content knowledge for teaching: What makes it special?, *Journal of Teacher Education*, 59, pp. 389–407.

Heid, M. K., Wilson, P. S., & Blume, G. W. (Eds.), (2015). *Mathematical understanding for secondary teachers: A framework and classroom-based situations*, Charlotte, NC: Information Age Publishing.

Ní Ríordáin, M., & Hannigan, A., (2009). *Out-of-field teaching in post-primary mathematics education: an analysis of the Irish context*.

Ní Ríordáin, M., & Hannigan, A., (2011). Who teaches mathematics at second-level in Ireland?, *Irish Educational Studies*, 30(3), pp. 289–304.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching, *Educational Researcher*, 15(2), pp. 4–14.

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Students' Positive Engagement with Euler's Mathematical Demonstrations

Introduction

Exposure to and engagement with the history and development of mathematical ideas is very important for learners of mathematics at all levels. Over many years, conversations with teachers, and with colleagues in Maynooth University, have revealed an appetite for opportunities to incorporate the history of mathematics and statistics (HoM) into their teaching. However, although there are extensive resources available, incorporating the HoM into teaching in a substantial way can be perceived as adding extra material that is not required. If HoM material is seen as separate to curricula, then it simply increases the existing teaching burdens and time pressures that all teachers are under.

In this paper, I present two examples which can be included in an existing curriculum and give students both insight into and experience of how one of the most remarkable results in mathematics, Euler's Identity, was first established. At the end of the paper, I will briefly describe another initiative to include the HoM as part of transition year teaching; some teachers may be interested in becoming involved.

Background

I have been involved in teaching the HoM, a final year module, in the Department of Mathematics and Statistics in Maynooth University for many years, both as tutor and lecturer. The course was originally created with future secondary teachers of mathematics in mind, and its purpose was to reveal the origins of mathematical ideas and consider the mathematicians who discovered the mathematics and statistics we see today in classrooms at all levels. In other parts of my teaching, for example while lecturing different mathematics courses or teaching in the Maynooth University Mathematics Support Centre (MSC), I have witnessed the impact that including HoM materials can have on the student learning experience. It can allow them to get a sense of where the material they are working on came from, why it was developed in the first place, and its subsequent progression. Particularly in the MSC, it allows students to see that mastering concepts in mathematics and statistics is not instantaneous. It is normal for insight to develop over a period of time, and the struggle to understand what is going on is a key part of the learning process.

For those looking for resources in the HoM, in addition to a wide range of textbooks, for example [i, ii], there are also extensive online resources. The *MacTutor* website [iii] is an excellent starting point, and on the Maynooth University MSC website (iv) I have gathered together links to a range of resources, which include podcasts and links to other resource websites. Materials on the history of many topics in the second level mathematical curriculum are also available from the *Project Maths* website (v).

The HoM is such a broad subject area, that it can be difficult to cover a wide range of material in great depth. However, the breadth of potential course material provides an opportunity to be flexible, and each year I make slight adjustments to the content covered. While student feedback on my HoM module is generally very positive, the feedback received when I included some examples of Euler was exceptional. Students reported that they were fascinated by his approaches to establishing and demonstrating the results that we covered. The mathematical techniques that he used in these examples were very familiar to students from school, but a common feature of his work was Euler's *creative* interpretation and application of mathematical rules. Several students

considered that Euler was conducting a mathematical experiment, which made the material all the more accessible. These examples do not present a final rigorous proof of results, but rather give insight into how the results were established and approaches to solving them were developed.

Euler

Leonhard Euler [1707–1783] was a remarkable individual and mathematician; he was born in Switzerland but spent most of his mathematical career in St. Petersburg and Berlin. He was blind in one eye from an early age, and blind in both from about the age of 59, after which his productivity increased. Euler's contributions to mathematics are extensive and significant, including calculus, analysis, differential equations, infinite series and number theory. For those interested in further detail, the entry on Euler on the MacTutor website is a good place to start. For the purposes of this paper, we consider Euler's inspired treatment of complex numbers and the mathematical constant e.

Both Jakob Bernoulli [1654–1705], while working on compound interest, and John Napier (1550–1617) who first worked on logarithms, were aware of the mathematical constant $2.7182\ldots$ before Euler. However, Euler was the first to use the symbol e for this number, and he considered the exponential function. Indeed, Euler was the first to introduce the modern idea of a function and to write $f(x)$ for f a function of x. One key aspect of Euler's remarkable advances in mathematics was his skill in manipulating functions, and many authors highlight his ability to write mathematics in a more modern and accessible manner. Euler started the use of Σ for sum, and is also responsible for the widespread use of the symbol π in its current context, although he is not the first to have used it for this purpose.

Complex numbers and e

The origins of complex numbers (from solutions to cubic, not quadratic, equations) in the first half of the 16^{th} century is a remarkable story, for another article perhaps. Imaginary numbers were being used since the time of Rafael Bombelli (c 1526–1573), but it is safe to say that their users were rarely comfortable with the material. In his 1637 book *La Géométrie*, René Descartes (1596–1650) stated that *Neither the true nor the false roots are always real*, true referring to positive and false to negative. Gottfried Leibniz (1646–1716) called $\sqrt{-1}$ *that amphibian between being and non-being*.

Euler, who was the first to regularly use $i = \sqrt{-1}$, introduced $\sqrt{-1}$ in his 1770 *Elements of Algebra* as *neither nothing, nor greater than nothing, nor less than nothing* ..., he continued *we are led to the idea of numbers, which from their nature are impossible; and therefore they are usually called imaginary quantities, because they exist merely in the imagination*. Fortunately, he added ... *notwithstanding this, these numbers present themselves to the mind; they exist in our imagination and we still have a sufficient idea of them;* ... *nothing prevents us from making use of these imaginary numbers, and employing them in calculation*.

Make use of imaginary numbers he certainly did. In his famous 1748 work *Introductio in analysin infinitorum*, Euler considered $\cos(\theta) + i\sin(\theta)$ which appears, for example, in

$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1 = (\cos(\theta) + i\sin(\theta))(\cos(\theta) - i\sin(\theta)).$$

Euler demonstrated, essentially using proof by induction, that

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta), \tag{1}$$

where $\theta \in \mathbb{R}, n \in \mathbb{N}$. This also gives us

$$(\cos\theta - i\sin\theta)^n = (\cos n\theta - i\sin n\theta). \tag{2}$$

This is now known as de Moivre's Theorem. Abraham de Moivre (1667–1754) had a version of this before Euler, however, it was Euler who used the result to its fullest. Before we consider two examples of this, we must briefly look at some of the *bending* of mathematical rules that Euler frequently employed.

- **[A].** Let $x = n\theta$, where n is infinitely large, thus θ is infinitely small. Euler noted, that in these cases, $\cos(\theta) = 1$ and $\sin(\theta) = \theta = \frac{x}{n}$.
- **[B].** If n is infinitely large, then there could be no difference between n and $n-1, n-2, n-3, \ldots$. Euler would replace all such terms with n.
- **[C].** Euler let $e^{\omega} = 1+\omega$, for ω infinitely small. So, if a is a finite number, and n is infinitely large, then

$$e^a = (e^{\frac{a}{n}})^n = \left(1 + \frac{a}{n}\right)^n.$$

I encourage students to look at these through the lense of limits, which were not rigorously established until after Euler's death. They consider these interpretations of rules as *experiments* but can establish that Euler's intuition, as was often the case, was correct.

We now return to the two examples, which students can work through using techniques they have seen in school.

