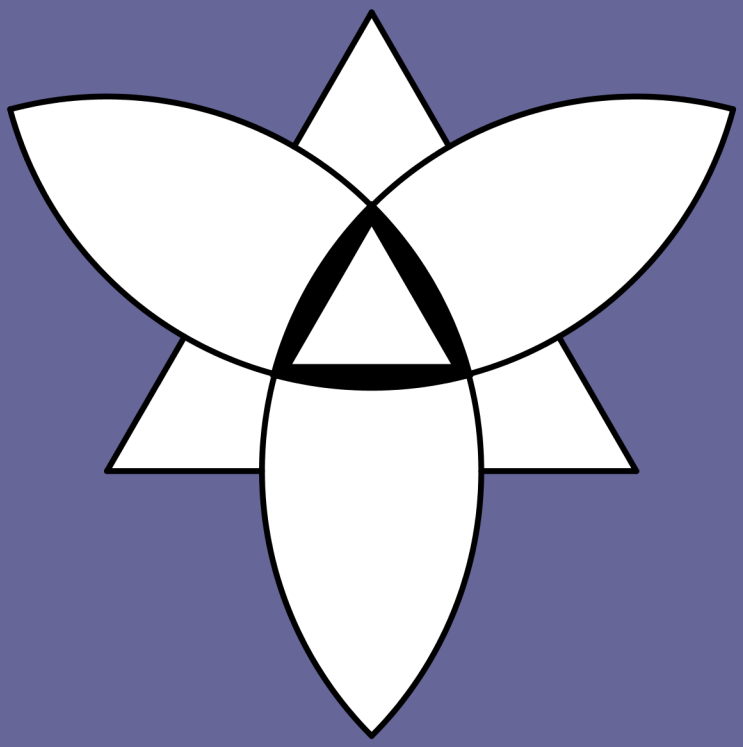
*Cumann Oidí Matamaitice na hÉireann*

# Newsletter

*Irish Mathematics Teachers' Association*



2017

Number 115

# Maths Development Team

## Important update

From September 1st 2017, the Maths Development Team(MDT) will form part of the Professional Development Service for Teachers. We will continue to provide professional development to post-primary teachers of Mathematics through the following supports:

- The effective use of Lesson Study: All new participants are asked to sign up at (https://goo.gl/forms/GJRS7GLuHdehVZqz1) for one of our three-day Induction Programmes, which will be held in Dublin, Carrick-on-Shannon and Limerick on September 27th , 28th and 29th , 2017. Closing date for receipt of applications: Monday 11th September 2017.
- A series of six Lesson Study meetings to be held from October 2017 to February 2018. These meetings will be conducted in over 50 locations nationally. Contact Administrator Gráinne Haughney: email grainneh@ecdrumcondra.ie Phone: 01-8576422 for venues.
- Ongoing participation in Lesson Study for all mathematics teachers.
- On-line support and resources available at www.projectmaths.ie
- Half-day school visits to support the on-going professional development of teachers. The primary focus will be on teaching and learning using reflective approaches with particular emphasis on:
  i. Algebra through the Lens of Functions,
  ii. Geometry through the Lens of Enquiry,
  iii. School-identified needs.

**Please note that substitution will not be provided for teachers availing of these school visits.**
If you wish to apply for a member of the team to visit your Mathematics Department during the 2017-2018 school year, please contact grainneh@ecdrumcondra.ie or racheld@ecdrumcondra.ie for an application form.

The MDT will host a fifth National Conference; *Maths Counts 2018: Engaging Post-Primary Teachers in Lesson Study* on March 2nd & 3rd , 2018.

We look forward to working together to improve teaching and learning of Mathematics.

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The IMTA would like to thank **Folens Publishers** for sponsoring the cost of producing this *Newsletter*.

Information about the IMTA is available at `www.imta.ie` and also on Facebook.

Note that the views expressed in this Newsletter are those of the individual authors and do not necessarily reflect the position of the IMTA. While every care has been taken to ensure that the information in this publication is up-to-date and correct, no responsibility will be taken by the IMTA for any errors that might occur.

**Thanks:**
To all the authors — without you there would be no Newsletter.
To Neil Hallinan, immediate past editor, who could not have been more helpful in giving me information, guidance and advice and who always responded promptly to my queries.
To Brendan O'Sullivan, Chair of IMTA, for his helpfulness and his work on obtaining sponsorship for the printing of the Newsletter.
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To Frances O'Regan, former editor, who went to a lot of trouble to find information for me about the IMTA emblem.
To Folens Publishers for sponsoring the printing of the Newsletter.
To Jane Campbell who helped with the design of the cover.

If you would like to have an article published in the Newsletter or have an idea for an article then email the editor at imta.newsletter@gmail.com. Michael O'Loughlin (editor)

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# Peter's Problem 2015

A family of two adults and three children have their water supplied by a private water scheme. They wish to minimise the cost of water in their home by harvesting rain water from the flat roof on the shed at the back of the house.

The shed measures $8$ metres by $8$ metres, and they believe that the amount of rainfall on the shed would best be measured as the average of the last three year's (i.e. $2012$, $2013$ and $2014$) rainfall at Mullingar as per the Met Eireann web site.

They can use all of the water that is harvested, although they acknowledge that at most $3\%$ of the water is lost through leakage and condensation and a further $2\%$ is used to wash their car and water the garden annually.

The family monitor their usage of water as follows:

Kitchen

Dishwasher is used $5$ times a week, using $10$ litres each time.

Washing machine is used 6 times weekly using 46 litres each time.
The water tap in the kitchen sink dispenses 7 litres of water per minute and is run daily for a total of 33 minutes.

Bathroom

The cistern has a full flush using 15 litres and a mini flush using 6 litres. The full flush is used on average 8 times a day and the mini flush 28 times daily.
The bath is run 3 times weekly using 120 litres of water each time. The shower is used 45 times a week.
Each shower lasts an average of 3 mins and 45 seconds, and the shower dispenses water at a rate of 9 litres per minute.
The hand basin taps in the bathroom flow at the rate of 5 litres per minute and are run every day for a total of 28 minutes.

Questions:

1. Using the above information, calculate the number of litres of water required by the family annually from the water scheme for household use.
   For the purposes of the problem, there are 365 days and 52 weeks in the year.
   The charges from the water scheme are as follows:
   Annual standing charge: €50
   Annual allowance: 30000 litres plus 10000 litres per person per household.
   Additional water usage is charged as follows:

   - .05 cent per litre for the first 80000 litres
   - .09 cent per litre for the next 50000 litres
   - .14 cent per litre for the next 80000 litres
   - .42 cent per litre for the next 60000 litres
   - .65 cent per litre for all other usage.

2. Calculate the amount of the savings made by harvesting the water. NOTE: There is no water from the water scheme used in the garden or for car washing.

3. If a leak in the system caused the annual bill to become €353.68, how many litres of water were lost through leakage?

4. If the water bill came to €240, calculate the actual rainfall for the year.

You may enhance your solution with charts and/or illustrations as you deem appropriate.

Dominic Guinan and Neil Hallinan

# Peter's Problem 2015 — The Solution

1. **Rainfall in Mullingar:**
   See: `http://www.met.ie/climate/monthly-data.asp?Num=875`
   $2012 : 1024.2\,\text{mm}$
   $2013 : 963.9\,\text{mm}$
   $2014 : 1203.5\,\text{mm}$

   Average: $\dfrac{1024.2 + 963.9 + 1203.5}{3} = \dfrac{3191.6}{3} = 1063.9\,\text{mm}$
   (correct to 1 decimal place) or $\mathbf{106.39}\,\text{cm}$.

2. **Water harvesting:**
   Roof Area: $8 \times 8 = 64\,\text{m}^2 = 640000\,\text{cm}^2$.
   Volume gathered per $1\,\text{cm}$ of rainfall: $640000\,\text{cm}^3 = 640$ litres.
   Water gathered: $640 \times 106.39 = 68089.6$ litres gathered in $1$ year.
   Wastage: $3\%$ evaporation + $2\%$ car/garden usage = $5\%$ wastage.

   $$\begin{aligned}\text{Water harvested:} &= 68089.6 - 5\% \text{ of } 68089.6\\ &= 68089.6 - 3404.48\\ &= 64685.12 \text{ litres}\\ &= \mathbf{64685} \text{ litres correct to the nearest litre.}\end{aligned}$$

3. **Household water usage:**
   Note: Use $52$ weeks per year for weekly usage figures; $365$ days per year for daily usage figures.

   **Kitchen**:
   Dishwasher: $5$ litres; $10$ times per week: $5 \times 10 \times 52 = 2600$ litres.
   Washing machine: $46$ litres; $6$ times per week: $6 \times 46 \times 52 = 14352$ litres.
   Water tap: $7$ litres per min.; $33$ min. per day: $7 \times 33 \times 365 = 84315$ litres.

   **Bathroom**:
   Cistern full flush: $15$ litres; $8$ times per day: $15 \times 8 \times 365 = 43800$ litres.
   Cistern mini-flush: $6$ litres; $28$ times per day: $6 \times 28 \times 365 = 61320$ litres.
   Bath: $120$ litres: $3$ times per week: $120 \times 3 \times 52 = 18720$ litres.
   Shower: $9$ litres per min.; $45$ times per week; $3$ min. $45$ sec. each:
   $9 \times 45 \times 3.75 \times 52 = 78975$ litres.
   Hand basin taps: $5$ litres per min.; $28$ min. per day: $5 \times 28 \times 365 = 51100$ litres.

   Total household usage: $\mathbf{355182}$ litres.

   **Q1. Answer: The total household usage is 355 182 litres.**

4. **The Costs:**
   Domestic Usage: $355182$ litres.
   Free usage Allowance: $30000$ litres basic; $5$ person-household – $10000$ litres each: $50000$ litres.
   Total: $80000$ litres.

Harvested: $64685$ litres.
Billable: $355182 - 80000 - 64685 = 210497$ litres.

First $80000$ litres @ $0.05$ c per litre: $80000 \times \dfrac{0.05}{100} = €40$.
Total billed so far: $80000$ litres.
Next $50000$ litres @ $0.09$ c per litre: $50000 \times \dfrac{0.09}{100} = €45$.
Total billed so far: $130000$ litres.
Next $80000$ litres @ $0.14$ c per litre: $80000 \times \dfrac{0.14}{100} = €112$.
Total billed so far: $210000$ litres.
Next: $210497 - 210000 = 497$ litres @ $0.42$ c per litre:
$497 \times \dfrac{0.42}{100} = €2.0874 = €2.09$ to the nearest cent.

Total bill:
Standing Charge €50 + Usage bill $(€40 + €45 + €112 + €2.09) = \mathbf{€249.09}$.

5. **The Savings:**
Harvested amount: $64685$ litres.
The final billable amount $(497$ litres$)$ was within the 'next $60000$ @ $0.42$ c per litre' range.
$60000 - 497 = 59503$ litres @ $0.42$ c per litre:
$59503 \times \dfrac{0.42}{100} = €249.9126 = €249.91$ to the nearest cent.

Remainder: $64685 - 59503 = 5182$ @ $0.65$ c per litre:
$5182 \times 0.65 = €33.683 = €33.68$ to the nearest cent.

Total: $€(249.91 + 33.68) = €283.59$
$[€(249.9126 + 33.683) = €283.5956 = €283.60$ to the nearest cent.]

**Q2. Answer: The total amount saved through harvesting water is €283.59.**

6. **A bill involving leaks:**

$$\begin{aligned} \text{New Bill amount} &= €353.68 \\ \text{Old Bill} &= €249.09 \\ \text{Increase in bill amount} &= €353.68 - €249.09 \\ &= €104.59 \end{aligned}$$

The final $497$ litres of the Old Bill were within the 'next $60000$ litres at $0.42$ c per litre' band. The remaining $59503$ litres @ $0.42$ c per litre would cost €$249.09$ extra. Therefore, the increase in the bill does not bring the usage of water beyond the '$0.42$ c per litre' band.

Then $104.59 \div \dfrac{0.42}{100} = 104.59 \times \dfrac{100}{0.42} = 24902$ litres have been lost due to leaks (to the nearest litre).

**Q3. Answer: 24 902 litres have been lost due to leaks.**

7. **A change in the bill due to a change in rainfall:**
New bill: €$240$.

Old bill: €$249.09$.
Reduction in bill: €$249.09 -$ €$240 =$ €$9.09$.

Rainfall figure of $106.39$ cm in a year gives the bigger bill. Therefore there must have been an increase in rainfall to allow more harvesting of water in order to reduce the bill.

Looking at the bands:
The final $497$ litres @ $0.42$ c per litre cost €$2.09$ which was saved.
€$9.09 -$ €$2.09 =$ €$7.00$ was further savings.

The previous band of $80000$ litres @ $0.14$ c per litre cost €$112$ in total. Therefore a fraction $(\frac{7}{112})$ of this amount has been saved.

$80000 \times \dfrac{7}{112} = 5000$ litres have been saved.

Total litres reduced from the billable amount: $497 + 5000 = 5497$ litres.
Total increase in harvesting of water: $5497$ litres which is $95\%$ of water gathered.

$$\begin{aligned}\text{Total increase in water gathered} &= 5497 \div \frac{95}{100} \\ &= 5497 \times \frac{100}{95} \\ &= 5786.3 \text{ litres} \\ &= 5786 \text{ litres correct to the nearest litre.}\end{aligned}$$

Since $640$ litres are gathered per cm of rainfall this means that
$5786 \div 640 = 9.040$ cm of extra rain fell.

Total rainfall: $106.39 + 9.04 = 115.43$ cm of rainfall.

**Q4. Answer: The actual rainfall for the year was $115.43$ cm or $1154.3$ mm.**

Neil Hallinan and Dominic Guinan

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# The Maths of a Rigged Roulette Wheel

This project was created for the Young Scientist and Technology Exhibition 2017. From experience, we find it easier to understand something when using an example and when we get a chance to 'do it ourselves'. Our $1^{st}$ year Maths Teacher has used a roulette wheel to explain to $1^{st}$ years the basic concepts of probability (and bias!) for many years. Being the curious little first years we were, we were intrigued by the wheel's bias and asked our teacher would he loan it to us to investigate further.

We decided to start our investigation at the start of $4^{th}$ year as we had finally found the time we needed; we also decided to enter the wheel as a project in the Young Scientist Competition.