Taylor Series

Adding (1) and (2) yields

$$\cos n\theta = \frac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{2}. \tag{3}$$

Expanding (3) using the Binomial Theorem, gives

$$\cos n\theta = \frac{1}{2}[(\cos\theta)^n + \frac{(i)n(\cos\theta)^{n-1}\sin\theta}{1!} - \frac{n(n-1)(\cos\theta)^{n-2}(\sin\theta)^2}{2!}\cdots$$

$$+(\cos\theta)^n - \frac{(i)n(\cos\theta)^{n-1}\sin\theta}{1!} - \frac{n(n-1)(\cos\theta)^{n-2}(\sin\theta)^2}{2!}\cdots]$$

$$= \frac{1}{2}[2(\cos\theta)^n - \frac{2n(n-1)(\cos\theta)^{n-2}(\sin\theta)^2}{2!}$$

$$+\frac{2n(n-1)(n-2)(n-3)(\cos\theta)^{n-4}(\sin\theta)^4}{4!}\cdots].$$

So $\cos n\theta$

$$= (\cos\theta)^n - \frac{n(n-1)(\cos\theta)^{n-2}(\sin\theta)^2}{2!}$$

$$+\frac{n(n-1)(n-2)(n-3)(\cos\theta)^{n-4}(\sin\theta)^4}{4!}\cdots \tag{4}$$

Using **[A]** in (4), yields that $\cos x$

$$=(1)^n-\frac{n(n-1)}{2!}(1)^{n-2}\left(\frac{x}{n}\right)^2+\frac{n(n-1)(n-2)(n-3)}{4!}(1)^{n-4}\left(\frac{x}{n}\right)^4-\cdots \tag{5}$$

Now, using **[B]** in (5) reveals that

$$\cos x=(1)^n-\frac{n^2}{2!}(1)^{n-2}\left(\frac{x}{n}\right)^2+\frac{n^4}{4!}(1)^{n-4}\left(\frac{x}{n}\right)^4-\cdots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

Which is, remarkably, the Taylor Series expansion for $\cos x$. Subtracting (1) and (2), and following a similar process will yield the Taylor Series expansion for $\sin x$. Euler exploited these series with incredible success in his other works, for example, students get to work out his first demonstration of a solution to the *Basel Problem*. Further examples are available from (vi). For this article, we return to Euler's 1748 work.

Euler's Identity

Using **[A]** on (3),

$$\cos n\theta=\frac{(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n}{2},$$

yields

$$\cos x=\frac{(1+i\frac{x}{n})^n+(1-i\frac{x}{n})^n}{2}. \tag{6}$$

Recall **[C]**. $\mathrm{e}^{\omega}=1+\omega$, for ω infinitely small, and if a is a finite number, and n is infinitely large, then

$$e^a=\left(e^{\frac{a}{n}}\right)^n=\left(1+\frac{a}{n}\right)^n.$$

Euler * used $a=ix$ and $a=-ix$. When applied to (6), we get

$$\cos x=\frac{e^{ix}+e^{-ix}}{2}. \tag{7}$$

A similar process on $\sin n\theta$ establishes that

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i}. \tag{8}$$

Adding (7) to (8), gives Euler's Identity

$$\cos x+i\sin x=\frac{e^{ix}+e^{-ix}}{2}+i\left(\frac{e^{ix}-e^{-ix}}{2i}\right)=e^{ix}.$$

Conclusion and other projects

I believe that there is an opportunity to include some of Euler's methods outside of HoM courses. The examples shown in this article include methods that students at a certain level in school will be familiar with. Furthermore, the way Euler manipulated mathematics and *bent* the rules gives students added insight. Students who have seen this material, although it is a small sample, are generally very positive about it. It has led to in-class discussions on how mathematical results are established, what does proof actually mean, and what can you assume and not assume. It also allows students an opportunity to pursue further investigations on their own, for example, * Euler's use of a as a finite number in one line and letting $a = ix$ and $a = -ix$ shortly after.

I am involved in a separate initiative which aims to make the HoM more applicable to a wider range of students. I am developing HoM resource materials into small sections that can be easily incorporated into the teaching of mathematics to all levels in transition year. This project is flexible, as it aims to fit in with how mathematics is being taught in transition year in different schools. The idea is that material from the HoM can be used to introduce students to topics in mathematics, and suitable examples and materials can allow them to explore the relevance of these topics, where they came from, what they are used for in everyday life and why it is important that they study them. These ideas were piloted in two schools in 2017–18, Le Chéile Secondary School in Tyrellstown, and Kishoge Community College, Lucan. Initial feedback from both students and teachers was very encouraging. The resources are currently being reviewed, and if anybody is interested in getting involved, do not hesitate to contact me.

References

(i) *A History of Mathematics*, Carl Boyer, 2nd Edition, Wiley and Sons, 1991.

(ii) *An Introduction to the History of Mathematics*, Howard Eves, 6th Edition, Thomson-Brooks Cole, 1990.

(iii) The MacTutor History of Mathematics archive
`http://www-history.mcs.st-and.ac.uk/`

(iv) The History of Mathematics resource page, on the Maynooth University Mathematics Support Centre website
`http://supportcentre.maths.nuim.ie/hom`

(v) The Maths Development Team website `http://www.projectmaths.ie/`

(vi) *Euler: The Master of Us All*, William Dunham, The Mathematical Association of America, 1999.

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MathsFest 2017

MathsFest originated in Cork in 2010 in response to a growing need for information and knowledge by teachers in light of the new Project Maths curriculum. Under the reins of the Cork IMTA, MathsFest continued to grow in 2011, culminating in 2012 with approximately 500 educationalists attending the conference in UCC, travelling from all corners of Ireland and further afield. The international and national selection of speakers was very impressive and the day was hailed a roaring success. MathsFest moved to a new home for the following two years under the direction of the national IMTA and then went into hibernation until 2017 when it returned to the branch in Cork.

Five speakers gave presentations on different areas of relevance to mathematics teaching while several exhibitors had stand displays in the background. I first met Elisabet Mellroth at a summer school in Kassel, Germany and we became firm friends. Her work in the area of highly able students, shed light on a cohort of students that can often be neglected or overlooked.

The Educational Research Centre is an internationally recognised centre of excellence in research, assessment and evaluation in education. Research is undertaken at the request of the Department of Education and Skills, at the request of other agencies and on the initiative of the ERC itself and its staff. An assessment service is also provided through, for example, the supply of standardised tests to schools. This was something that I knew was of growing importance to teachers and we were lucky enough to have Rachel Cunningham from the ERC come and speak to us about it.

I first came into contact with Majella Dempsey when I was preparing to give a presentation to her pre-service teachers on the analysis and design of tasks. Suzanne Graham was not familiar to me personally before she came to Cork but I had noticed that many of her well-being workshops were regularly booked out. I got to know Aoife Kelly when she was an Education Officer with the NCCA and was delighted to invite her to give a presentation when I heard of her work involving the teaching of Geometry through Art.

A brief outline of each presentation at Mathsfest 2017 follows:

Highly able students — theories and implementation
by Elisabet Mellroth, University of Karlstad, Sweden.

Highly able students are not necessarily high achievers. The “worst” student in your classroom might actually be the most highly able. Perhaps he or she is sleeping through all your lessons, maybe he or she always interrupts you and argues with you or the other students? He or she might be at risk of becoming a drop-out. You as a teacher are just as important for the highly able as you are for the students with the most severe learning difficulties. In this seminar, theories on giftedness were presented along with what research suggests students with high ability in mathematics need in order to be able to develop their knowledge. Elisabet then focused on the diverse classroom and spoke about how and what Swedish teachers find possible to implement of these theories; some interesting concrete cases were given.

Drumcondra Online Testing System (DOTS). Use of standardised tests of mathematics at the end of Second Year by Rachel Cunningham, Educational Research Centre, Drumcondra.

Earlier in 2017, the Educational Research Centre launched its new online testing platform for schools. The Drumcondra Online Testing System (DOTS), is a cloud-based, fully integrated, secure system for the delivery of standardised tests to students in Ireland. Currently, there are three new standardised tests available to post-primary schools on the DOTS. This presentation focused on the Drumcondra Post-Primary Mathematics Test (DPPT-Maths), which is a computer-based, curriculum-linked assessment for students at the end of Second Year. The test was standardised in 2016 among a large, representative sample of Second Years. Each test item is linked to a learning outcome in the Junior Certificate syllabus and is further categorised by strand and by cognitive process. Class reports and individual student reports are available to schools almost immediately after testing. The presentation began with an overview of the DOTS, including how the system can be accessed and used by schools. This was followed by details of the main characteristics of the DPPT-Maths, including examples of real test items and the different types of reports available.