### The Wheel

The wheel in question is actually an old 'fairground' roulette wheel. That is, it was commonly found in many Irish fairs and is rigged in such a way that was common amongst other biased fairground wheels.

The wheel has two parts: the wooden bowl in which the ball spins and the metal dish in which the ball lands. The metal dish spins upon an axle inside the bowl.

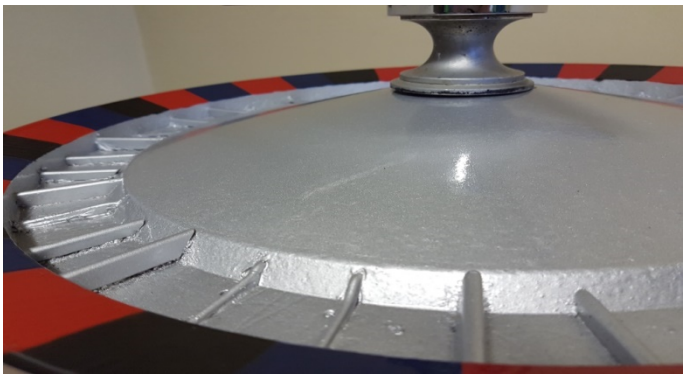

Figure 1: Roulette wheel showing different sized frets

The wheel has 36 pockets in total: 12 Red, 12 Black, 12 Blue.

The wheel can be biased in two ways:

(i) in favour of Red and Black

(ii) in favour of Blue.

This is because the metal frets are of different heights, allowing the ball to pass over some easily but stop at others. At each 'Large' fret, there is a Red/Black side and a Blue side meaning:

throwing the ball clockwise is biased in favour of Red/Black
throwing the ball anticlockwise is biased in favour of Blue

This is the basic cause of the bias, however we still could not statistically prove that the wheel was biased and were unsure of how much of an effect the frets truly had on the bias.

### Our Investigation of the Wheel

The wheel, like all other fairground roulette wheels, is handmade. As each wheel is unique no previous studies on biased roulette wheels are of any use to us. Therefore, the only way we could learn about the wheel was through obtaining our own primary data.

- We recorded the results of over 7,000 spins in a Microsoft Excel spreadsheet.
- We altered the size and density of the balls used to see how it affected the bias.

Figure 2: roulette ball types: cue, metal, big, green, pink, small

- We changed the direction we spun the wheel and ball to compare the wheel's behaviour.

**Results**

B = Ball, W = Wheel, C = Clockwise, A = Anti-Clockwise

1. Using confidence intervals, we can say with 99% confidence that the wheel is in fact, biased.
   - Big Ball: B.C./W.A.
   - The percentage relative frequency range for Blue when spinning for Red/Black is 13.76% – 20.20%.
   - Big Ball: B.A./W.C.
   - The percentage relative frequency range for Blue when spinning for Blue is 48.94% – 57.5%.
   - As there are an equal number of Red, Black and Blue pockets, the percentage relative frequency of each colour should be 33.3%, however that is not the case so we can reject the null hypothesis and accept the alternative hypothesis.
2. We found that a bigger ball makes the wheel more biased, but with a denser ball it becomes less biased. We used correlation coefficients to compare the results of the wheel spinning for Blue and against Blue:
   - Big Ball: $-0.6825638074$ Small Ball: $-0.319480728$
   - Big Ball: $-0.6825638074$ Metal Ball: $-0.3989640031$
     (Results closer to $-1$ mean the wheel is more biased.)
3. We found that the direction the ball spins is the most important in determining whether the wheel is biased in favour of Red/Black or Blue. Even if the wheel is spinning in the same direction as the ball, the effect it has is negligible.
   - B.C./W.A. v B.C./W.C. $-0.6825638074$
   - B.C./W.C. v B.A./W.A. $-0.7064821424$

**Conclusions**

We have statistical proof that the wheel is in fact biased. We know that the wheel is biased primarily because of the different sized frets. However, we also have evidence that both ball size and density

can increase or decrease the bias.

Most importantly, throughout the course of this project, we've vastly improved our understanding of probability and statistics. Not only do we understand, but we know where, when and how we can use our knowledge in real life, something a textbook cannot teach.

Of course this method of teaching has one major caveat — time, or in this case the lack thereof. As we all know, the sheer volume of the L.C. Maths syllabus often forces teachers to use the quickest methods for fear of not completing the course in advance of the L.C. examinations!

**Perhaps though, a focus on the quality of the learning over the pure volume learned would be a better approach**.

Thank You for taking the time to read this article about our project; we hope it was just as interesting and enjoyable to read as it was for us to make.

Calum O'Donnell, Patryk Lis and Kieran Murphy
Students at Ardscoil Uí Urmoltaigh, Bandon, Co. Cork

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# Mathematics at the Edward Worth Library

Very little is known about Dr. Edward Worth (1676–1733), an early eighteenth-century Dublin physician, who decided, in 1723, to leave his library to Dr Steevens' Hospital. We know that he was a son of John Worth (1648–1688), Dean of St Patrick's Cathedral, Dublin and that Worth junior studied for his B.A. in Merton College, Oxford, and his M.D. in the University of Leiden. Around the same time (1699), he became a member of the Royal Society, and it is clear from the collection of books he left to Dr Steeven's Hospital that he was deeply committed to all things scientific. His library offers students of the history of science a rich cornucopia of early modern texts, covering many areas. In particular, his collection suggests that he was especially interested in mathematical works, collecting not only texts he might have read as an undergraduate in the halls of Merton, but also works published in the later decades of his life, such as James Stirling's *Lineae tertii ordinis Neutonianae* (Oxford in 1717), Colin MacLaurin's *Geometria organic* (London, 1720) and, finally, a 1732 edition of Sir Isaac Newton's *Arithmetica universalis* (Leiden, 1732).

Some of these texts are investigated in an online exhibition, created in 2016, which continues to be accessible via the 'Exhibitions' tab of `www.edwardworthlibrary.ie`. This exhibition was curated by Dr Maurice O'Reilly, Senior Lecturer in the School of STEM Education, Innovation and Global Studies, DCU and formerly Head of the Department of Mathematics in St Patrick's College, Drumcondra, and his students, with the help of Mr. Fionnán Howard (DCU), myself, and Mr. Antoine Mac Gaoithín of the Edward Worth Library.

Although Worth was not a corresponding member of the Royal Society, it is clear that he was an avid follower of its publications, and, in particular, collected the many Newtonian commentaries emanating from the Society at that time (the latter are examined in the online exhibition 'Newton at the Worth Library' which includes a discussion of his copies of Stirling's and MacLaurin's texts). But Worth did not limit his scientific interests to the activities of Newton and his English coun-

Figure 1: Portrait of Edward Worth – Courtesy of the Trustees of the Edward Worth Library

terparts. His set of volumes of the *Acta Eruditorum*, the Leipzig equivalent of the *Philosophical Transactions of the Royal Society*, bears witness to his commitment to science in all its forms. Like the *Philosophical Transactions* it contained material on all areas of science, including mathematics. As Dr O'Reilly's recent paper on 'Mathematics from the Imprimerie Royale, Paris, at the Edward Worth Library' (presented at IHOM4 in June 2017), demonstrates, Worth was equally interested in mathematical works emanating from the Académie des Sciences in France.

Equally, Worth's mathematical collection includes many works by Jesuits. The importance of the Jesuit nexus of mathematicians in the early modern world cannot be over-estimated. Michael John Gorman has drawn attention to the role of mathematics, natural philosophy and experimentalism in Jesuit culture during the period 1580–1670 and, as Feingold (2003) reminds us, more generally 'we are still far from being fully conscious of the enormous contribution of Jesuit teachers to the formation of Catholic secular culture during the early modern period'. Undoubtedly one of the most famous Jesuit mathematicians was Christopher Clavius (1538–1612). Known primarily for his astronomical textbooks and his work on the new Gregorian calendar, he was the author of a number of works specifically devoted to mathematics, such as his 1574 commentary on Euclid. Worth owned the famous 5 volume 1612 Mainz edition of his mathematical works and to this he added the 1608 Genevan edition of Clavius' seminal edition of the astronomical works of Johannes de Sacro Bosco (c. 1195 – c. 1256). Clavius' textbooks were the result of years of experience teaching mathematics at the Academy of Mathematics at the Collegio Romano. However, as Baldini (2003) reminds us, his books reflected his many roles in the Academy, and therefore included not only the normal elementary course of mathematics available in other Jesuit colleges, but also the advanced research of the Academy. Both Clavius' work and Claud François Milliet Dechales' *Cursus* demonstrate that the Jesuit curriculum viewed mathematical education in its broadest sense — i.e. it did not limit itself to arithmetic and geometry but also incorporated related disciplines, such as astronomy, statics and optics. However, as Baldini notes, archimedean statics were excluded from the scheme, as were mechanics, since they were viewed by Jesuits as being part of natural philosophy. Dechales' (1621–1678) *Cursus*, first published in 1674, was bought by Worth in a 1690 Lyon edition. In effect, it was a course covering all areas of mathematics in the Jesuit sense, and proved to be immensely popular, but even it was outshone by Deschales'

translation of the *Elements* of Euclid (which appeared in 1660), which became a best-seller.

There were many reasons for this — not least the popularity of the *Elements* as a core mathematical text, not only in France but across Europe. The central role of geometry, one of the four subjects of the quadrivium[1], is attested by the sheer number of editions of Euclid which appeared in early modern Europe. This is reflected in Worth's own collection, which includes not one but three editions, dating primarily from the seventeenth and early eighteenth century. His earliest edition was by Federico Commandino (1509–1575), published in London in 1620; his second text was printed at the same venue some thirty-nine years later by the renowned Cambridge Fellow, Isaac Barrow (1630–1677); and, finally, his last edition of the *Elements*, by the Newtonian inspired David Gregory (1659–1708), was printed at Oxford in 1703. Likewise, the text (and commentaries on it) could also be found in Worth's many compilatory mathematical works such as Samuel Marolois' *Mathematicvm opus absolvtissimvm: continens geometriae, fortificationis, architectur, & perspectiv theoretic ac practic regulas* (Amsterdam, 1633) or the Jesuit Christopher Clavius' *Opera mathematica* (Mainz, 1611), not to mention commentaries specifically on it such as Peter Ryff's *Quaestiones geometricae: in Euclidis et P. Rami Stoicheiosin. in vsum scholae mathematicae* (Frankfurt, 1621). The predominance among these commentators of links with universities was no coincidence for Euclid was *the* geometrical text studied at universities by early modern mathematicians.

Figure 2: Euclid 1703 frontispiece – Courtesy of the Trustees of the Edward Worth Library

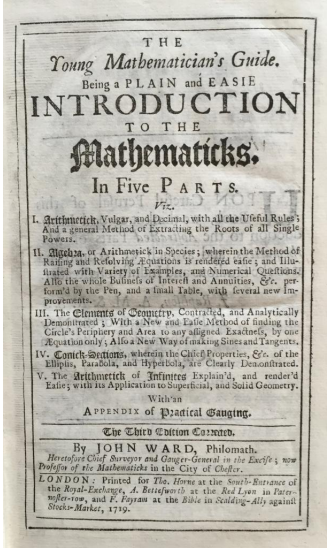
The crucial importance of mathematics generally, and geometry specifically, is vividly illustrated on the frontispiece of Worth's copy of Gregory's 1703 edition of Euclid's *Elements*. Picturing the Socratic philosopher Aristippus (c. 435 – c. 356 BC), arriving on dry land following his shipwreck, it depicts him pointing out signs on the ground and accompanies the image with a caption from Vitruvius: 'We can hope for the best for I see the signs of men' — the signs in this case being geometrical designs. As Remmert (2010) notes, the frontispiece thus not only drew attention to the antiquity of geometry but also to its civilising role: for Gregory, being a mathematician was intrinsic to being a civilised human being — the message to the reader was clear. Written c. 250 BC, Euclid's *Elements* sought to bring together all known theorems of plane and solid

[1] arithmetic, geometry, music and astronomy

geometry. The fact that no ancient Greek copies survived ensured that its textual history was long and complex. Worth's editions clearly reflect not only the complexity of the task of editing such a work but also the pre-eminent place of geometry in the mathematical sciences.

That printing mathematical textbooks in the vernacular, as well as in Latin, was considered lucrative by some early modern printers may be seen in the history of a small, and relatively insignificant book in the Worth Library: John Ward's *The Young Mathematician's Guide. Being a plain and easie Introduction to the Mathematics* (London, 1719). This small book became a mathematical bestseller in Georgian England and Ireland. As Benjamin Wardhaugh (2015) points out, the text was printed no fewer than twelve times at London alone and he estimates that by 1725 there were c. 25,000 copies of it in existence. It was reprinted at Dublin in 1731, 1755 and 1769, and a French translation was published in 1756. Ward's five-fold mathematical division of 'Arithmetic', 'Algebra', 'Geometry', 'Conic Sections' and 'Infinites', outlined on his title page, demonstrates the essential simplicity of his approach, a simplicity which proved immensely appealing to the mathematical marketplace.

Ward considered that his chief selling point was simply that, though many other mathematical authors had published on the subject, they all too often over-estimated the mathematical learning of their audience. By contrast, Ward's text was 'purely for the Use and Benefit of such as are wholly Ignorant of the very first *Rudiments (Or have not the least Notion) of Mathematicks*.' Ward's aim was to ensure a good grounding of the basics which would be a handy guide for aspiring mathematicians. For Ward, *The Young Mathematician's Guide* was a book which should be constantly referred to as an essential aid. Ward started with 'the very First Foundation or First Principles' (i.e. Arithmetic and Geometry), and gradually moved the reader forward to deal with more complex formulations. The aim was to provide the young reader with a basic mathematical education which would enable him to 'either Qualify him for Business; or fit and prepare him for higher Speculations without the Trouble and Charge of other Books.'



THE
*Young Mathematician's Guide.*
Being a PLAIN and EASIE
INTRODUCTION
TO THE
Mathematicks.
In Five PARTS.
*Viz.*

I. Arithmetick, Vulgar, and Decimal, with all the Useful Rules; And a general Method of Extracting the Roots of all Single Powers.