The problem with homework! by Majella Dempsey, Maynooth University.

Research tell us that homework in mathematics improves academic achievement. However, how do we design, implement and mark homework so we get the best from the experience for our learners? This workshop explored how we develop homework tasks, how students work on the tasks and how we correct and respond to that work within a 40-minute class. This is something that is not easy to do but it can be done! Majella looked at what research has to say about homework. What kind of homework is best? How might we mark homework? How might we design homework tasks? She accompanied this with a few examples from mathematics education research.

Teacher well-being and self-care by Suzanne Graham, PDST

Suzanne looked at the importance of teacher well-being and ensuring that teachers take care of themselves. The job of teaching is undoubtedly rewarding but it can be stressful too. She demonstrated ways that teachers could reflect on the way that they work and how to listen to what your body is telling you. In addition, she offered examples of mindfulness techniques and guided teachers towards ways of building up their emotional resilience. Make sure that you allow yourself time to relax after work as this is essential to maintain long-term physical and psychological health. The presentation concluded with tips on getting your work-life balance correct and helping teachers to feel in control and meet the daily challenges of the job.

Craft it with Maths — Teaching Geometry through Art by Aoife Kelly, St. Kilian's Community School, Bray.

This presentation explored the use of Art as a way of developing geometrical skills such as constructing angles, using geometrical tools and measuring and constructing shapes. By creating graphic art work, students develop understanding of theorems, angles and line and shape

properties. They use maths as a problem-solving tool and look at patterns in 2D polygons. This series of lessons makes connections across areas of learning and provides students opportunities to learn maths outside of the textbook, while meeting multiple Junior Cert. and Leaving Cert. syllabus outcomes. The Key Skills of the Junior Cycle and a number of statements of learning are integrated and developed through this approach.

Fr. Ingram Memorial Lecture

The Fr. Ingram memorial lecture was delivered by Leo Creedon from Sligo Institute of Technology. Dr. Leo Creedon is a mathematics lecturer in IT Sligo's School of Engineering and Design and is the manager of the Mathematical Modelling Research Group. The Fr. Ingram memorial lecture is a special lecture that is given in conjunction with the annual conference. It commemorates the late Fr. Richard Ingram S.J. (1916–1967) who greatly encouraged the founding of the IMTA. He was an Associate Professor of Mathematics (Modern Algebra) in UCD at the time of his passing, as well as being a member of the Royal Irish Academy. Leo spoke about a Cork mathematician named Robert Murphy.

Robert Murphy was born in Mallow in 1806, to a large family. He learned mathematics while recovering from a childhood accident, studying Euclid's Elements and a textbook on algebra, as well as reading several articles on astronomy, navigation and surveying. By the time he reached the teenage years, he had become an accomplished mathematician. A local schoolmaster went about arranging a more formal education for him.

In 1823, Murphy applied to Trinity College Dublin but was refused admission. With no other opportunities available in Ireland, he would have to look elsewhere. Murphy managed to gain admission to Gonville and Caius College, Cambridge. The people of Mallow were impressed with his mathematical aptitude and raised funds for him to study in Cambridge. Upon entry, he won several awards and honours, and graduated in 1829 as third wrangler — third best in mathematics in the entire university.

He produced twenty papers and two books, all of a high standard. It has been stated that he had a profound influence on George Boole and laid foundations for the development of aspects of modern algebra. Of course, it all went wrong and he fell into gambling and heavy drinking and sadly died (in 1843) before his time. He left a lot of incomplete work behind him. It is certain that he had a lot of unrealised potential but certainly left a mark after his short time as a mathematician.

Dr. Brendan O'Sullivan, IMTA Chairperson, Cork Branch.

Generalisations of n-Space Geometric Projections and Transformations

This article is based on our 2017 Young Scientist project. Both of us have a strong interest in the mathematics of higher dimensions. We found that it helps to have visualisation techniques to build up intuitions. This in turn helps when dealing with the more rigorous mathematics of shapes and points in these higher dimensions. In the course of our research, we noticed an absence of such visualisation techniques, especially for dimensions exceeding four.

With this in mind, we set out with the goal of developing a clear, implementable method of visualising shapes in n dimensions with a piece of online software. Our project's central mathematical aims were to generate formulae that would work in an arbitrary number of dimensions to express transformations and projections, and use this information to visualise n-dimensional shapes. In service of this goal, we set out to generalise the transformations of translation, scaling, and rotation. We also determined through our research that we would need to generalise the projection methods of perspective and orthographic projection to fulfil this goal.

When we first investigated transformations, we found that many of the solutions involved matrices and vectors — the core notations used in linear algebra. In these solutions, the vectors represented points, and matrices represented transformations. We also found that these transformations are divided into two categories, namely linear and affine transformations.

Linear transformations are transformations that manipulate space in such a way that preserves the location of the origin. These are the category of transformation that linear algebra is most naturally adapted to dealing with, and derives its name from. These transformations can be represented using square matrices with n elements in each row and column, where n is the number of dimensions being considered. However, we found that translation fell into the second category — it is an affine transformation. Affine transformations involve some composition of translation with a linear transformation, and thus do not preserve the location of the origin. Similarly, affine transformations cannot be easily represented with the use of a square matrix with n columns and rows. Instead, the matrices that correspond to affine transformations like translations can make use of what is known as a homogeneous coordinate system. To understand what a homogeneous coordinate system is, it is worth looking at the definitions and notations imperative to the mathematics of this project.

Definitions and Notation

Our project's notation and language is often esoteric and bemusing, so this section aims to explain the terms used to the reader.

1. **Vectors:** A vector is a geometric object that possesses only magnitude (length) and direction. A vector is frequently represented as a line segment, from the point A to the point B, and is denoted $\overrightarrow{AB}$. In our project, all vectors have an initial point at the origin $O = (0, 0, 0)$. Therefore, the coordinates $v = (0, 0, 1)$ is represented by the vector $\overrightarrow{OV}$, where V is the point $(0, 0, 1)$. Three of the most relevant operations that can be applied to vectors are outlined below.

   The simplest vector operation is vector addition. To add two vectors, $v = (a, b, c)$ and $t = (d, e, f)$, corresponding coordinates are added. Thus, $v + t = (a + d, b + e, c + f)$.

The second relevant operation used on vectors is scalar multiplication. This is the multiplication of a vector by a scalar. To multiply a vector by a scalar, each coordinate of the vector (x, y, z) is multiplied by the scalar a giving the vector (ax, ay, az). It is also possible to multiply the individual coordinates of the vector (x, y, z) by separate scalars (a, b, c) giving the vector (xa, yb, zc).

The third relevant operation is the dot product or scalar product of two vectors. The corresponding coordinates of the two vectors (x, y, z) and (p, q, r) are multiplied and then the products are added giving a single number, (a scalar), $xp+yq+zr$ as the result. Dot products for vectors as well as matrices are denoted by the symbol $\cdot$

2. **Matrices:** A matrix (plural matrices) is an array of numbers/expressions arranged in rows and columns. A matrix with 3 rows and 4 columns is termed a 3×4 matrix. When a matrix has the same number of rows and columns, it is called a square matrix. One primary function of matrices is to solve linear systems of equations. For example, the linear system of equations

$$x + 2y = 5$$
$$3x + 5y = 14$$

can be represented as a matrix equation as

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \end{pmatrix}$$

Matrices also find application in representing linear transformations. When you view the above example geometrically, the vector on the left, $\begin{pmatrix} x \\ y \end{pmatrix}$, represents a point, the matrix, $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, represents a linear transformation, and the resulting vector on the right, $\begin{pmatrix} 5 \\ 14 \end{pmatrix}$, represents the transformed point.

There are several common types of matrices which are useful and come up frequently in our project. The first of these is the identity matrix, which corresponds with the identity linear transformation. This matrix doesn't change the vector it is being multiplied with — it has the same effect as multiplying a number by one. It is given the name I or Id, and has the form

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Another common type of matrix which is used in this article is the diagonal matrix. The diagonal matrix is one which has non-zero elements only in the diagonal running from the upper left to the lower right of the matrix. A diagonal matrix can be used to multiply a vector by a scalar, for example, to multiply the vector $v = (x, y)$ by the scalar s we can do:

$$s \begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} sx \\ sy \end{pmatrix}$$

This can be extended to multiply the coordinates of a vector by separate scalars, for example, to multiply the coordintes of the vector $v = (x, y)$ by the separate scalars (a, b) we can do:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

3. **Matrix Operations:** There are two matrix operations which are used in this article. The first is Matrix Multiplication, which is the multiplication of two matrices together to yield a new matrix. To multiply two matrices, the number of columns in the first matrix must match the number of rows in the second.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

The product, C, of two matrices A and B is $C_{i,k} = A_{i,j}B_{j,k}$, where j is summed over the number of columns in A (or rows in B). In other words, the i^{th} row in the first matrix multiplies the k^{th} column in the second matrix and the products are added. Matrix multiplication is non-commutative, meaning the order the matrices are written in affects the answer. Matrix multiplication is also associative (the order of groupings does not matter) and distributive over matrix addition. Hence, for any matrices M_1, M_2 and M_3

$$M_1(M_2) \neq M_2(M_1), \quad M_1 \neq M_2 \neq O \neq I \tag{1}$$

and

$$M_3(M_2(M_1 v)) = (M_3 M_2 M_1)v \tag{2}$$

and

$$(M_1 + M_2)v = M_1 v + M_2 v \tag{3}$$

This associative property forms part of the reason why matrices are so useful in computer graphics. Instead of applying (i.e. multiplying) the matrices by the vector(s) one by one, the matrices can all be multiplied together. Therefore, all these matrices can be simplified into only just one matrix, considerably improving the efficiency of calculations. GPU's (Graphics Processing Units) in particular are used in computers to multiply matrices together very quickly.

The second matrix operation used in our project is the matrix inverse. The matrix inverse corresponds to 'undoing' the operation done by a matrix. The inverse of a matrix M is M^{-1}. Formally, $M(M^{-1}) = I$, the identity matrix.

The calculation of matrix inverses is more complicated, though, again, it is used heavily in computer graphics.

4. **Homogeneous Coordinates:** Homogeneous coordinates are defined formally as a set of coordinates (x_1, x_2, x_3) which correspond to a point (x, y), for which

$$\frac{x_1}{x_3} = x$$
$$\frac{x_2}{x_3} = y$$

In an arbitrary number of dimensions, n, the point $(x_1, x_2, \ldots, x_n)$ in Cartesian (standard) coordinates, corresponds to a set of homogeneous coordinates $p = (p_1, p_2, \ldots, p_n, p_{n+1})$. These homogeneous coordinates fulfil the property that

$$\begin{aligned} \frac{p_1}{p_{n+1}} &= x_1 \\ \frac{p_2}{p_{n+1}} &= x_2 \\ &\vdots \\ \frac{p_n}{p_{n+1}} &= x_n \end{aligned}$$

Homogeneous coordinates are used over standard Cartesian coordinates for a number of reasons. Formulae that make use of homogeneous coordinates are often simpler than the corresponding formula in Cartesian coordinates. They also allow for affine transformations to be easily represented as a matrix in a similar way to linear transformations.

In our project, we used a fairly simple definition of homogeneous coordinates. By this definition, vectors corresponding to Cartesian coordinates can be converted to homogeneous coordinates by adding an extra row to the bottom, with a value of 1. A similar process is performed on matrices, which have an extra row and column added to their bottom and to the right. All of these new elements have the value of 0 except for the bottom right value, which has the value of 1. For the sake of consistency, all coordinates in this article are represented as homogeneous coordinates. Homogeneous coordinates were also used in the visualisation software.

5. **Vector Basis and Vector Spaces:** A vector basis is the set of vectors that define what is known as a vector space. The vector basis of a vector space are the vectors that, when multiplied or added, can yield any other vector in the space. This means that the space is "closed" under the operations of vector addition and scalar multiplication. The vector space that can be made up by a vector basis is known as the "span" of the basis vectors. For example, if you have a set of basis vectors which includes unit vectors in the x and y directions, the span of the basis vectors will be the xy plane, which is the vector space closed under vector addition and scalar multiplication with the x and y unit vectors. Now consider the 3D vector space formed with the basis vectors of the x, y, and z unit vectors. The span of the x and y unit vectors is contained within this vector space and shares a common origin (the 0 point of all vector bases). Therefore, the xy plane is said to be a vector subspace of the larger vector space defined by the span of the x, y, and z unit vectors.

   A vector space with n basis vectors is identical to the real space of dimension n if and only if its basis vectors are all $linearly\ independent$. This means none of the basis vectors can have the same direction, and thus none can be formed from the addition or multiplication of other basis vectors. If some or all of the basis vectors are linearly dependent, the span will have fewer dimensions than the real space. One possible set of vector basis for Euclidean 3D-Space, $\mathbb{R}^3$, are the unit vectors. These are the linearly independent vectors that travel one unit in the x, y and z directions. These are commonly denoted $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$. The unit vectors are also known as the standard basis for any real space $\mathbb{R}^n$. The span of a standard basis in a real space $\mathbb{R}^n$ is always that space $\mathbb{R}^n$.

6. **Definitions:**

A variety of different terms were used in our project to indicate the number of dimensions being considered. It is important to note that all of these dimensions refer to Euclidean space, which is a standard geometric space where parallel lines never meet and the angles in a triangle sum to $180°$. Theoretical geometries can also be constructed where the angles in a triangle sum to less than or more than $180°$, which are known as hyperbolic and elliptic geometry, respectively. However, due to embedding theorems, any elliptic or hyperbolic geometric space can be embedded within a higher dimensional Euclidean space. In the project, we use the terms n*D*, n*-Space*, and n*-Dimensional* interchangeably to denote the idea of a Euclidean Space that has n dimensions.

When referencing matrices and vectors, the notation $\mathbb{R}^n$ is used to denote the Real Vector Space, denoted by n real numbers, which corresponds to the n-Dimensional Euclidean Space.

The term *Polytope* is used as a generalised version of polygon and polyhedron in n Dimensions. It is any flat-sided shape in n*-Space*.

The term *Higher Dimensional* indicates that the number of dimensions being considered is greater than 3. The prefix *Hyper* is also used alongside another term to indicate that it is in Higher Dimensional Euclidean space.

The term *Space* is used to refer to a Euclidean space or a real vector space, depending on the context.

Transformations in 2D

1. **Translation:** A translation is a geometric transformation that moves every point of a figure or a space by the same distance in a given direction. Translation is the same as the addition of a constant to a vector.

Definition: If v and p are arbitrary vectors, the translation operator T_v will work as

$$T_v \cdot \mathbf{p} = \mathbf{p} + \mathbf{v}$$

Using homogeneous coordinates, the operation is very simple to represent as a matrix. In two dimensions, to translate a point a distance v_x in the x direction and a distance v_y in the y direction, the translation matrix is: (see [Vin05, section 1.10.3])

$$T_v = \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix}$$

So to apply this translation to the arbitrary point (p_x, p_y)

$$T_v \cdot \mathbf{p} = \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + v_x \\ p_y + v_y \\ 1 \end{pmatrix} = \mathbf{p} + \mathbf{v}$$

The bottom row with the value of '1' due to homogeneous coordinates is ignored, meeting the definition of translation.

2. **Scaling:** Scaling is the geometric transformation that involves the sizing and resizing of objects. The transformation of scaling is divided into two categories, uniform scaling and non-uniform scaling.