II. Algebra, or Arithmetick in Species; wherein the Method of Raising and Resolving Æquations is rendered easie; and Illustrated with Variety of Examples, and Numerical Questions. Also the whole Business of Interest and Annuities, &c. perform'd by the Pen, and a small Table, with several new Improvements.

III. The Elements of Geometry, Contracted, and Analytically Demonstrated; With a New and Easie Method of finding the Circle's Periphery and Area to any assigned Exactness, by one Æquation only; Also a New Way of making Sines and Tangents.

IV. Conick Sections, wherein the Chief Properties, &c. of the Ellipsis, Parabola, and Hyperbola, are Clearly Demonstrated.

V. The Arithmetick of Infinites Explain'd, and render'd Easie; with its Application to Superficial, and Solid Geometry.

With'in
APPENDIX of Practical Gauging.

The Third Edition Corrected.

By JOHN WARD, Philomath.
*Heretofore Chief Surveyor and Ganger-General in the Excise; now Professor of the Mathematicks* in the City of *Chester*.

LONDON: Printed for *Tho. Horne* at the *Sun's-Entrance* of the *Royal-Exchange*, *A. Bettesworth* at the *Red Lyon* in *Paternoster-row*, and *F. Fayram* at the *Bible* in *Scalding-Ally* against *Stock-Market*, 1719.

Figure 3: Ward's Title Page – Courtesy of the Trustees of the Edward Worth Library

In his *Young Mathematician's Guide* Ward was interested in two things: mathematical innovation

and the provision of a clear synthesis of what was already known. It is evident that he was interested in expanding mathematical knowledge as much as he could: Cohen and Shannon (1981) draw attention to his attempt in his *Young Mathematician's Guide* 'to use geometric methods to calculate pi' — the last attempt of its kind, where Ward indicated his desire to find $2\pi$ to 15 places (16 places of figures). His treatment of the subject demonstrates that he was aware of earlier work on the topic by mathematicians such as Ludolph Van Ceulen (1540–1610) and John Wallis (1616–1703). Works by both mathematicians were likewise collected by Worth and these proved to be only the tip of Worth's mathematical iceberg.

For Worth was as interested in innovation as Ward had been, and therefore purchased books not only on the mathematical staples such as arithmetic and geometry, but also on areas such as probability, infinities, notation, algebra and the application of mathematics. To find out more see 'Mathematics at the Worth Library' which may be found on the 'Exhibitions' tab at `www.edwardworthlibrary.ie`. If you would like to visit the Worth Library, you are very welcome — just check out our contact information on our website and book a tour!

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Dr Elizabethanne Boran, Librarian of the Edward Worth Library

# Mathematics comes to the aid of Cell Biology I

Much of biology is descriptive, how large? what shape? what colour? etc. so the role of mathematics in biology is not immediately apparent. Here we will see that a knowledge of maths is essential for analysing images that are obtained by microscopy. This is the first of two articles that will explore work that has been done in this area. We will introduce the biological context and show how maths can provide some assistance with solving a general problem. The second article will give a specific example and show how the data obtained revealed an important facet of cell activity in all organisms.

Everyone is aware that all organisms are made up of communities of cells each containing a library of information stored in DNA. Life depends on the ability of cells to express ('read') this genetic information and undertake specific tasks. We are striving to understand the structure and functioning of cells to help us deal with major issues facing the human race. These range from diseases and cancers to food production and the care of our planet.

Understanding the biology of cells is an immense task. Each cell is very small, typically less than one tenth of a millimetre ($100\,\mu m$) across, yet contains thousands of different complex molecular machines which either promote mechanical functions or perform thousands of different, specific, chemical reactions (enzymes). This activity results in the formation and maintenance of cellular structures, and in many cases enables the cell to grow and divide to form two new cells.

The small size of cells means that we can only visualise their internal structures with the aid of a microscope. Since these structures are often less than one tenth of a micron ($100\,\eta m$) across we need to use an electron microscope, as the optical systems in conventional light microscopes cannot visualise objects smaller than about $200\,\eta m$ across. As cells are packed with structures we need to cut the cell up into very thin slices (called sections, typically $70 - 120\,\eta m$ thick) so that we can see them. Think of trying to look at the internal structure of a loaf of bread with a magnifying glass; it is easier if you first cut it into slices. Observing thin slices through cells shows that they contain an array of enclosed compartments along with expanses of membrane surfaces. Molecular machines and enzymes are held in ordered arrays within the compartments and located on the membrane surfaces. This internal organisation serves to isolate the thousands of biochemical reactions from each other.

In looking at our slices of cells we see that we have reduced the 3D complexity of the whole cell to a series of 2D images. This process does, of course, kill the cells, so we see no movements, none of the teeming activity seen inside a living cell. With practice you learn to reassemble in your mind the original 3D structures that gave rise to these 2D images. You can even build an accurate 3D model of your cell from successive 2D slices taken through a single cell, much like reassembling a loaf of bread from the individual slices. Modern electron microscopes allow you to tilt the sections of cells, so you can take a series of pictures through a relatively thick slice at different tilt angles and build a 3D digital image of your cells.

We have learned that the internal structure of cells reflects their function; they are individually specialised for their role in life. Your kidney cells have very highly convoluted surfaces to maximise the area of cell surface available for the production of urine. Green leaf cells have huge areas of membrane surfaces, carrying molecules of the green pigment chlorophyll, which act as solar

panels collecting the sunlight energy required for plant growth.

Following such observations cell biologists came to believe that if a cell has more of a particular component it will have a greater capacity for a particular function (think of a car with a larger engine). Assessing 'more' or 'less' from the 2D images collected from cell sections is not straightforward. We can just look, and compare images, to make a qualitative assessment of 'more/less'. Clearly it would be preferable to have a quantitative method, so that a statistically-defined value could be assigned to each structure. These quantitative evaluation methods fall under the discipline known as 'Stereology'.

This is not just some ivory-tower aspiration. For example, thousands of experiments are being carried out by pharmaceutical companies on our behalf, searching for ways to alleviate our illnesses. Some of these experiments are assessed by examining the effects of new drugs on cells, so the investigators are faced with the need to make a statistically-sound judgment about the extent, if any, of changes to the cell structures induced by the treatment.

**Stereology:** the recovery of 3D information from 2D images

The problem of recovering 3D information from 2D plane samples through 3D structures is not confined to cell biology. In fact materials scientists had developed sound mathematical methods to tackle this problem many decades earlier. These were applied to flat smooth surfaces cut through such structures as metals and conglomerate rocks, often viewed by reflected light microscopy. Note that for these structures the data is derived from true 2D planes, whereas the microscope images are 2D projections of the thin 3D cell sections.

By applying these methods of stereology to microscope images of cells it is possible to obtain estimates of all the main parameters: numbers, sizes, volumes and surface areas. Many cell functions occur on membrane surfaces within the cell so, just like a factory floor, the greater the area the greater the capacity for production. For example, the membrane surfaces in your liver would cover a football pitch if spread out in a single layer (perhaps more if you overindulge); that is a lot of molecular machines!

Retrieving quantitative information from such 2D plane samples through solid objects is relatively simple if the inclusions are all discrete units, such as stones in concrete. Counting the number of stone profiles on the cut, flat, concrete surface enables us to calculate the number per unit area, $N_A \left(\frac{\text{no. counted}}{\text{area surveyed}}\right)$. If we know the average size ('height') of one stone, D, we can estimate the number of stones per unit volume of concrete,

$$N_V = \frac{N_A}{D} \tag{1}$$

Cell structures can be complex, so some caution is needed when applying the above to the image of a cell section. Some cells contain a 3D reticulate structure, think of a 3D fish-net lattice. Cell sections reveal many profiles cut through this net, but counting them would be erroneous as there is only one net in each cell.

## Spherical structures

Spherical structures pose special problems. If we take a series of random 2D planes through a population of identical spheres embedded in a matrix we will see circles scattered across our image. The largest corresponds to spheres sliced through their centres. Smaller circles result from slices that are off-centre. The mean diameter of all the circles will be smaller than the sphere diameter by a constant ratio, $4/\pi$.

For spheres it is possible to construct a theoretical relationship for the relative frequency of encountering circles of different sizes that can be derived from cutting sections through a population of spheres all of the same size. Problems arise, of course, if spheres of several different sizes are present in our 2D plane or cell section. We don't know which circles are median sections through smaller spheres and which are off-centre sections through larger spheres.

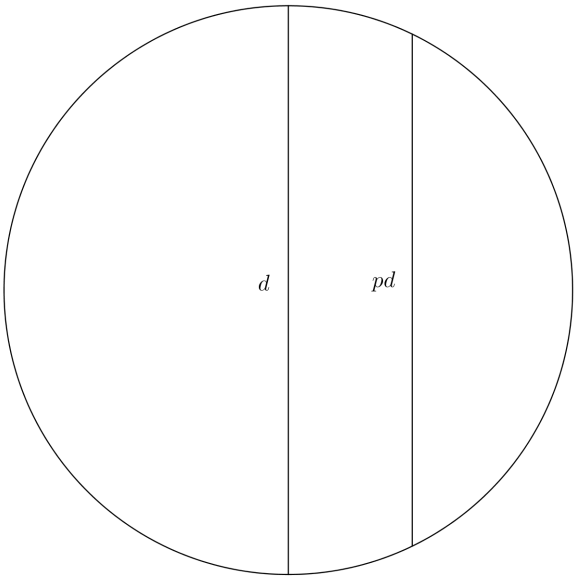


Figure 1: section through a sphere that does not pass through its centre

Consider a section through a sphere that does not pass through its centre as in figure 1. The section produces a circular section whose diameter is some fraction $p$, of the diameter of the sphere, where $0 < p < 1$.
Let the sphere have diameter length $d$ and radius length $r$.
The circular section has diameter length $pd$ and radius length $pr$.

Using Pythagoras' theorem we can calculate how far the circular section is from the centre of the sphere. Let this distance be $\ell$.

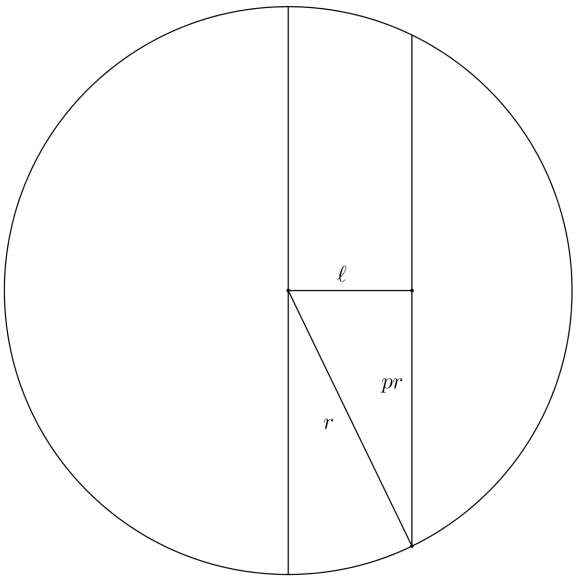


Figure 2: calculating the distance from the section to the centre of the sphere

From figure 2 we have:

$$\begin{aligned} \ell^2 + (pr)^2 &= r^2 \\ \ell^2 &= r^2 - p^2 r^2 \\ \ell^2 &= r^2(1 - p^2) \\ \ell &= r\sqrt{1 - p^2} \end{aligned}$$

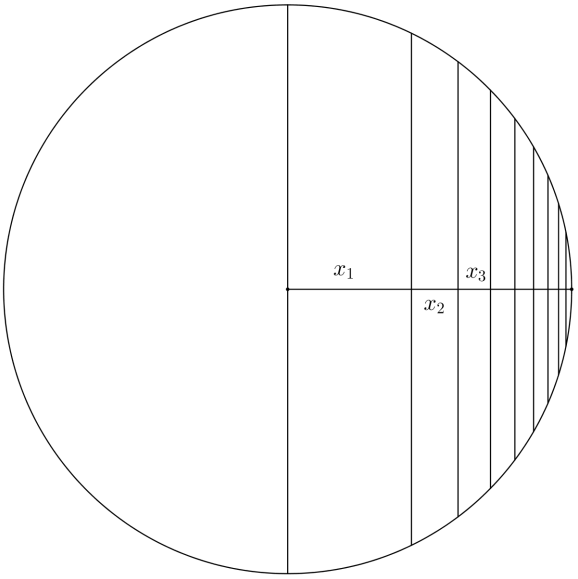


Figure 3: radius $\perp$ to the sections is divided into unequal intervals

Next consider several sections through the sphere. These have diameter lengths $pd$ for different values of $p$ and are shown in figure 3. These sections divide up the radius that is perpendicular to

them into intervals of unequal length, $x_1,\ x_2,\ x_3, \ldots$ as shown in figure 3. These lengths can be calculated by first calculating the distance of each section from the centre of the sphere and then subtracting each distance from its nearest neighbour to its right. For example if $\ell_1 (= x_1)$ is the distance of section 1 from the centre, $\ell_2$ the distance of section 2 and $\ell_3$ the distance of section 3, then

$$\begin{aligned} x_1 &= \ell_1 \\ x_2 &= \ell_2 - \ell_1 \\ x_3 &= \ell_3 - \ell_2 \end{aligned}$$

Letting $d_1$ be the length of the diameter distant $\ell_1$ from the centre, $d_2$ the length of the diameter distant $\ell_2$ and so on, then all those diameters which fall in the first interval ($d \geq$ diameter $> d_1$) have relative frequency $\dfrac{x_1}{r}$. Those which fall in the second interval ($d_1 \geq$ diameter $> d_2$) have relative frequency $\dfrac{x_2}{r}$ and so on.

Each $d_1$, $d_2$, $d_3, \ldots$ is of the form $pd, 0 < p < 1$.
Let $d_1 = p_1 d,\ d_2 = p_2 d$ and so on. Then

$$\begin{aligned} x_1 &= \ell_1 = r\sqrt{1 - p_1^2} \text{ from our earlier calculation.} \\ x_2 &= \ell_2 - \ell_1 = r\sqrt{1 - p_2^2} - r\sqrt{1 - p_1^2} \\ x_3 &= \ell_3 - \ell_2 = r\sqrt{1 - p_3^2} - r\sqrt{1 - p_2^2} \text{ and so on.} \end{aligned}$$

Hence the relative frequencies are:

$$\begin{aligned} \frac{x_1}{r} &= \frac{r\sqrt{1 - p_1^2}}{r} = \sqrt{1 - p_1^2} \\ \frac{x_2}{r} &= \frac{r\sqrt{1 - p_2^2} - r\sqrt{1 - p_1^2}}{r} && = \sqrt{1 - p_2^2} - \sqrt{1 - p_1^2} \\ \frac{x_3}{r} &= \frac{r\sqrt{1 - p_3^2} - r\sqrt{1 - p_2^2}}{r} && = \sqrt{1 - p_3^2} - \sqrt{1 - p_2^2} \\ &\vdots \end{aligned}$$

Notice that the relative frequencies are independent of r, the length of the radius of the sphere, so that if we did the same analysis on any other sphere we would get the same results.

We can use the frequency of encountering circles from one size class of spheres to help us analyse the total circle population recorded from a population of spheres of mixed sizes. If we place our circles into a series of size classes in a table, the largest size class will correspond to sections through the centres of the largest spheres. Since the spheres are not all the same size, each smaller size class will contain circles derived from several different size classes of spheres. Starting with the largest size class of circles we can calculate the number of smaller circles in each of the smaller size classes that are derived from these largest spheres and deduct them from each of the remaining size classes of circles in the table.

Now we move to the next smallest size class of circles in our table of observations and repeat the process, continuing iteratively until we have accounted for all the circles in our observations. This will give us the size distribution of the spheres in the cells. We can calculate from equation (1) the number of spheres per unit volume ($N_V$) from the number of circles per unit area ($N_A$) of cell examined. If we know the volume of the sample we can then find the total number of spheres present.

$$\text{number of spheres present} = \text{volume of sample} \times N_V$$

The next article will look at a specific example of this process.

Martin Steer, Professor Emeritus in the School of Biology and Environmental Science, UCD

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# Mathematics comes to the aid of Cell Biology II

In the previous article we provided an introduction to cell biology and showed how maths can be used to extract quantitative data about the 3D structures in cells from 2D information contained in pictures of thin slices of cells viewed in an electron microscope. Here we will describe the successful application of these methods to enhance our understanding of a fundamental cell process: growth.

All cells, animal and plant, are contained within a surrounding membrane. To grow, the area of this membrane must be increased. New membrane is made inside cells in the form of small spherical balloons called vesicles, about $80$ – $200\,\eta m$ in diameter. These migrate to the cell surface and fuse with the outer cell membrane. This provides additional surface area to the cell, equivalent to the surface area of the spherical vesicle. It also releases the contents of the vesicles to the world outside of the cells. This is the origin of the enzymes that digest your food, for example.

How rapidly do vesicles arrive at the cell surface?

We cannot see vesicles moving, the cells are dead and sliced up for microscopy. Our images are a snapshot of an instant in time. Vesicles move continuously from their sites of formation through the cell interior and disappear when they fuse with the cell surface. The problem of determining the rate was solved by halting the movement of vesicles for a few minutes before the cells were killed for microscopy. In that time vesicles accumulated in the cell. So we can repeat our microscopy and use stereology to find the extra number of vesicles present after a fixed period of time (usually 5 or 10 minutes). Think of a photograph of a constant stream of cars travelling along a road. Counting them in the photo will not tell us how many cars are using the road. If there is a traffic light that turns red we can take another photo of the queue after 2 minutes and then count the cars in this photo.

In this example we are using data obtained from growing pollen tubes. These are thin tubes that grow from pollen grains when a flower is pollinated. They grow down into the ovary and deliver male nuclei to the egg for fertilisation. Each tube grows by addition of new membrane to the dome at the growing tip. As only one pollen tube can fertilise each egg it is a race, the fastest growing tubes are likely to be successful. Hence there is a strong evolutionary pressure for the development

of fast growing pollen tubes.

Pollen from some plants can be persuaded to germinate and grow in an artificial medium. So pollen was germinated and allowed to grow for a short period and the rate of tube extension was recorded. Then one batch was treated with a drug to stop vesicle movements for a few minutes and both batches were processed for microscopy.

Sections of the pollen tube tips were scanned for vesicle profiles and the number and diameter of all the circles was recorded, along with the area of cell surveyed. This data was then analysed using the methods described in the preceding article to find the rate of arrival of vesicles at the tips of the growing pollen tubes.

Recall from the previous article that we had worked out the distances of the various slices from the centre of the sphere and the relative frequencies of diameters within class intervals. Now consider sections through the sphere which give rise to diameters whose lengths form a geometric sequence: $1,\ pd,\ p^2d,\ p^3d, \ldots$ In that case the distances of the sections from the centre of the sphere are:

$$\begin{aligned}
\ell_1 &= r\sqrt{1-p^2} \\
\ell_2 &= r\sqrt{1-p^4} \\
\ell_3 &= r\sqrt{1-p^6} \text{ and so on.}
\end{aligned}$$

The relative frequencies are then:

$$\begin{aligned}
\frac{x_1}{r} &= \sqrt{1-p^2} \\
\frac{x_2}{r} &= \sqrt{1-p^4} - \sqrt{1-p^2} \\
\frac{x_3}{r} &= \sqrt{1-p^6} - \sqrt{1-p^4} \\
&\vdots
\end{aligned}$$

It was the materials scientist S.A. Saltykov who developed the idea of using a geometric sequence; this obviated the need for using look-up tables.
In the Saltykov method the value chosen for $p$ is $10^{-0.1}$. Everything is now set up to do the calculations in a spreadsheet. Figure 1 shows the spreadsheet.
Column 2 is just the geometric sequence: $1, p, p^2, \ldots$ with $p = 10^{-0.1}$.
Column 4 is the sequence: $\ell_1,\ \ell_2,\ \ell_3,\ \ldots$ shown above and which has been divided by $r$. It can be easily calculated from column 2. Note that it is the relative distance of the **lower** bound of the interval.
Column 5 is just the sequence: $\ell_2 - \ell_1,\ \ell_3 - \ell_2,\ \ldots$ divided by $r$ and is calculated from column 4 just by doing successive subtractions.

The relative frequencies of each class interval are now used to work out the number, $N_V$, of vesicles per $\eta m^3$ in each class interval. The formula to do so was worked out by Saltykov and involves a set of coefficients that are derived from the relative frequencies. Recall the equation $N_V = \frac{N_A}{D}$. This equation is used in each class interval with an adjusted $N_A$ and a proportion of each of the higher class $N_A$ stripped away to remove the circles that belong to larger spheres. The working formula

| Group No. | Relative Section Diameters | Relative Intervals | Relative Distance from centre | Relative frequency |
|---|---|---|---|---|
| 1 | 1.0000 | 1.0000 - 0.7943 | 0.60749 | 0.60749 |
| 2 | 0.7943 | 0.7943 - 0.6310 | 0.77582 | 0.16833 |
| 3 | 0.6310 | 0.6310 - 0.5012 | 0.86534 | 0.08952 |
| 4 | 0.5012 | 0.5012 - 0.3981 | 0.91734 | 0.05200 |
| 5 | 0.3981 | 0.3981 - 0.3162 | 0.94868 | 0.03134 |
| 6 | 0.3162 | 0.3162 - 0.2512 | 0.96794 | 0.01925 |
| 7 | 0.2512 | 0.2512 - 0.1995 | 0.97989 | 0.01195 |
| 8 | 0.1995 | 0.1995 - 0.1585 | 0.98736 | 0.00747 |
| 9 | 0.1585 | 0.1585 - 0.1259 | 0.99204 | 0.00468 |
| 10 | 0.1259 | 0.1259 - 0.1000 | 0.99499 | 0.00294 |
| 11 | 0.1000 | 0.1000 - 0.0794 | 0.99684 | 0.00185 |
| 12 | 0.0794 | 0.0794 - 0.0631 | 0.99801 | 0.00117 |
| | 0.0631 | | | |

Figure 1: Saltykov area analysis: group limits and relative frequencies

is:

$$\begin{aligned}(N_V)_i = \frac{1}{D_i}[&1.6461\,(N_A)_i - 0.4561\,(N_A)_{i-1} - 0.1162\,(N_A)_{i-2} \\ &- 0.0415\,(N_A)_{i-3} - 0.0173\,(N_A)_{i-4} - 0.0079\,(N_A)_{i-5} \\ &- 0.0038\,(N_A)_{i-6} - 0.0018\,(N_A)_{i-7} - 0.0010\,(N_A)_{i-8} \\ &- 0.0003\,(N_A)_{i-9} - 0.0002\,(N_A)_{i-10} - 0.0002\,(N_A)_{i-11}]\end{aligned}$$

where $i$ is the index of the class interval. The formula varies according to the index, e.g.

$$(N_V)_1 = \frac{1}{D_1}[1.6461\,(N_A)_1]$$
$$(N_V)_2 = \frac{1}{D_2}[1.6461\,(N_A)_2 - 0.4561\,(N_A)_1] \text{ and so on.}$$

The value of $D_i$ to use is the **upper** bound of interval$_i$.
Figure 2 shows a sample vesicle size distribution. The second column is just column 3 from figure 1 multiplied by the largest diameter of the circular cross sections. The third column is also derived from raw data; the circular cross sections from a set of random selections of sections are measured and counted and allocated to their respective class intervals. Then the numbers are divided by the total area of the sections to give the density per unit area. The numbers for the density per unit volume are worked out from the working formula above. For example the fourth size class figure

| Group No. | Diameter of circles (ηm) | No. of circles per μm sq. | Mean class diameter of vesicles (ηm) | Number of vesicles per μm cubed |
|---|---|---|---|---|
| 1 | 333 - 265 | 0.085 | 298.8 | 0.42 |
| 2 | 265 - 210 | 0.456 | 237.3 | 2.69 |
| 3 | 210 - 167 | 1.237 | 188.5 | 8.65 |
| 4 | 167 - 133 | 1.609 | 149.7 | 12.15 |
| 5 | 133 - 105 | 1.601 | 118.9 | 13.11 |
| 6 | 105 - 84 | 1.199 | 94.5 | 9.46 |
| 7 | 84 - 66 | 0.240 | 75.0 | 0.00 |
| 8 | 66 - 53 | 0.085 | 59.6 | 0.00 |
| 9 | 53 - 42 | 0.008 | 47.3 | 0.00 |

Figure 2: Vesicle sizes and frequencies

was calculated as follows:

$$\begin{aligned}(N_V)_4 &= \frac{1}{0.167}[1.6461\,(1.609) - 0.4561\,(1.237) \\ &\quad - 0.1162\,(0.456) - 0.0415\,(0.085)] \\ &= 12.14\end{aligned}$$

Remember that the diameter is measured in $\eta m$ and must be converted to $\mu m$ in the calculation. Also note that the discrepancy $(12.14$ v $12.15)$ is because the spreadsheet stores the figures to a greater level of accuracy than it shows in the display of them. It is worthwhile to work through one or more of the calculations.

It is possible to calculate the rate of additional membrane surface required to maintain the growth of the pollen tubes from the growth rate and the tube size. This rate can now be compared with area of membrane delivered by the arrival of vesicles at the rate determined above. The results were very surprising, vesicles deliver a much greater area of new membrane than needed for tube growth. In fast growing tubes only about $30\%$ of the delivered vesicle membrane is required for growth; for slow growing tubes only about $10\%$ is used. So there is no control mechanism that regulates vesicle production to match tube growth rate. This was a surprise to the research workers at the time, who were hoping to study the control of vesicle formation rates. Clearly the nature of the race to the eggs is such that there is no point in conserving vesicle production, it is literally a do or die situation!

As the membrane surface remains smooth, not all folded up, this additional area must be retrieved. Some years later it was shown that this retrieval process occurs further back up the pollen tube, behind the tip.

For cells that are not growing in size, in both plants and animals, vesicles deliver enough new cell surface area to make a whole new surface once every 10 minutes. This is called ‘turnover’ and it recycles the additional area delivered back into the cell. All this was discovered with the aid of mathematics.

These results were arrived at over 30 years ago, and contributed a little to our modern understanding of cell biology. These turnover results are typical of all cell components, except the DNA. So you are, quite literally, not the person you were a few weeks, months or years ago. Your cell structures have been renewed and replaced many times in that period.

*Addendum*

Human liver

For readers who may have been sceptical, or just interested, here is the stereology data to back up the contention that your liver contains sufficient membrane to cover a football pitch. The values given are 'means' or 'typical' for the parameters given:

Volume of an adult liver: $1862\,cm^3$

From microscopy and stereology

Proportion of liver occupied by liver cells: $77.8\%$

Proportion of liver cells occupied by cytoplasm (the part containing the membranes): $87\%$

Surface density of membranes in cytoplasm: $7.0\,\mu m^2\ \mu m^{-3}$

[Area of a FIFA pitch: $8,250\,m^2$
Area of a GAA pitch: $10,500\,m^2$ ]

$1\,\mu m = 10^{-6}\,m$

When you do the calculations you should find that the surface area of membrane lies between the area of a FIFA pitch and that of a GAA pitch!

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# Unsolved problems in mathematics and their educational value

## 1. Introduction

I believe that unsolved problems in mathematics have a role to play in mathematics education at second level. I will give a variety of reasons for this belief shortly after I discuss some examples of unsolved problems. I think it is also relevant to state here (and in any discussion on mathematics) that mathematics consists of an abundance of ideas and that beauty, freedom and practical power are important features of mathematics (see [1] and [2]). A discussion about unsolved problems could be done very quickly in any class in second level and one could spend longer in transition year. Some of the material in the examples below appear in certain chapters that I wrote in [3]. Two free copies of [3] were sent to every second level school in the country by the Department of Education in 2006. There are thousands of unsolved problems in mathematics. Some of them can be stated using concepts that can be understood by second level students.

## 2. Examples of unsolved problems

(a) **Is there an odd perfect number?**

(b) **How many perfect numbers are there?**

A positive integer $x$ is called a perfect number if $x$ is the sum of all its positive factors excluding $x$ itself. Pythagoras and his followers were the first to study perfect numbers around $500$ BC. They found perfect numbers aesthetically pleasing and that is why they called them perfect. For example, $6$ is a perfect number since its positive factors are $1,\ 2,\ 3$ and $6$. We see that $6$ is the sum $1 + 2 + 3$ of all its positive factors excluding $6$ itself. One can check that $6$ is the smallest perfect number. Ask the students to find the next perfect number. Solutions to selected questions for students are given later.