   Uniform scaling increases or decreases the size of objects by a scale factor that is the same in all directions. The resulting object is geometrically similar to the original, and is congruent (identical) when the scale factor is 1. Non-uniform scaling is the opposite, where at least one of the scaling factors is different from the others. The resulting shape is not similar to the original.

   To scale an object in 2D by the scaling factor v_x, v_y, each point p needs to be multiplied by the 2D scaling matrix: (see [Vin05, section 1.10.1])

$$S_v = \begin{pmatrix} v_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Giving the result

$$S_v\mathbf{p} = \begin{pmatrix} v_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} v_x p_x \\ v_y p_y \\ 1 \end{pmatrix}$$

   The above equation describes scaling only about the origin. To scale a point p around an arbitrary point a, a translation matrix that moves the point a to the origin is applied to both points. Then, the point is scaled by the scaling factor, and finally the reverse of the original translation is applied.

3. **Rotation:**

   A rotation is a transformation which keeps a given point fixed and moves all other points at a fixed distance from that point.

   In two dimensions, only one angle is needed to specify rotation. This angle specifies how much the whole plane is rotated around the point. This angle is typically specified as the anticlockwise rotation from the positive x direction. In our project, any angle θ is specified in radians, which makes calculations easier.

   The matrix that describes a rotation must preserve the areas of any rotated figures. This is intuitive, as rotating a shape does not affect its area or size. The factor by which the area of a shape changes as it undergoes a linear transformation is known as its determinant. So, in other words, the determinant of a rotation must be equal to 1. There is technically another category of rotations known as improper rotations, which have a determinant equal to -1, however these were not explored in our project.

   In a rotation, the distance from the point being rotated around must be preserved also. This implies that the rotation's matrix inverse is the same as its transposed matrix, which is the matrix you get by turning the 1^{st} row of the old matrix into the 1^{st} column of the new matrix, the 2^{nd} row into the 2^{nd} column, etc. This means the rotation matrix is an orthogonal matrix. With this in mind, the rotation matrix is described as: (see [Vin05, section 1.10.4])

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

Multiplying this matrix by a vector describing the point p gives

$$R_\theta \cdot \mathbf{p} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_x\cos\theta - p_y\sin\theta \\ p_x\sin\theta + p_y\cos\theta \\ 1 \end{pmatrix}$$

This matrix corresponds with rotating the xy plane. It is important to note that this rotation matrix, like the scaling matrix, describes rotation about the origin. To describe rotation about an arbitrary point, a similar process as was done with scaling is applied, whereby a translation moves the points so that the centre of rotation is moved to the origin, the rotation is applied, and then the reverse translation of that is applied.

There is another rotation that instead of rotating the xy plane, keeps it fixed and rotates the coordinate frame (the axes) instead. The matrix for doing this is

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

If we were to apply this matrix to the point p it would not rotate it but instead would give the coordinates of p as viewed from the newly rotated coordinate frame. It turns out that these coordinates are the same as if p had been rotated backwards, i.e through an angle $-\theta$, in the original coordinate frame. That is, if we were to replace θ in (4) with $-\theta$ we would get (5).

Transformations in Higher Dimensions

1. **Translation:** If v be the n-dimensional vector $v = (v_1, v_2, \ldots, v_n, 1)$, then the n-dimensional translation matrix T_v is

$$T_v = \begin{pmatrix} 1 & 0 & \ldots & 0 & v_1 \\ 0 & 1 & \ldots & 0 & v_2 \\ & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & v_n \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \tag{6}$$

Multiplying by a point p makes it clear that this meets the definition of a translation in nD.

$$T_v \cdot \mathbf{p} = \begin{pmatrix} 1 & 0 & \ldots & 0 & v_1 \\ 0 & 1 & \ldots & 0 & v_2 \\ & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & v_n \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{pmatrix} = \begin{pmatrix} p_1 + v_1 \\ p_2 + v_2 \\ \vdots \\ p_n + v_n \\ 1 \end{pmatrix} = \mathbf{p} + \mathbf{v}$$

2. **Scaling:** Scaling, too, is trivial to generalise into n-dimensions. The matrix S_v becomes

$$S_v = \begin{pmatrix} v_1 & 0 & \dots & 0 & 0 \\ 0 & v_2 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & v_n & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{7}$$

Multiplying by a vector p in homogeneous coordinates

$$S_v \cdot \mathbf{p} = \begin{pmatrix} v_1 & 0 & \dots & 0 & 0 \\ 0 & v_2 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & v_n & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 p_1 \\ v_2 p_2 \\ \vdots \\ v_n p_n \\ 1 \end{pmatrix}$$

As before, the scaling is uniform if and only if $v_1 = v_2 = \cdots = v_n$. Also as before, this equation describes scaling only about the origin. To scale about an arbitrary point, that point would be translated to the origin by T_{-a}, the scaling matrix, S_v, would be applied, and the translation would be reversed by T_a.

For scaling around an arbitrary point, multiple transformation matrices are used — the scaling and the two translations. Hence, these matrices can be multiplied together to get one matrix which performs the entire transformation (in order, because, as stated before, matrix multiplication is not commutative). So, for an arbitrary centre of scaling, a, described as a vector and a scaling vector v, the multiplied matrices give

$$T_a S_v T_{-a} = \begin{pmatrix} v_1 & 0 & \dots & 0 & -a_1 v_1 + a_1 \\ 0 & v_2 & \dots & 0 & -a_2 v_2 + a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & v_n & -a_n v_n + a_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{8}$$

3. **Rotation:** Rotation in three dimensions (or higher) is non-commutative, unlike that in two dimensions. Rotations in three dimensions also require three angles to be fully specified. These angles specify the rotation of each of the three 2D planes that are formed by the three axes of the 3D space. These are the xy, xz, and yz planes. It seems natural, therefore, that any nD rotation can be performed through the composition of 2D rotations formed by its n axes.

   In n dimensional space, these 2D planes that compose rotations are rotated around an $n-2$ dimensional subspace. So, in two dimensions, rotations are performed around a $(2-2=0)$ dimensional space, i.e. a point. In three dimensions, rotations are performed around a $(3-2=1)$ dimensional space i.e. a line, and so on. Using this method of rotating 2D planes around subspaces at the origin to specify any rotation, $\binom{n}{n-2}$ angles must be specified. This notation stands for n choose $n-2$, and it describes all of the ways in

which $n-2$ objects (in this case, axes) can be chosen from n objects. The "choose" function is proportional to the factorial of n, meaning it balloons out in size rapidly — much more so than exponential growth. In 2 dimensions, 1 angle needs to be specified, in 3 dimensions 3 angles are required, in 4 dimensions 6 angles, in 5 dimensions 10 and by the time you have reached 10 dimensions, 45 separate angles must be specified. This approach is generalised from a paper written by Steven Hollasch [Hol91].

As mentioned before, the vector space $\mathbb{R}^n$ is specified by n linearly independent basis vectors. These are used as the axes.

Labelling the standard basis vectors $\{X_1, X_2, X_3, \ldots, X_n\}$, any standard plane ($\mathbb{R}^2$) can be specified by 2 of these vectors. An $\mathbb{R}^{n-2}$ space, on the other hand, would be specified by $n-2$ vectors. In a 1994 paper, [DB94], written by Kirk Duffin and William Barrett, there is a description of a rotation matrix, for rotating an $\mathbb{R}^2$ subspace, specified by 2 axes (standard basis vectors), through θ radians in the direction from axis a towards axis b:

$$R_{a,b}(\theta) = \left[\begin{array}{c|l} & r_{a,a} = \cos(\theta) \\ & r_{b,b} = \cos(\theta) \\ & r_{a,b} = -\sin(\theta) \\ r_{i,j} & r_{b,a} = \sin(\theta) \\ & r_{j,j} = 1, \quad j \neq a, \quad j \neq b \\ & r_{i,j} = 0, \quad \text{elsewhere} \end{array}\right] \tag{9}$$

This matrix can be used repeatedly to specify any arbitrary rotation around the origin. To do this, the values of a and b are changed to the indexes of the axes which specify the $\mathbb{R}^2$ subspace being rotated. This is the generalisation of rotation involving compositing 2D planes that was featured in Hollasch's paper.