Perfect numbers are extremely rare and only 49 have been found. Philosophers and mathematicians have been very interested in perfect numbers since Pythagoras' time. For a long time, philosophers were more concerned with the mystical or religious significance of perfect numbers than with their mathematical properties. Many ancient philosophers thought the reason it took around $28$ days for the moon to orbit the earth was because $28$ is a perfect number.

By $100$ AD only four perfect numbers were known. These were

$$6,\ 28,\ 496,\ 8128.$$

At that time Nicomachus conjectured that the $n^{th}$ perfect number contains exactly $n$ digits and that the even perfect numbers end alternately in $6$ and $8$. These two conjectures seemed reasonable at the time given that the only known perfect numbers were the four above. However, both assertions are wrong. There is no perfect number with five digits as the next perfect number, first given in an anonymous fifteenth century manuscript, is $33,550,336$. Also, the next largest perfect number is $8,589,869,056$ which means perfect numbers do not alternately end in $6$ and $8$, since these two consecutive ones both end in $6$. Nevertheless, Nicomachus' second conjecture is almost correct as every even perfect number will end in

either $6$ or $8$, but not necessarily alternately.

There are still some unsolved problems concerning perfect numbers. For example, **it is not known if there are any odd perfect numbers**. Just because none have been found does not mean they do not exist. Also, **it is not known how many perfect numbers there are**. Again, just because only 49 have been found does not mean that there are only finitely many. There may still be infinitely many that have not yet been found. By the way, if you start looking for an odd perfect number, then you should probably know that all numbers up to $10^{1500}$ have been checked and no odd perfect number has been found. However, again this does not mean there is no odd perfect number. Ask the students to construct a mathematical statement that is true for all numbers up to $10^{1500}$ but is not true for all numbers.

How do we find perfect numbers? It would be very tedious if we had to sum up all the positive factors of a number $n$, excluding $n$ itself, and check if the answer equals $n$. Fortunately, with the help of some work by Euclid and Euler, we have a much more convenient method. Prime numbers play an important role in this method. Over two thousand years ago, in Book IX of the *Elements*, Euclid proved that if $2^k - 1$ is prime, for an integer $k > 1$, then $n = 2^{k-1}(2^k - 1)$ is a perfect number. In the eighteenth century, the Swiss mathematician, Euler, proved that every even perfect number is of the form $n = 2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime, for some integer $k > 1$. These two results can be combined in the following statement:

*If $2^k - 1$ is prime, for an integer $k > 1$, then $n = 2^{k-1}(2^k - 1)$ is a perfect number, and every even perfect number is of this form.*

Consequently, the problem of finding even perfect numbers is reduced to finding primes of the form $2^k - 1$, with $k > 1$. These are the Mersenne primes and will be discussed in more detail in example (c) below. It transpires that if $2^k - 1$ is prime, then $k$ must be prime. However, the converse is not true, in the sense that if $k$ is prime, then it does not necessarily follow that $2^k - 1$ is prime. For example

$$\begin{aligned} 2^{11} - 1 &= 2047 \\ &= 23 \times 89 \end{aligned}$$

Thus, we search for primes $k$ for which $2^k - 1$ is also prime. When this search is successful, then $n = 2^{k-1}(2^k - 1)$ will be a perfect number. Here are some examples:

$$\begin{aligned} 2(2^2 - 1) &= 6 \\ 2^2(2^3 - 1) &= 28 \\ 2^4(2^5 - 1) &= 496 \\ 2^6(2^7 - 1) &= 8128 \\ 2^{12}(2^{13} - 1) &= 33,550,336 \\ 2^{16}(2^{17} - 1) &= 8,589,869,056 \end{aligned}$$

These are the first six perfect numbers. The sixth one above was found by Pietro Cataldi in $1603$.

(c) **How many Mersenne primes are there?**
Currently, on July 23, 2017, the largest known prime is

$$2^{74,207,281} - 1$$

This was found by Curtis Cooper on January $7,\ 2016$, in Missouri, and it has $22,338,618$ digits. So, it's kind of a big number! Of course, there is no largest prime since Euclid proved that there are infinitely many primes. The website `www.utm.edu/research/primes/` provides information regarding primes. Observe that the above prime is of the form $2^n - 1$. A number of the form $2^n - 1$ is called a Mersenne number, after the French monk, Mersenne $(1588–1648)$, who studied them. If a Mersenne number is prime, then it is called a Mersenne prime. Often, the largest known prime is a Mersenne prime. So far, 49 Mersenne primes have been found, but **it is not known whether there are finitely many or infinitely many Mersenne primes**. Note that any time a new Mersenne prime is found, then a corresponding new perfect number is found by the discussion on perfect numbers above.

Cooper is one of thousands of GIMPS (Great Internet Mersenne Prime Search) volunteers. GIMPS was founded in $1996$ and is a collaborative group of volunteers who use freely available software to search for new Mersenne primes. Anybody can join GIMPS.

In $1978$ two second level students, Landon Curt Noll and Laura Nickel in California, found the largest known prime at the time, which was $2^{21,701} - 1$. Their achievement was reported all over the world. I feel it can be interesting for second level students to see that students their own age can make contributions to new mathematics that was not known before.

Another example of a second level student's contribution to new mathematics appears in the area of friendly numbers. Two positive integers $x,\ y$ are called friendly (or amicable) if $x$ is the sum of the positive factors of $y$ excluding $y$ itself and $y$ is the sum of the positive factors of $x$ excluding $x$ itself. Pythagoras and his followers were the first to study friendly numbers around $500$ BC. For example, they were aware of the friendly pair $220,\ 284$. Students can check this is a friendly pair. This pair $220,\ 284$ had a mystical aura and later on superstition led to the belief that two talismans containing these numbers would mean a perfect friendship between the wearers. Apparently, no new friendly numbers were found until al-Banna $(1256–1321)$ found the pair $17,296$ and $18,416$. In the eighteenth century Euler gave a list of sixty new pairs of friendly numbers. However, in $1866$, a sixteen year old Italian boy, Paganini (not the famous violinist!), found a new pair of relatively small friendly numbers, $1184,\ 1210$ that had not been found by the great mathematicians for over two thousand years.

Large prime numbers now have important practical applications. For example, prime numbers have recently become fundamental in security systems for electronic transmissions. Thus, primes have relevance to credit cards, telecommunications, international security etc. The study of encoding and decoding messages is called cryptography and depends heavily on all sorts of advanced mathematics. However, at the heart of the electronic security system is the fact that it is easy to compute the product of two large primes but it is incredibly difficult to factor the resulting product into its two prime factors. One needs

to know the prime factors in order to decode the encoded message and so it is extremely difficult for an outsider to decode a message.

In 1977 Ronald Rivest, Adi Shamir and Leonard Adleman, a group of mathematicians and computer scientists at MIT, discovered that prime numbers provided a perfect setting for a simple-scramble/hard-unscramble security system. This system is called the RSA system and is now widely used in electronic security systems. The search for large primes is also used as a test for computer hardware. For example, the search for large primes has uncovered flaws in certain CPUs (Central Processing Units) and detected hardware problems in many individual PCs.

(d) **How many twin primes are there?**
There is no simple pattern to the distribution of primes in the set of all numbers. Consecutive prime numbers may differ by only two or may be arbitrarily far apart. For example, 3 and 5, 5 and 7, 22,271 and 22,273, are all consecutive primes that are two apart. This is the closest that two consecutive primes can get, except in the case of 2 and 3. Ask students to prove this. A pair of consecutive primes, that differ by two, are called 'twin primes'. Thus 3 and 5, and 5 and 7 are examples of twin primes. Ask students to find all twin primes less than 100.
A triplet of consecutive primes, that each differ by two, are called 'triple primes'. Ask the students to prove that there are no triple primes except for the case 3, 5, 7.

**It is not known how many twin primes there are**. It is not known if there are finitely many or infinitely many twin primes. This is a famous longstanding unsolved problem in mathematics. The Twin Prime Conjecture states that there are infinitely many twin primes but it has not been proved yet. Recently, there has been some major progress in trying to prove there are infinitely many twin primes. In 2013 Yitang Zhang proved that there is some integer $N$ less than seventy million, such that there are infinitely many primes that differ by $N$. If one could prove that $N$ could be 2, then there would be infinitely many twin primes. However, it has not been proved yet that $N$ can be 2. Subsequently, Terence Tao proposed a Polymath project collaborative effort to improve on this bound of seventy million. In 2014, they improved this bound to 246 and the work appeared in two papers published under the pseudonym, D.H.J. Polymath. So, now there is some integer $N$ less than 246, such that there are infinitely many primes that differ by $N$. Note that 246 is a big improvement on seventy million. The plan is to try and prove that $N$ can be 2.

(e) **Irish connections and who wants to be a millionaire?**
If one wants to see some important unsolved problems which require advanced concepts, then check out the website `www.claymath.org`. In 2000, the Clay Mathematics Institute announced that it was offering a reward of one million dollars for the solution of any of its seven millennium prize problems. One of these celebrated problems involves finding the general solution to the Navier-Stokes equations, which are named after the two mathematicians, Navier and Stokes, who developed the theory. These equations have significant practical applications and are the fundamental equations for the motion of incompressible fluids and play a crucial role in aeronautical engineering, ship design and meteorology. George Stokes (1819–1903) was born in Skreen, Co. Sligo. A memorial was erected at his

birthplace in 1995.

Ireland's greatest mathematician, William Rowan Hamilton, has a connection with another one of the seven millenniun prize problems. The famous Travelling Salesman Problem (TSP) can be phrased (in less technical terms) as: Given $n$ cities, how do you find a shortest route that visits each city exactly once and returns to the starting city? In more technical terms it can be phrased as: How do you find a shortest Hamiltonian circuit? A Hamiltonian circuit is named after Hamilton. The TSP is currently unsolved. There are many important practical applications of the TSP in logistics, planning, the manufacture of microchips and astronomy. The TSP is closely related to the millennium prize problem called P versus NP. Santa Claus is an example of somebody who has to try and solve the TSP on Christmas Eve!

Hamilton was born in Dublin and spent his early youth on the banks of the Boyne in Trim. There is a Hamilton plaque at St. Mary's Abbey in Trim across the Boyne from the spectacular ruins of Trim Castle. I organise the annual Hamilton walk on October 16 which commemorates his famous Eureka moment when, in a flash of inspiration, he created a seemingly strange new number system called Quaternions on the banks of the Royal Canal on October 16, 1843. The mathematical world was shocked at his audacity in creating a system of numbers that did not satisfy the usual commutative rule for multiplication ($xy = yx$). Hamilton has been called the Liberator of Algebra because his quaternions smashed the previously accepted convention that a useful algebraic number system should satisfy the rules of ordinary numbers in arithmetic. His quaternions opened up a whole new mathematical landscape in which mathematicians were now free to conceive new algebraic number systems that were not shackled by the rules of ordinary numbers in arithmetic. Hamilton's motivation for doing mathematics was the quest for beauty. The beauty in mathematics typically lies in the beauty of ideas because mathematics comprises an abundance of ideas. Hamilton was also a poet and once wrote:

*Mathematics is an aesthetic creation, akin to poetry, with its own mysteries and moments of profound revelation.*

One can read more about Hamilton and the walk in [4] and [5].
Many transition year groups come on the walk every year. If you are interested in coming on the walk, then you should contact me at (01) 7083763 or fiacre.ocairbre@nuim.ie. Quaternions now have many important practical applications in areas like space navigation, computer animation, computer games and special effects in movies. Vector analysis, which is indispensable in Physics, is an offspring of quaternions. Over the years there has been an abundance of art works, media events and activities related to Hamilton. One can read about these art works etc. in [6].
One of the seven millennium prize problems has been solved. Perelman proved the Poincare Conjecture.

(f) **The Goldbach conjecture.**
The Goldbach conjecture is another famous unsolved problem in mathematics. In 1742, Prussian-born mathematician Christian Goldbach, tutor to the teenage Czar Peter II, con-

jectured that every even integer greater than 4 can be written as a sum of two odd primes. It is not known if this conjecture is true, in the sense that a general proof for all even integers greater than 4 is not known, and also no counterexample, to show it is false for a single even integer, has yet been found. Students can check that the conjecture is true for even integers less than 100. It has been shown that the conjecture is true for all even integers up to $4 \times 10^{18}$. The following are some examples:

$$\begin{aligned} 6 &= 3 + 3 \\ 8 &= 3 + 5 \\ 10 &= 5 + 5 \\ 12 &= 5 + 7 \\ 21,000 &= 17 + 20,983 \end{aligned}$$

In 2000, the Irish Times reported that the publisher Faber and Faber was offering a reward of one million dollars to anyone who can prove the Goldbach conjecture within two years. The reward was part of the publicity surrounding the publication of the novel *Uncle Petros and Goldbach's Conjecture* by Apostolos Doxiadis. Nobody proved the conjecture within two years.

(g) **The Hailstone sequence.**
Pick any positive integer and consider it the first number in your hailstone sequence. If it is even, then divide by two, and if it is odd, then multiply by three and add one. This produces the second number in your hailstone sequence. If this number is even, then divide it by two and if it is odd, then multiply it by three and add one. This produces the third number in your hailstone sequence. Continue like this to obtain each number in your hailstone sequence. For example, suppose you started with 15. Then your hailstone sequence will be

$$15,\ 46,\ 23,\ 70,\ 35,\ 106,\ 53,\ 160,\ 80,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,$$
$$2,\ 1,\ 4,\ 2,\ 1\ldots$$

Observe that the sequence eventually hits 4 and then repeats 4, 2, 1 forever. The word, hailstone, is used because the numbers in the sequence can go up and down just like a hailstone in a cloud before it falls to earth. It transpires that every starting number ever tested produces a hailstone sequence that eventually hits 4 and thus repeats 4, 2, 1 forever. All starting numbers up to $5 \times 2^{60}$ have been tested. One might therefore be led to believe that this behaviour occurs for any starting number. However, this conclusion is not necessarily correct. As before, just because something is true for a large finite collection of numbers (namely all numbers tested so far in this case) does not mean it is true for all numbers. One needs a mathematical proof based on deductive reasoning in order to prove that something is true for all numbers. So **it is not known if every starting number produces a hailstone sequence which eventually repeats 4, 2, 1?**

(h) **The most famous unsolved problem that is now solved.**
In 1637 Pierre de Fermat was reading a translation of Diophantus' *Arithmetica* which discussed Pythagorean triples. Diophantus was a Greek mathematician who lived around

250 AD. A Pythagorean triple is a set of three positive integers $x$, $y$ and $z$ such that

$$x^2 + y^2 = z^2$$

For example, 3, 4 and 5 form a Pythagorean triple, since

$$3^2 + 4^2 = 5^2$$

One can see how Pythagorean triples are related to Pythagoras' famous theorem. Ask the students to find another Pythagorean triple. Euclid had actually proved that there are infinitely many different Pythagorean triples and Fermat was aware of this.
Fermat began to wonder if there were any triplets of positive integers $x$, $y$ and $z$ such that

$$x^3 + y^3 = z^3$$

This equation only differs from the previous equation in the fact that the power 2 has been replaced by 3. He was now seeking three positive integers such that the sum of the cubes of two of them is equal to the cube of the third one. However, Fermat could not find any solutions to this new equation. Could it be that this seemingly minor alteration to an equation with infinitely many solutions turned it into an equation with no solutions. He then generalised the problem to finding three positive integers $x$, $y$ and $z$ such that

$$x^n + y^n = z^n, \text{ where } n \text{ is a number greater than } 2$$

Again, he could not find any solutions. In the margin of the *Arithmetica*, next to problem 8, Fermat wrote

*Cubem autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere.*

meaning

*It is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as the sum of two fourth powers or, in general, for any number which is a power greater than the second to be written as a sum of two like powers.*
Fermat then added the following celebrated comment which would intrigue mathematicians for centuries:
*Cuius rei demonstrationem mirabilem sane detex hanc marginis exiguitas non caparet.*

meaning

*I have a truly marvellous demonstration of this proposition which this margin is too narrow to contain.*

Fermat died 28 years later and never told anyone about the proof. Fermat was essentially an amateur mathematician and was isolated from the Parisian school of mathematics. His enormous contribution to many areas of mathematics (he cofounded coordinate geometry

with Descartes, he cofounded probability theory with Pascal, he laid the groundwork for the development of calculus and he made huge advances in number theory) would probably have been lost forever if it was not for his son, Clément-Samuel, who realised the importance of his father's pastime. Clément-Samuel gathered all his father's mathematical notes and published them. One such publication appeared in 1670 in Toulouse and was entitled *Diophantus' Arithmetica containing observations by P. de Fermat*. This publication contained 48 observations made by Fermat. Many of his observations did not contain proofs even though he said he had proofs for all of them. Thus, for him they were all theorems. However, technically speaking, some were conjectures in the mathematical community until proofs were given. Gradually, one by one, Fermat's conjectures were proved to be true until one remained. This observation, whose proof eluded the greatest mathematicians in the world, was the one mentioned in the marginal note above. It became known as **Fermat's Last Theorem** and can be stated as follows:

*There are no triplets of positive integers $x$, $y$ and $z$ such that $x^n + y^n = z^n$, where $n$ is any integer greater than 2.*

For over three hundred years mathematicians attempted in vain to prove or disprove Fermat's Last Theorem: it became the most famous unsolved problem in mathematics. One way to attempt to disprove Fermat's observation would be to seek a counterexample, i.e. an example of a triplet of positive integers $x$, $y$ and $z$ and an integer $n$ greater than 2, such that $x^n + y^n = z^n$.

Fermat's Last Theorem was finally proved to be true by Andrew Wiles, in 1995. Wiles instantly became a celebrity for closing such a famous chapter in the history of mathematics. [7] was number one on the best seller's list in Ireland for some time. Wiles' proof is quite complicated and uses very heavy mathematical machinery. It still is possible, of course, that Fermat really did have *a marvellous demonstration of this proposition which this margin is too narrow to contain*. Who knows?

Fermat's Last Theorem appears in the Simpsons! In fact a lot of mathematics pops up in the Simpsons and the main reason for this is that some of the writers for the show are actually mathematicians. Other topics related to this paper that appear in the Simpsons are perfect numbers, Mersenne primes and the TSP. Students may find it amusing to see these on youtube. See [8] for more on mathematics in the Simpsons.

3. **Reasons why I believe unsolved problems have a role to play in mathematics education.**

(1) Students may be pleasantly surprised that they can understand the statements of problems in mathematics that remain unsolved after the greatest minds in mathematics have worked on them for hundreds (and sometimes thousands) of years.

(2) It shows that there is some wonder and mystery in mathematics because nobody knows the answers to these unsolved problems.

(3) Unsolved problems provide a new experience for many students. I have never seen an unsolved problem in a second level textbook. I feel a discussion (which could be very brief) about unsolved problems would provide a positive new experience for many students.

(4) It highlights the fact that not everything is done in mathematics. Many people think everything is done in mathematics but this is not the case. Research is vibrant in mathematics with mathematicians trying to solve unsolved problems and develop new theories.

(5) There are some interesting stories related to unsolved problems.

(6) Students will see that there are Irish connections to unsolved problems.

(7) Some of the unsolved problems have important practical applications as discussed above.

(8) Students could think of unsolved problems as being like homework for the greatest minds in mathematics.

(9) Students get to hear about some of the history of mathematics when unsolved problems are discussed. Consequently, students will hear about some of the famous characters in mathematics.

(10) A discussion about unsolved problems can lead to a discussion about contributions made by second level students to new mathematics that was not known before as in example (c) above.

(11) Students can get to see youtube clips from the Simpsons after hearing about unsolved problems as in example (h) above. Students may find this amusing.

4. **Solutions.**

1. 28 is the next perfect number.
2. The statement $x \leq 10^{1500}$ is true for all numbers up to $10^{1500}$ but it is not true for all numbers.
3. Any two consecutive primes (both bigger than 2) have to both be odd. Hence, the closest they can get is two apart.
4. The twin primes less than 100 are: (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73).
5. Suppose $x > 3$. Now 3 divides exactly one of $x, x+1, x+2$. If 3 divides $x$, then $x$ is not prime and so $x, x+2, x+4$ is not a triple prime. If 3 divides $x+1$, then 3 divides $x+4$ and so again $x, x+2, x+4$ is not a triple prime. Finally, if 3 divides $x+2$, then $x, x+2, x+4$ is not a triple prime. So, there is no triple prime $x, x+2, x+4$, for $x > 3$.
6. Another Pythagorean triple is 5, 12, 13.

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# Lesson Study — A Mechanism for Maximizing Teaching Mathematics through Problem Solving

## Introduction

*Lesson Study* is a translation of the Japanese phrase *jugyoukenkyuu*, which refers to a set of practices that have been used in Japan to improve teaching and learning for over 100 years (Makinae, 2010). *Lesson Study* was introduced to the international education community primarily by the publication of *The Teaching Gap* (Stigler & Hiebert, 1999). Since then, many mathematics educators around the world have been involved in introducing *Lesson Study* as an approach to teacher professional development. *Lesson Study* is credited with enabling profound changes in Mathematics and Science teaching in Japan in recent decades — a fact evidenced by Japan's high levels of achievement in international assessments over the past 20 years (TIMSS 2015, 2011, 2007). While the focus of this document is on utilising *Lesson Study* to improve mathematics teaching and learning, Japanese teachers use *Lesson Study* to hone teaching in all content areas, including Physical Education and modern foreign languages. In Mathematics, *Lesson Study* is used as the mechanism to improve:

- teaching of Mathematics through problem solving
- students' confidence and experience as problem solvers

*Lesson Study* was introduced nationally to Ireland in 2014 by The Maths Development Team and since then the number of mathematics teachers choosing it as a means of professional development has increased year on year. This article serves as an introduction to what *Lesson Study* is, how it is used to promote the teaching of Mathematics through problem solving and how mathematics teachers and their schools can get involved.

## What is Lesson Study?

*Lesson Study* is a structured process where teachers work together to formulate solutions to challenges they encounter in relation to teaching and learning. At its heart, it aims to improve teaching and learning by developing successful approaches to teaching.
The major characteristics of *Lesson Study* are: collaborative planning of a research lesson, teaching the research lesson, collecting data by observing the lesson as it unfolds and reflecting on what

has been learned to inform the design of future lessons. As these lessons respond to identified needs and inevitably involve student-centred learning, inquiry and problem solving, *Lesson Study* contributes in a practical and meaningful way to the implementation of *School Self-Evaluation*[2], *The Literacy & Numeracy Strategy*[3], *Junior Cycle Key Skills & Statements of Learning*[4] and *The Digital Strategy for Schools*[5].

**Why Lesson Study?**

Mathematical content knowledge is an important facet of being a mathematics teacher, however this alone is not enough to be a highly effective teacher. Japanese mathematics educators typically characterise teacher expertise according to three levels (Sugiyama 2008):

- Level 1: Teachers *tell* students important basic ideas of Mathematics such as facts, concepts, and procedures.
- Level 2: Teachers *explain* the meanings of and rationale for the important basic ideas of Mathematics in order for students to understand them.
- Level 3: Teachers *provide* students with opportunities to understand these basic ideas, and support their learning so that the students become independent learners.

Borrowing from Takahashi's description (Takahashi, 2010), the differences between Level 3 teaching and other levels can be understood by looking at how teachers might use a problem in a textbook. At Level 1, the teacher presents the problem and shows the steps for solving it. At Level 2, the teacher shows the steps and explains why they are correct and useful. At Level 3, the teacher, presents students with the same problem, providing structure and guiding the conversation so that students arrive at a new understanding as a result of their own efforts in solving it. The philosophy behind Level 3 teaching is that students should engage in reasonable independent work, such as problem solving, in order to develop knowledge, understanding, and skill of Mathematics (Lewis & Tsuchida, 1998; J. Stigler & Hiebert, 1999; Akihiko Takahashi & Yoshida, 2004; Yoshida, 1999).

The difference between Level 1 and Level 3 teaching may also be thought of as the difference between "teaching the Mathematics textbook" and "teaching Mathematics using the textbook". Level 3 teaching clearly requires greater knowledge and expertise that goes beyond knowing, to include the application of Mathematics in practical and context-rich situations. *Lesson Study* is an approach to continuing professional development that can help teachers to develop such knowledge and expertise.

**Characteristic components of *Lesson Study***

Based on our experience of teachers participating in *Lesson Study* in Ireland, certain aspects of Lesson Study are critical for improving teaching and learning. If any of these are missing, then the activity will not be as effective as it ought to be and, strictly speaking, should not be called

[2] *Looking at our School 2016 — A Quality Framework for Post-Primary Schools*
[3] *The National Strategy to Improve Literacy and Numeracy among Children and Young People 2011–2020*
[4] *A Framework for Junior Cycle — Statement of Learning and Key Skills of Junior Cycle*
[5] *Digital Literacy Strategy*

*Lesson Study*.

The characteristic components of *Lesson Study* are:

A. Clearly defined research themes and learning goals

B. *Meitheal Machnaimh (Irish translation of Kyouzai kenkyuu)*

C. A written research lesson proposal

D. A live research lesson

E. A post-lesson discussion

F. Contributions by knowledgeable others

G. A process for sharing outcomes.

Figure 1 shows how these components fit together to form a complete *Lesson-Study* Cycle. We will now provide an explanation of each key component of *Lesson Study*.

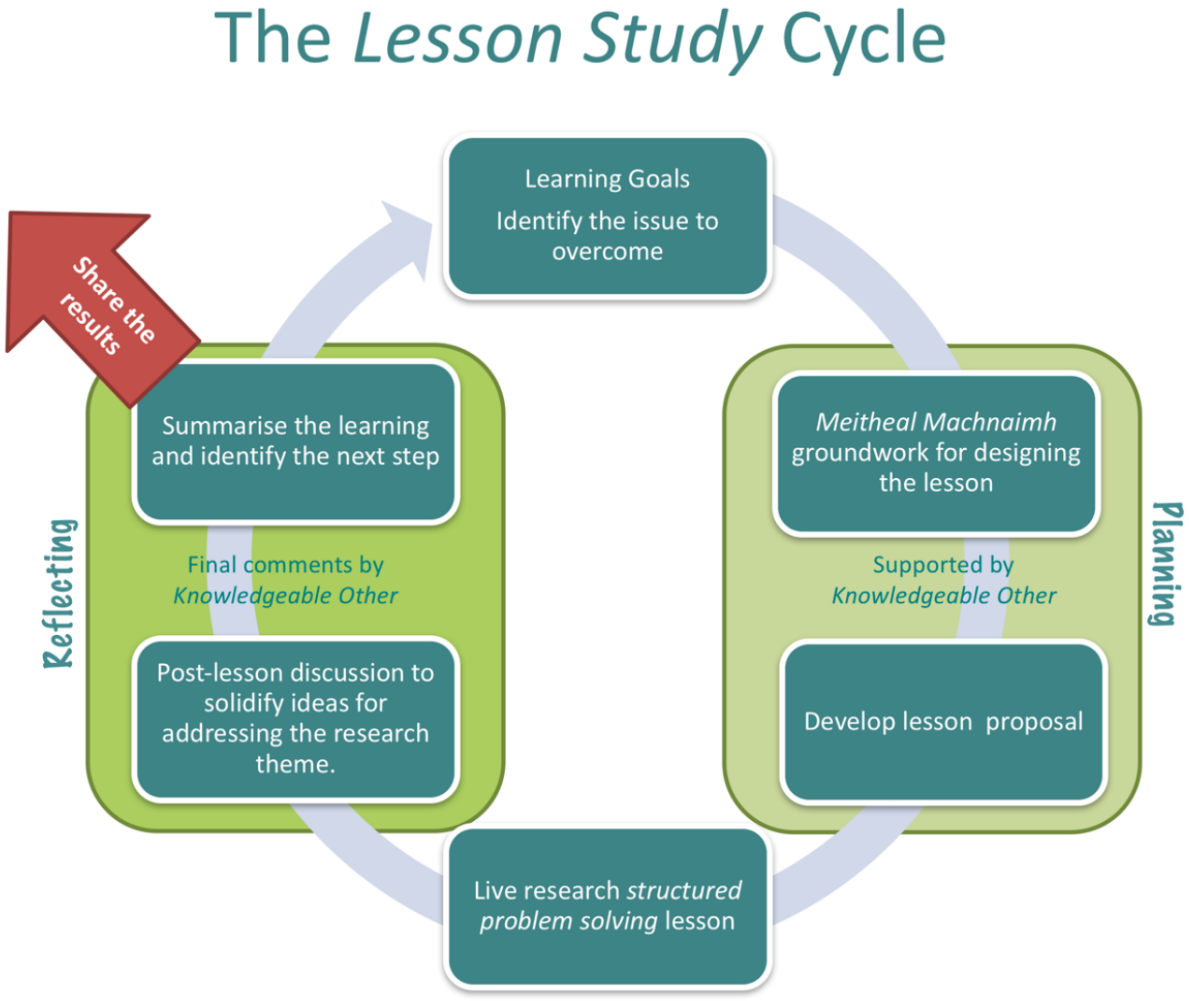


Figure 1: Lesson Study Cycle

**A. Clearly defined research themes and learning goals:**

To participate in a *Lesson Study* is to conduct research; to seek a solution to a teaching/learning problem. *Lesson Study* usually has two research objectives for student learning and development.