To rotate around an arbitrary $\mathbb{R}^{n-2}$ subspace, let this subspace be specified by the set of vectors $V = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-2})$, where $\mathbf{v}_m$ is a vector. Let $\mathbf{v}_{m,n}$ represent the n^{th} coordinate of the m^{th} vector in V.

(a) First, the vector $\mathbf{v}_1$ must be translated to the origin, with the matrix $T_{-\mathbf{v}_1}$.

(b) Apply this matrix $(T_{-\mathbf{v}_1})$ to all of the vectors (which represent the points) on which you intend to perform the rotation.

(c) Now the rest of the vectors in V need to be aligned with the plane made by the n^{th} and $(n-1)^{\text{th}}$ basis vectors as follows:

(d) For each of the vectors in V, you must apply the matrix $R_{n,n-1}\left(\operatorname{atan2}\left(v_{m,n}, v_{m,n-1}\right)\right)$ to the vectors on which you want to perform the rotation, for each coordinate n in $\mathbf{v}_m$. (Here $atan2(x, y)$ is used instead of $atan(\frac{x}{y})$ as it gives the principal value of the arctan of $\frac{x}{y}$).

(e) This has the effect of making $\mathbf{v}_{m,n} = 0$.

(f) With everything oriented, the actual rotation through the angle θ can be performed with the matrix $R_{n-1,n}(\theta)$ on each of the vectors on which you want to apply the rotation.

(g) The inverses of all of the matrices previously used (excluding the rotation through the angle θ) can be applied, completing the rotation.

This procedure has been implemented in software by Jos De Jong using $Javascript$ and the math library *mathjs*.

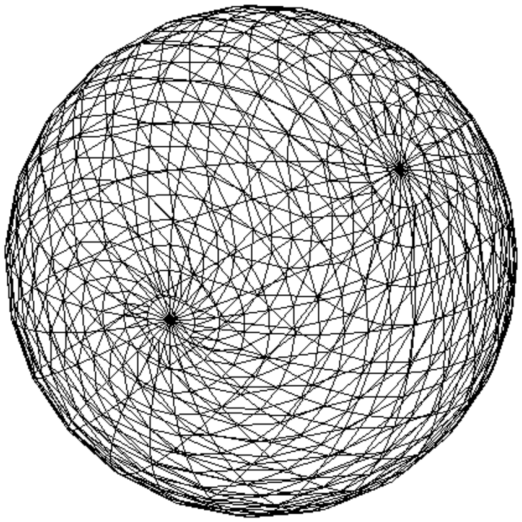
The Remainder of the Project

The remaining parts of the project involved finding a way to visualise n-dimensional objects and then developing software to achieve the visualisation. Space constraints do not allow us to describe the work involved here and is perhaps for another article so a brief outline will be given. We used two projection methods to achieve the visualisation — perspective projection and orthographic projection.

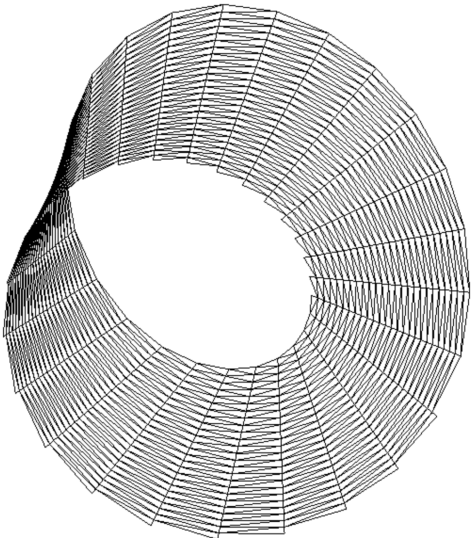
Perspective projection is the projection category that takes into consideration depth and distance from the observer to the object. In perspective projection, parallel lines appear to "meet" at a vanishing point, but distances are not preserved. Perspective projection is the most familiar projection type to us in everyday life, as it has been used for centuries by artists to represent 3D scenes on a 2D surface like a page.

Orthographic projection is the projection category that preserves parallel lines and distances but not depth. This type of projection is frequently used in technical drawing and graphics as well as CAD software. In everyday life we come across it when viewing the drawing plans of a house. We get a sense of what the house looks like by viewing say, the front elevation.

Once we had derived general matrices and formulae that described the transformations and projections we needed, we moved onto the last part of our project — writing a piece of software that would allow for visualisation of higher dimensional shapes. For this we used the programming language *Javascript* along with the mathematics library *math.js* for handling matrices and matrix operations and the graphics library *Three.js* for constructing the visualisations. The method for visualising shapes in 3D that we opted to use was wireframing, where the vertices and edges of an object are displayed by the computer. Wireframing, while losing some surface information, shows all edges of an object, and can also show the relationships between them. Our finished software is freely available online at `https://adamisntdead.github.io/higher-shapes` where you can view the visualisations. For those interested, the source code is available at `https://github.com/adamisntdead/higher-shapes`.



Wireframe of a sphere
generated by the software



Wireframe of a Möbius Strip

References

[DB94] Kirk L. Duffin and William A. Barrett. "Spiders: A New User Interface for Rotation and Visualization of N-dimensional Point Sets". In: *Proceedings of the Conference on Visualization '94*. VIS '94. Washington, D.C.: IEEE Computer Society Press, 1994, pp. 205–211. ISBN: 0-7803-2521-4. URL: http://dl.acm.org/citation.cfm?id=951087.951126.

[Hol91] Steven Richard Hollasch. "Four-Space Visualization of 4D Objects". MA thesis. Arizona State University, Aug. 1991.

[Vin05] John Vince. *Geometry for Computer Graphics: Formulae, Examples and Proofs*. Springer London, 2005.

Adam Kelly and Sam Enright, students at Skerries Community College, Co. Dublin.

John Courlander (1920–2018)

We regret to inform readers of the recent death of John Courlander (Dublin Branch). John was a regular contributor to the IMTA Newsletter, occasionally using pseudonyms such as 'I.M.A. Worm', 'A.E. Mouse' and 'John of Dalkey' especially for his poetic endeavours. Also, he made a number of informative and enthralling presentations to the Dublin Branch. As recently as 2012 he was the speaker at the Branch AGM, giving a talk on 'Discovering Pythagoras' — one of the many topics in which he had a deep and passionate interest. In 2014, he was present at the IMTA Golden Jubilee Celebration, being the oldest person at the event.

Born in England in 1920, he obtained some of his education in France. He flew with the RAF during WW2, for which service he was decorated with the Arctic Star medal in 2014. An engineer by training, he went on to teach for many years — notably at Rathdown school — after making his home in Ireland.

John was predeceased by his wife, Rosemary, and is survived by his children Charles, Lucy and Emma, grandchildren Rory, Grace, Sarah, Ally, Alice, Alex and Ben, and great-grandchild Anía, and sons-in-law Kieran and Brian. To all his family we extend our sympathies. May he rest in peace.

John at the celebration marking the close of the 40^{th} session of the Dublin Branch, 2005

Picture from Vimeo interview by Jonny Somerville
`https://vimeo.com/144036839`

The Dublin Branch.