> One objective involves a broad teaching/learning goal that is shared by the *Lesson Study* group (e.g. all of the school's mathematics teachers), and that goes beyond any particular topic or year level and may even be cross-disciplinary. This broad objective is chosen from the standards outlined in *Looking at our Schools 2016* and is referred to as the *long-term goal*.

> The second goal relates to the teaching of a specific topic; How can a lesson be designed so that students learn a given concept or skill better than they did in the past? Thus, the topic of the research lesson should usually be one with which students or teachers have previously struggled.

Ideally, a research theme describes (a) a desired outcome for students, and (b) practical steps for teachers to take on a day-to-day basis for achieving that outcome in the mathematics classroom.

**B Meitheal Machnaimh:**

*Meitheal Machnaimh* translates roughly as "the collaborative study and research of problems and materials for the research lesson". It is a routine part of lesson planning for many teachers, but in *Lesson Study* it is done with extra intensity. *Meitheal Machnaimh* may also be thought of as an in-depth discussion of the key factors which determine the effectiveness of a lesson. One such factor is an understanding of the learning paths related to the topic, that is to say an understanding of students' prior knowledge relating to a topic and an understanding of how this topic will be utilised in future learning. This is achieved through study of the relevant mathematics curricula (including the primary-school curriculum) and should also include research into common misunderstandings around the topic.

Since we, in Ireland, are using *Lesson Study* to implement the teaching of Mathematics through structured problem solving, a major focus of *Meitheal Machnaimh* is in choosing problems which are appropriate to achieving the goals of the research lesson. This process involves the study of a range of textbooks and lesson proposals. In choosing a problem, it is commonplace for teachers to study ten problems but to only choose the one which they deem most suitable for the goals of the lesson.

*Meitheal Machnaimh* also includes consideration of possible tools e.g. dynamic software packages, manipulatives, or materials that may aid students' understanding. In our experience, there is a distinct lack of quality materials to support teachers in Ireland partaking in *Meitheal Machnaimh*. Whilst there is a need to develop more, some progress is already underway as *Lesson Study* continues to grow in Ireland. Already there is a library of lesson proposals available on `www.projectmaths.ie`, developed by Irish mathematics teachers teaching through problem solving.

**C Written Research Proposal:**

A *Lesson Study* team – usually 3 to 6 teachers – creates a written document, called the research *lesson proposal*, the goals of which are to communicate what the team learned from their *Meitheal Machnaimh*, and to explain their pedagogical thinking. In addition to a detailed teaching/learning proposal for one particular lesson (the *research lesson*), the research proposal includes an overview of the unit in which the research lesson will reside, the rationale for the design of the unit and research lesson, and a clear statement of how the research lesson aims to address the research theme and the learning goals of the lesson. In our experience, a thorough research lesson proposal may be nine pages long. To guide teachers in writing a lesson proposal, the Maths Development Team have designed an appropriate template, this template is based on Japanese Lesson Proposals but adapted for the Irish context.

**D & E Live research lesson and post-lesson discussion:**

Based on the lesson plan in the research lesson proposal, one member of the team teaches the research lesson, observed by the entire planning team and by additional members of the *Lesson Study* community. Observers from the research team are responsible for collecting data on how the lesson impacts the students, relative to the research theme and the learning goals (a free iPad application called LessonNote is available to help observers collect valuable data during a research lesson). As soon as possible after the research lesson, observers share the data and discuss its implications, especially with respect to the learning goals of the lesson and the research theme. The primary goal of the discussion is to gain insights into teaching and learning and to inform the design of future lessons, not to revise the lesson plan. These discussions generally benefit from a moderator (someone not on the planning team) who helps focus the discussion on important issues and keeps the conversation grounded in the evidence gathered during the lesson observation.

**F Contributions by knowledgeable others:**

A knowledgeable other is someone from outside of the planning team with experience of *Lesson Study*, who provides a broader perspective and challenges the team to extend their thinking. Ideally a *Lesson Study* community needs two knowledgeable others: one for supporting development of the research proposal and another for providing the final comments at the end of the post-lesson discussion. To date in Ireland, Associates working with the Maths Development Team (MDT) support the development of the lesson proposal.

During planning, a knowledgeable other such as an Associate may help the team identify teaching examples to review, valuable resources from, amongst others, the MDT website and may give feedback on the lesson proposal.

Another knowledgeable other is needed at the research lesson. In final comments at the end of the post-lesson discussion, he or she is expected to highlight important events from the research lesson that were not discussed, and make connections between the lesson and new knowledge from research and requirements of the syllabus. The knowledgeable other also provides suggestions to the research team of possible steps they could take toward accomplishing their research theme. Where possible, personnel from MDT provide this final commentary.

**G Process for sharing outcomes:**

*Lesson Study* can be informative not only for the team of teachers who plan the research lesson but for other teachers as well. For others to benefit from a *Lesson Study*, a structure or process for disseminating what is learned from each research lesson to the larger community is required.

A simple way to enable others to learn from a *Lesson Study* is to invite teachers from a school outside of the planning team or teachers from neighbouring schools to observe and discuss the research lesson. This can also benefit the research team, through insightful and critical feedback provided by these guest teachers. In addition, the research lesson proposal encapsulates the team's research and written reflection completed within a few days after the research lesson. This reflection describes what was learned from the live-lesson observation and post-lesson discussion about the effectiveness of the lesson in terms of the team's research and content goals, students' thinking, etc.

Each component in the *Lesson Study* Cycle is closely related to the others, and component D in the *Lesson Study* Cycle is based on Components A, B and C. Participating in *Lesson Study* is not like attending a workshop — it is a continuing, locally based, professional development programme which focuses on developing expertise in teaching Mathematics. This includes the expertise needed to develop lessons to achieve both short and long-term goals, to facilitate the implementation of SSE, JC Key Skills, Digital Strategy for Schools and Numeracy & Literacy (Component A). It also develops the expertise to use questioning techniques, to implement assessment for learning, to anticipate student responses to questions and tasks (Component B and Component C) and to make purposeful observations during a lesson (Component D). For teachers to develop such expertise for Level 3 teaching, they need opportunities to reflect on their teaching and learning based on careful observation (Component D and Component E). Lesson Study provides these opportunities through an intense process; one cycle typically spans 6 evening meetings. You may ask: "How can you spend so much time on just one lesson?" *Lesson Study* is not about just one lesson; a good research theme ensures that what is learned from *Lesson Study* applies to many lessons, perhaps to all lessons. Teachers who have gone through *Lesson Study* with us often say that the process leads them to think differently in planning their everyday lessons.

**A Structured Problem-Solving Lesson**

When teaching the problem-solving research lesson, the aim of the lesson is not only to develop or deepen students' content knowledge but also to foster students' mathematical thinking and disposition towards Mathematics. The critical pattern and features for a structured problem-solving lesson is seen as a teaching method. These lessons can be described as having a clear style with the following framework and structure: (Fujii, 2017)

1. Presenting the "problem for the day". (From the students' point of view, this activity could be renamed "introducing the problem") (5 to 10 minutes)
2. Problem solving by students (10 to 20 minutes)
3. Comparing and discussing (Ceardaíocht) (10 to 20 minutes)
4. Summing up by the teacher (5 to 10 minutes)

What the teacher's and students' roles are in each of these stages is as follows:

**1. Presenting the problem for the day**

At the start of a problem-solving lesson the teacher presents a problem to the students without demonstrating how to solve the problem. The teacher helps the students understand the context of the task then stops, even though the task has an unfamiliar aspect to it. Students are expected to use their prior knowledge to find the solution. The processes the students use in working on the problem, rather than the solution itself, is the focus of the lesson. During Cearadaíocht students are expected to learn new Mathematics. This is why a structured problem-solving lesson focuses on a single task, allowing for the important new mathematical idea(s) identified as the goal of the lesson to emerge during Ceardaíocht. Ideally the task incorporated into a problem-based mathematics lesson should:

i. be appropriate and mathematically valuable in terms of the aims of the lesson

ii. capture students' interest and generate curiosity

iii. be at the appropriate level of difficulty

iv. be capable of being solved in several ways

v. have the potential to elicit valuable basic wisdom (Fujii, 2015)

For students to appreciate that mathematical processes are more important than just getting the answer, they need to experience strategies that differ from their own. This is why a problem-based mathematical lesson includes a comparison and discussion (Ceardaíocht) phase for students to compare or experience their peers' strategies and discuss similarities, differences and merits among strategies in a whole-class setting. It is important to note that Ceardaíocht is where students' appreciation for the processes inherent in mathematics is cultivated. Thus, the task chosen, and prepared during the critical step of *Meitheal Machnaimh*, is crucial as it must elicit various strategies from the students to have an effective Ceardaíocht (see iv. and v. above).

**2. Problem-solving by students**

While students are working individually to solve the problem, the teacher performs *in-between desk work*. This means circulating, checking and monitoring students' work in preparation for Cearadaíocht. In-between desk work entails the teacher using a seating chart to identify which solution method(s) each student is using (including both correct and incorrect approaches). These solution methods have already been anticipated during Meitheal Machnaimh by the research group when planning the lesson proposal. The teacher determines which solution methods are favoured by the majority of the class and gives a hint to students who are having difficulty without divulging the complete solution. Students who have solved the problem are encouraged to find another way of solving it. The teacher does not spend long with any one student as the intention of the in-between desk work is to plan what to do during Ceardaíocht. Sometimes a student may produce a method that was not anticipated by the research group. If so, while doing in-between desk work, the teacher should ask the student to explain this method and record it on the seating chart. The teacher must now decide which solution methods to utilise and the order in which these should be

presented during Ceardaíocht so that students reach the goal of the lesson.

**3. Comparing and discussing (Ceardaíocht)**

Ceardaíocht is where students' mathematical processes and ways of thinking are fostered. This phase of the lesson focuses on comparing and discussing the different solution methods arrived at by students. In a classroom, students' thinking is often invisible and a structured problem-solving lesson allows the invisible become visible. According to Fujii (2017), "the teacher who teaches the lesson needs to grab a tail of thought when it first becomes visible, and use that tail as a concrete example to teach or suggest how to think mathematically". A mathematical way of thinking cannot be demonstrated by the teacher showing and telling the students "how to do", it can only be accomplished through students actively engaging through and in problem solving activities. Therefore, problem-solving lessons allow students to practise mathematical thinking, and promote independent learning. This is not a new idea:

> *Ten pages of mathematics understood are better than a hundred memorised and not understood, and one page actually worked out independently is better than ten pages clearly passively understood. The question is not how much? But how? The object is mastery, attainment of the spirit of the subject, and not to train the memory, or to ingest a large bulk of mathematical fact and formulas.* *(Young, 1908)*

Students present solutions at the whiteboard starting with the most naïve and working up to the most sophisticated method. The whiteboard is a key tool for Ceardaíocht as it is used to record the thought processes of the students and progression of the lesson. Boardwork is planned very carefully by the research group before the lesson is taught.

**4. Summing Up**

This phase of the class is for a short period only but critical to what students take from a lesson. The teacher should not conclude the lesson with a summary of a procedure. If so students may feel that instrumental knowledge is the most important element of Mathematics. Instead, the teacher should point out how the students constructed their knowledge by using prior knowledge and highlight for students that *how we think* is what is important and fundamental to Mathematics. Students should then write a brief reflection on the lesson itself where each student can express his/her own thoughts.

**How to get involved in *Lesson Study***

If you wish to join the growing number of teachers involved in *Lesson Study* please visit `www.projectmaths.ie` and pick from a wide variety of schools and Education Centres where you can access it locally to you. Teachers involved in *Lesson Study* are enabled to observe research lessons in their own schools and in other schools thanks to substitution provided by the Teacher Education Section of the Department of Education & Skills. All Lesson Proposals developed under the guidance of MDT are available at `www.projectmaths.ie` where they are organised by Junior and Senior Cycle topic. Our annual conference *Maths Counts: Engaging Mathematics Teachers in Lesson Study* showcases research lessons through live teaching and interactive workshops.

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# Mathematics Education at Croke Park: CERME 10

Even at less than 1% capacity, Croke Park was a lively place at the beginning of February this year, with 774 researchers in mathematics education from all over Europe and beyond. They were participating in CERME 10, the tenth *Congress of European Research in Mathematics Education*. They came from 29 European countries and 23 countries outside Europe, from as close as Marino and as far away as Hamilton, New Zealand.

What brings so many specialists in mathematics education to Dublin? How did CERME 10 come about? What happened at CERME 10? What was the extent of Irish participation? Will the work there be of any importance to IMTA members? These are the questions I will address in this short article. Rather than providing a conventional list of references, a list of relevant web links will be given at the end so that those interested in particular areas of the latest research in mathematics education can follow up at their leisure.

What brings so many specialists in mathematics education to Dublin?