Editor's note: Articles by John Courlander appeared in the following issues of the Newsletter:

No. 53 (Oct. 1983)	No. 58 (Apr. 1985)	No. 70 (May 1989)	No. 76 (June 1991)
No. 101 (Feb. 2003)	No. 106 (2006)	No. 109 (2009)	No. 110 (2010)
No. 111 (2011)	No. 112 (2012)	No. 113 (2013 - Special Archive Edition)	

Minus minus and minus by minus

When young learners have to handle negative numbers there are many difficulties that they encounter. Negative numbers can be a source of disenchantment with mathematics for them and can become a point of departure from believing that mathematics is real. Some of the difficulties are:

- The set of integers, $\mathbb{Z}$. They need to realise that integers have signs attached to them which indicate on which side of 0 on the number line they reside.
- The difficulty is that the signs are the same as those used for the operations of addition and subtraction. So when they see -2 they might read that as "subtract 2" but what are they to subtract 2 from?
- They also need to realise that the positive integers are the same as the set of natural numbers, $\mathbb{N}$, with which they are familiar. So when they see $5 + +2$ or $5 - +2$ they can drop the sign attached to 2 (but not the addition symbol) and write it as $5 + 2$ and $5 - 2$ respectively.
- Finally they will need to learn/discover that adding a negative number to another number is equivalent to subtracting the corresponding positive number. So when they see $5 + -2$ they can drop the sign attached to 2 and switch the addition symbol to subtraction to write it as $5 - 2$. i.e.

$$\begin{aligned} 5 + -2 &= 5 - +2 \\ &= 5 - 2 \end{aligned}$$

When introducing negative numbers we could consider everyday scenarios where it is appropriate to use negative numbers. Here are a few:

- The temperature of the weather — positive above freezing point, negative below.
- Floor level numbers in a lift in a building that has floors below ground level.
- Underground car parking that has several levels.
- A measuring post for sea or lake water levels above or below a mean level.
- A house in negative equity — the market value of the house less the amount outstanding on the mortgage.
- Bank balance on an overdrawn bank account.

We will now look at subtracting a negative number, multiplying a negative number by a positive number and multiplying any number by a negative number. What follows is by no means rigorous. All it attempts to do is to show the learner that operating with negative integers is a reasonable activity and not just dependent on following a set of made-up rules.

Subtracting a negative number

What is $5 - (-2)$? How do we show that subtracting a negative number is equivalent to adding a positive number? Calculating differences in temperature is ideal for doing this. What is the difference in temperature between (a) 7° and 3°, (b) 7° and 1°, (c) 41° and 13°, (d) 5° and -2°?

However we will not pursue this further but will use another scenario that although it is more contrived, will cover both subtracting a negative number and multiplying a negative number by a positive one. Suppose there is a shopkeeper who sells a particular type of lamp. It is his policy that when his stock of lamps falls to three lamps he re-orders to bring the stock level up to ten lamps. How much should he re-order? If the stock level has reached three then he orders $10 - 3 = 7$ lamps. Does he always re-order 7?

(a) Suppose he has 4 lamps in stock. A customer comes in and buys 2 lamps. His stock has fallen to 2 which is below the re-order level so he orders $10 - 2 = 8$ lamps.

(b) Again suppose he has 4 lamps in stock. Two customers arrive. The first customer buys 2 lamps and the second wants to buy 3 lamps. After selling to the first customer the stock level has fallen to 2 lamps but the second customer wants 3 lamps. The shopkeeper gives the 2 remaining lamps to the customer and promises to put aside the third lamp for her when the new stock arrives. What is the stock level now? It is not nought as the customer is owed 1 lamp. So the shopkeeper records the stock level as -1 to indicate that he was short 1 lamp for his customer.

How many does he re-order? He knows that it is ten less the number in stock so that is $10-(-1)$. He knows that he must order 1 for his customer for whom he was short one. That would bring the stock level back up to nought. Then he needs another ten to bring the stock level up to ten. That makes a total of $1 + 10 = 11$ lamps to be ordered.

So now we know that $10 - (-1) = 11$.
But we know that $10 + 1 = 11$.
Hence $10 - (-1) = 10 + 1$.
So it is reaonable to conclude that $-(-1) = +1$.

That is, subtracting a negative number is equivalent to adding the corresponding positive number, or more colloquially, ‘minus minus is plus’.

Subtracting a negative number has its counterpart in everyday speech through the use (usually unintentional) of the double negative. Sentences like “I don’t know no more”, “But I didn’t do nothing”, “I don’t know nothing about it” are sometimes heard. To say that you don’t know no more is to deny that you know no more and to deny that you know no more is to admit that you do know more! Using the symbol $\equiv$ for ‘is equivalent to’ we have that

I don’t know no more $\equiv$ I do know more.
But I didn’t do nothing $\equiv$ But I did do something.
I don’t know nothing about it $\equiv$ I do know something about it.

Multiplying a negative number by a positive number

Now on another occasion three customers arrive when the stock level is four. The first customer buys three which reduces the stock level to one. The second customer wants four lamps and the third three. The shopkeeper gives the one remaining lamp to the second customer but is short three. He is also short three for the third customer so he records the stock level as $(-3) \times 2$ to indicate that he owes three to both customers. His stock level is $(-3) \times 2$ which he works out as owing three to one customer and three to another. This makes a total of six that he owes. So it seems

reasonable to conclude that $(-3) \times 2 = -6$. ('minus by plus is minus')

How many does he re-order? According to his formula it is $10-(-3)\times 2$. He knows that he needs to order six for his two customers and another ten to bring the stock level up to ten. That makes a total of $6 + 10 = 16$ lamps to be ordered. So now we know that

$$\begin{aligned} 10-(-3)\times 2 &= 10-(-6) \\ &= 10+6 \\ &= 16 \end{aligned}$$

Notice that it is natural to multiply -3 by 2 before subtracting it from 10. (The shopkeeper needs to calculate how many he owes his customers *before* calculating how many to re-order).

Multiplying a number by a negative number

What does it mean to multiply a positive number by a negative number? What is $2 \times (-3)$? The shopkeeper's transactions are not much good to us here. There doesn't seem to be any good reason why he would need to multiply by a shortage of stock.

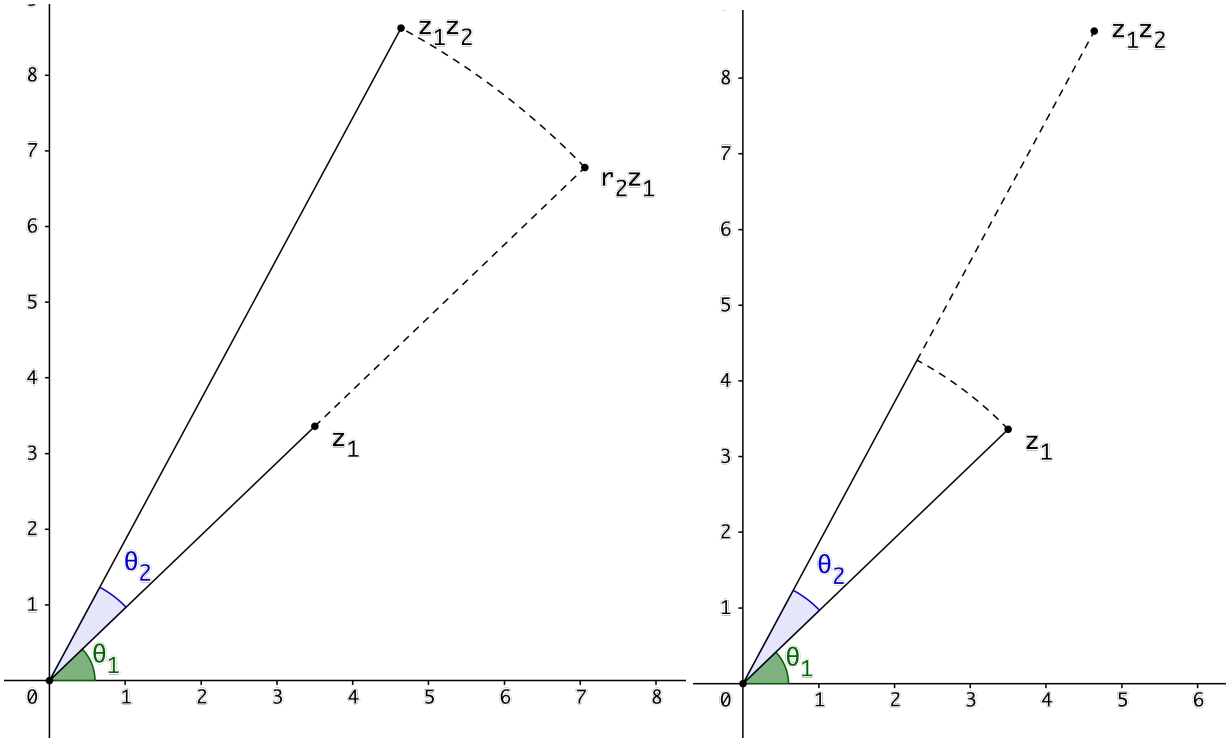
Let's turn to complex numbers and geometry to give us some inkling of what it could mean. Suppose there are two complex numbers z_1 and z_2. When expressed in polar form they are:

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + \imath \sin\theta_1) \quad r_1 = |z_1|, \quad \theta_1 = arg(z_1) \\ z_2 &= r_1(\cos\theta_2 + \imath \sin\theta_2) \quad r_2 = |z_2|, \quad \theta_2 = arg(z_2) \end{aligned}$$

Suppose we want to multiply z_1 by z_2.

$$\begin{aligned} z_1 \times z_2 &= r_1(\cos\theta_1 + \imath\sin\theta_1) \times r_2(\cos\theta_2 + \imath\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1 + \imath\sin\theta_1)(\cos\theta_2 + \imath\sin\theta_2) \\ &= r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \imath(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right) \text{ since } \imath^2 = -1 \\ &= r_1 r_2\left(\cos\left(\theta_1+\theta_2\right) + \imath\sin\left(\theta_1+\theta_2\right)\right) \text{ by using trigonometric formulae} \end{aligned}$$

If we draw a diagram of $z_1 \times z_2$ and consider z_2 as acting on z_1 we see that z_2 has caused z_1 to undergo an enlargement by a scale factor r_2 centred at the origin and a rotation about the origin through an angle θ_2. The enlargement arises from multiplying r_1 by r_2 and the rotation from adding θ_2 to θ_1.



$z_1 \times z_2$: Enlargement followed by rotation $z_1 \times z_2$: Rotation followed by enlargement

This two-part operation is known in geometry as a spiral similarity. The two operations — the enlargement and the rotation — can be done in either order, i.e. we can enlarge first and then rotate or rotate first and then enlarge as we can see in the diagram.
Either way the modulus ends up as $r_1 r_2$ and the argument as $\theta_1 + \theta_2$.

Now consider $2 \times (-3)$.

$$\begin{aligned} z_1 = 2 &= 2 + 0\imath \\ &= 2(\cos 0 + \imath \sin 0) \\ z_2 = -3 &= -3 + 0\imath \\ &= 3(\cos \pi + \imath \sin \pi) \\ z_1 \times z_2 &= 2 \times 3\left(\cos\left(0 + \pi\right) + \imath \sin\left(0 + \pi\right)\right) \end{aligned}$$

We see from this that -3 has acted on 2 by causing it to undergo an enlargement by a scale factor 3 and a rotation about the origin through π radians or $180°$.

So multiplication by a negative number is a two-part process:

(i) Rotate the first number through $180°$ about the origin (the position of nought on the number line).

(ii) Multiply the result by the size of the second number (its absolute value).

So to calculate $2 \times (-3)$

(i) Rotate 2 through $180°$ about the origin to get -2.

(ii) Multiply -2 by the size of -3 to get $-2 \times 3 = -6$.

What about -2×-3?

(i) Rotate -2 through 180° about the origin to get 2.

(ii) Multiply 2 by the size of -3 to get $2 \times 3 = 6$. ('minus by minus is plus')

If we consider multiplying by a positive number again, say -3×2.

$$\begin{aligned} z_1 = -3 = -3 + 0\imath = 3(\cos\pi + \imath \sin\pi) \\ z_2 = 2 = 2 + 0\imath = 2(\cos 0 + \imath \sin 0) \\ z_1 \times z_2 = 3 \times 2\,(\cos(\pi + 0) + \imath \sin(\pi + 0)) \end{aligned}$$

We see that 2 has acted on -3 by causing it to undergo an enlargement by a scale factor 2 and a rotation about the origin through 0° so effectively there is no rotation involved, only a scaling, when we multiply by a positive number.

To summarise: an everyday activity was used to illustrate subtracting a negative number and multiplying a negative number by a positive number but we had to resort to geometry to illustrate multiplying by a negative number.

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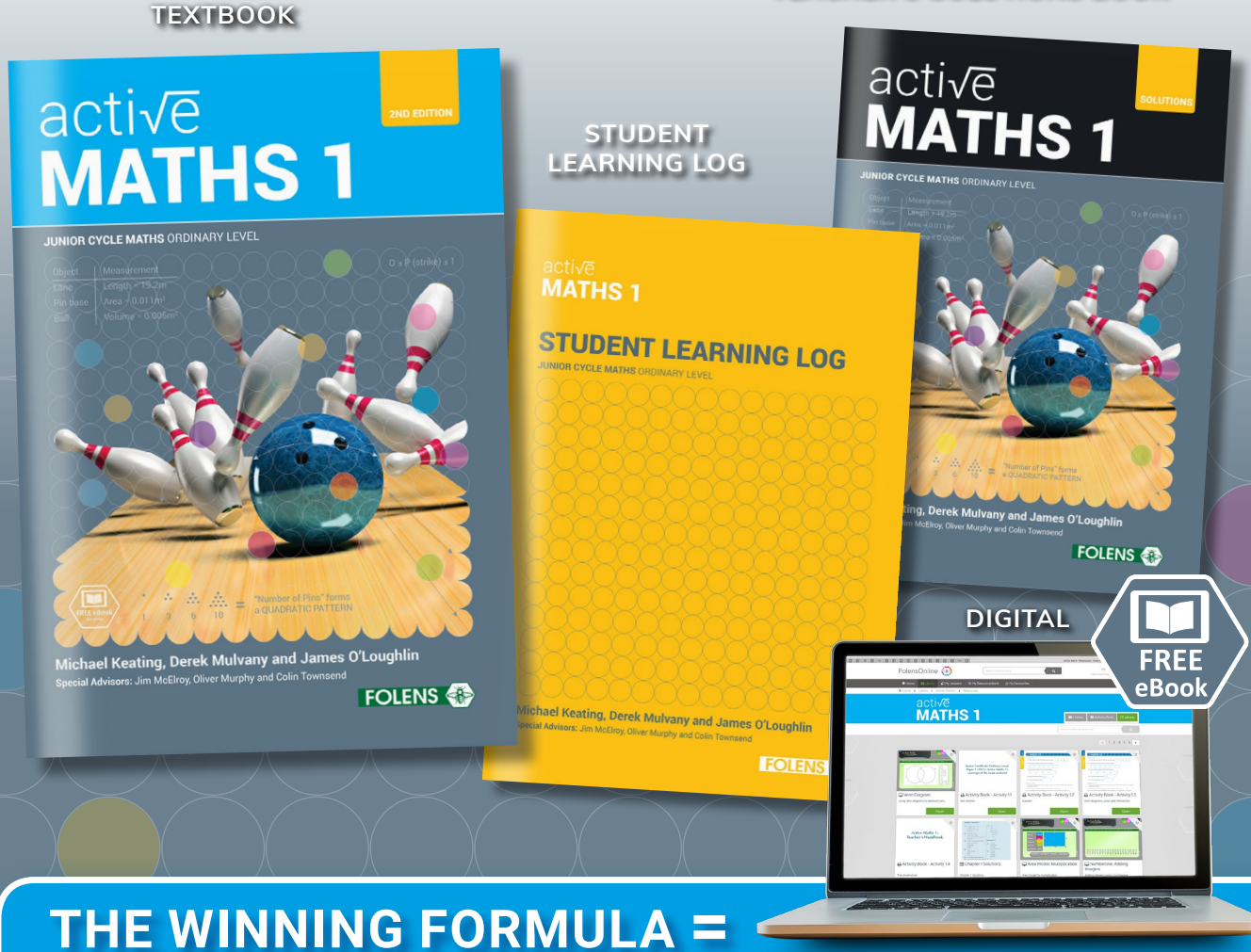

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