Twenty years ago, in May 1997, the *European Society for Research in Mathematics Education*, ERME, was founded in Osnabrück, Germany, with communication, cooperation and collaboration (the three Cs) as core principles in support of its objectives. In August of the following year the first CERME was held in that city. In 2010, ERME was affiliated to ICMI, the *International Commission on Mathematical Instruction*. ICMI itself was founded in 1908 in response to intense interest at that time in how mathematics was taught on the part of influential members of the *International Mathematical Union* (IMU), notably Felix Klein. Indeed Klein's interest in teaching mathematics is emphasised in Siddhartha Sen's Newsletter article (no. 113). To quote Klein (1923, from ICMI history site):

> *I believed that the whole sector of Mathematics teaching, from its very beginnings at elementary school right through to the most advanced level research, should be organised as an organic whole. It grew ever clearer to me that, without this general perspective, even the purest scientific research would suffer, inasmuch as, by alienating itself from the various and lively cultural developments going on, it would be condemned to the dryness which afflicts a plant shut up in a cellar without sunlight.*

I mention these organisations to draw attention to the strong and historic links between the research communities in mathematics and mathematics education, represented globally by the IMU and ICMI, and, at European level, by the EMS (*European Mathematical Society*) and ERME. It will be clear from what follows that the Irish community of researchers in mathematics education is now sufficiently large and active to put in place structures at a national level analogous to ERME and ICMI.

At the heart of ERME are the biennial CERME gatherings, while at the heart of each CERME are the Thematic Working Groups (TWGs), now numbering 23, to one of which each participant 'belongs' for the duration of a congress. The vast majority (over 88% for CERME 10) of participants have made a submission (paper or poster) by mid-September before the congress to the TWG of their choice (see list below). Submissions are open peer reviewed by at least two other members of that TWG and then modified in the light of the reviews. All submissions are made available for reading in advance of CERME; indeed, members of each TWG are expected to have read all the submissions (in their TWG) in time for discussion at seven substantial sessions during CERME. It is in this way that a strong commitment to the three Cs is copper-fastened. Discussion at CERME is intense, yet, in the main, very supportive — as the photos of the TWG sessions (taken by Paul Murphy, DCU Institute of Education) show:

As the online programme shows, there is a pre-CERME event, known as the YERME Day, involving workshops for 'young' researchers. Apart from the TWGs, CERME includes some plenary sessions (opening and closing ceremonies, two keynote presentations, a panel discussion and half a day of tours). It is a busy week attracting a lot of people, in the heart of the European winter!

How did CERME 10 come about?

A series of biennial national conferences on research in mathematics education (MEI), an initiative led by Seán Close (for many years a member of the IMTA Council) had taken place since 2005 in St Patrick's College Drumcondra. MEI always included research reports from the UK and beyond, as well as invited international keynote speakers. The peer-reviewed proceedings of the five MEI conferences to date have been published both electronically and in hard copy. Moreover, there has been much other activity in the field nationally during the past decade or so — of which readers will need no reminder.

It was in these national and international contexts that my colleague, Thérèse Dooley, and I submitted, at the end of October 2014, a bid to host CERME 10 in Dublin in February 2017. We got great support from our workplace, St Patrick's College (then in the process of incorporation with DCU), from DCU itself, as well as from the DES, the Royal Irish Academy, Dublin City Council, Fáilte Ireland and Keynote, the PCO (professional conference organisers) who worked with us from start to finish. Little did we know, at the time, how all-consuming the project would be and that we would need to seek a venue to accommodate almost 800 participants. The bidding process continued during the three months that followed, culminating at CERME 9, in Prague, in February 2015. It was here that we made the final presentation in support of our bid, and where the ERME Board confirmed that it was successful. The shocking news was that, having planned for not many more than 500 (and that at a tight squeeze), it seemed from the experience of our Czech friends that it was likely that many more would participate, and that it was difficult to predict numbers with any confidence. Of course the ERME Board was well aware of this predicament and so put in place new measures to ensure that the number would not spiral out of control in the weeks prior to the congress.

We worked closely with Keynote on the website (check out the link below), the budget, the venue (switching to Croke Park soon after our return from Prague), keeping numbers in check, accommodation and tours. We had the support of the Local Organising Committee which included six other colleagues from the 'new' DCU. In addition the International Programme Committee concerned itself with scientific matters, in liaison with the TWG leaders. At the top level, the ERME Board made executive decisions and advised on the big questions. It was convenient that the major world congress (ICME-13) was held in July 2016 in Hamburg. This series of *International Congresses on Mathematical Education*, attracting over 3500 participants in Hamburg, is held every four years under the auspices of ICMI. ICME-13 afforded many opportunities helpful for CERME including meeting with the ERME Board and valuable exposure to promote the Dublin congress. In addition, I worked (with Thérèse Dooley, Elizabeth Oldham and Gerry Shiel) on an Irish National Presentation on mathematics education (a summary of which will appear in the ICME-13 proceedings), while Eabhnat Ní Fhloinn did a terrific job of gathering about three dozen of the large Irish contingent at ICME for a meal one evening in Hamburg. There were still over six months to go before CERME ...

What happened at CERME 10?

As mentioned above, most of the work at CERME takes place in TWGs of which there were 23 in Dublin, as follows:

TWG1 Argumentation and proof

TWG2 Arithmetic and number systems

TWG3 Algebraic thinking

TWG4 Geometry

TWG5 Probability and statistics education

TWG6 Applications and modelling

TWG8 Affect and mathematical thinking

TWG9 Mathematics and language

TWG10 Diversity and mathematics education: social, cultural and political challenges

TWG11 Comparative studies in mathematics education

TWG12 History in mathematics education

TWG13 Early years mathematics

TWG14 University mathematics education

TWG15 Teaching mathematics with resources and technology

TWG16 Students' learning mathematics with resources and technology

TWG17 Theoretical perspectives and approaches in mathematics education research

TWG18 Mathematics teacher education and professional development

TWG19 Mathematics teacher and classroom practices

TWG20 Mathematics teacher knowledge, beliefs and identity

TWG21 Assessment in mathematics education

TWG22 Curricular resources and task design in mathematics education

TWG23 Implementation of research findings in mathematics education

TWG24 Representations in mathematics teaching and learning

The reader can see that that scope is very broad including particular topics (algebra, geometry, probability), levels of education (early years, university), teacher education and CPD, overarching matters (argumentation, history, representation). The details of the approach of each TWG and of the submissions (papers/posters) made can be seen in the links below. To give an indication of what is involved, here is an extract from the call for submissions to TWG4:

> The work of this group will be organized to address four main topics:
>
> 1. What is involved in "doing geometry" at school? Geometrical competencies, defining what geometry is about, the building of geometrical concepts, proof, the use of drawings, of tools, of technologies, etc. are concerned by this topic.
> 2. Learning geometry: what is at stake, what are the difficulties or obstacles, what is possible to propose or to improve it?
> 3. Teaching geometry: production, dissemination and assessment of resources, use of software, teaching practices, pre-service and in-service teacher trainings.
> 4. Geometry in the curricula: role, organization, aims, links with other mathematical domains.
>
> We invite prospective participants to submit papers or posters related to one or more of these topics, based on research work, either philosophical or experimental. We suggest them to make explicit their main scopes on their research problems: epistemological, cognitive, instrumental, semiotic, sociological, historical ...

In this TWG, there were 19 submissions (15 papers and 4 posters). There were 26 participants in the group, 24 of whom were authors. (Of course many submissions had several authors.) To illustrate the areas of discussion within TWG4, here are the titles of five of the submissions (including the country of the authors):

- Dialogic processes in collective geometric thinking: a case of defining and classifying quadrilaterals (UK)
- Learning to use a protractor: touch screen tablet or not? (France)
- The design of a teaching sequence on axial symmetry, involving a duo of artefacts and exploiting the synergy resulting from alternate use of these artefacts (Italy)
- Diagrams in students' proving activity in secondary school geometry (poster, UK & Japan)
- Hide and think: Limiting the obvious and challenging the mind in geometry (Germany)

The material of this TWG is of interest not only to researchers, but also to teachers in their practice. It seems to me that there is plenty of scope for a group of teachers from an IMTA branch to take a selection of papers from a TWG, to read them (in advance) and to discuss them at a workshop with support from the branch — in short, to replicate, in a sense, the CERME process and spirit locally!

What was the extent of Irish participation?

Irish researchers at CERME submitted 17 papers and seven posters to CERME. The Irish papers presented were:

TWG3: *Shape patterning tasks: An exploration of how children build upon their observations when asked to construct general terms*, by Aisling Twohill

TWG5: *Insights into the approaches of young children when making informal inferences about data*, by Aisling Leavy & Finbarr Sloane

TWG6: *Examining the role of group work as an effective instructional strategy when teaching problem solving*, by Eamonn Clohessy & Patrick Johnson

TWG8: *Students' images of mathematics: the role of parents' occupation*, by Ciara Lane

TWG9: *Ensuring methodological rigor for investigating the discourse development of bilingual mathematics learners*, by Máire Ní Ríordáin, Eilís Flanagan & Claire Brilly

TWG9: *The role of the classroom teacher in supporting bilingual primary pupils in the learning of mathematics*, by Miriam Ryan & Máire Ní Ríordáin

TWG10: *The context of workplaces as part of mathematics education in vocational studies: Institutional norms and (lack of) authenticity*, by Lisa Björklund Boistrup & John Keogh

TWG12: *Multiple perspectives on working with original mathematical sources from the Edward Worth Library, Dublin*, by Maurice OReilly

TWG14: *"What you see and don't see, shapes what you do and don't do": noticing in first year mathematics lectures*, by Maria Meehan, Ann O'Shea & Sinéad Breen

TWG18: *Analysing mathematics teacher learning in lesson study — a proposed theoretical framework*, by Aoibhinn Ní Shúilleabháin & Stéphane Clivaz

TWG19: *Considering research frameworks as a tool for reflection on practices: Grain-size and levels of change*, by Siún Nic Mhuirí

TWG20: *Is teacher knowledge affecting students' transition from primary to second level mathematics education?* by Niamh O'Meara, Mark Prendergast, Lorraine Harbison & Clare O'Hara

TWG21: *Ireland's Junior Certificate mathematics examination through the lens of the PISA and TIMSS frameworks: Has Project Maths made any difference?* by Rachel Cunningham, Seán Close & Gerry Shiel

TWG21: *Using assessment for learning to enhance mathematics education in the primary school: Irish students' perspectives*, by Ann Marie Gurhy

TWG21: *Examining difficulties in initial algebra: Prerequisite and algebra content areas for Irish post-primary students*, by Aoife O'Brien & Máire Ní Ríordáin

TWG22: *Children's performance on a mathematics task they were not taught to solve: A case study*, by Seán Delaney

TWG22: *Critical evaluation and design of mathematics tasks: pre-service teachers*, by Majella Dempsey & Ann O'Shea

The seven Irish posters were:

TWG14: *In what ways do lecturers receive and use feedback from large first year mathematics classes?* by Anthony Cronin & Maria Meehan

TWG14: *Investigating students' difficulties with differential equations in physics*, by Diarmaid Hyland, Paul van Kampen & Brien Nolan

TWG16: *Identifying the reasons behind students' engagement patterns*, by Emma Howard, Maria Meehan & Andrew Parnell

TWG16: *Technology enhanced Resources for Mathematics Education (TeRMEd)*, by Anne O'Shea, Sinéad Breen, Conor Brennan, Frank Doheny, Christine Kelly, Ciarán Mac an Bhaird, Eabhnat Ní Fhloinn, Caitríona Ní Shé, Brien Nolan, Fiona Lawless & Seamus McLoone

TWG19: *A study of readiness for the integration of 21st century practices in mathematics classrooms*, by Aibhín Bray, Elizabeth Oldham & Katriona O'Sullivan

TWG22: *Using variation theory to explore the reteaching phase of lesson study*, by Thérèse Dooley & Tugba Aysel

TWG22: *The application of the universal design for learning framework to task design in order to support lower attaining children in the primary school mathematics classroom*, by Caitríona Wynne & Lorraine Harbison

From these two lists the breadth of work being carried out in mathematics education in Ireland can be appreciated — 14 out of the 23 TWGs have contributions from Ireland.

Will the work there be of any importance to IMTA members?

It is somewhat arbitrary to choose aspects of the enormous work that leads to the publication of CERME proceedings (estimated at over 4000 pages) and suggest what might be of interest to members. Of the 24 Irish submissions to CERME 10 (most of which can be expected to appear in the proceedings), six arise in a *primary* setting (Twohill; Leavy & Sloane; Ryan & Ní Ríordáin; Gurhy; Delaney; and Wynne & Harbison), another eight arise in a *third level or workplace* setting (Ní Ríordáin, Flanagan & Brilly; Björklund & Keogh; OReilly; Meehan, O'Shea & Breen; Cronin & Meehan; Hyland, van Kampen & Nolan; Howard, Meehan & Parnell; and O'Shea, Breen et al.), while two concern *frameworks for professional practice* (Nic Mhuirí and Bray, Oldham & O'Sullivan).

The (Project) Maths Development Team is recently placing much emphasis on Japanese *lesson study*. Although there might be other contenders for "the most dynamic maths education event of the year" (see Maths Counts link), it is certainly the case that lesson study provides a powerful vehicle for exploring and transforming practice (Ní Shúilleabháin & Clivaz and Dooley & Aysel). Two other areas that attract much attention at home and abroad at present are *task design* (Dempsey & O'Shea) and *teacher knowledge* (O'Meara, Prendergast, Harbison & O'Hara), the former in the context of initial teacher education, the latter considering issues relating to *transition from primary to post-primary* school. The setting of Dooley & Aysel's work is also across this transition.

Finally let us turn to the remaining four papers, situated in a *post-primary* setting. O'Brien & Ní Ríordáin present and discuss a diagnostic test addressing issues in *Junior Cycle algebra*. Clohessy & Johnson report on how *problem-solving in small groups* has a positive effect on individual achievement in problem-solving (in Junior Cycle). Cunningham, Close & Shiel examine changes in the *Junior Certificate examination papers* in mathematics from 2003 to 2015, and report that these are less than might have been expected when viewed from the perspective of PISA and TIMSS, both in terms of higher-order cognitive demand and of readability. Lane reports on data *relating students' 'image of mathematics' to parental occupation*; her study indicates that parents having a positive attitude to the subject, create a cycle of positivity and engagement among their Senior Cycle children.

Conclusion

I have tried to give a flavour of what it was like to engage with the detail of working with the research community in mathematics education both nationally and internationally during the past three years through the lens of CERME 10. This period, and indeed the previous decade, has seen in Ireland an enormous broadening and deepening of engagement in mathematics education in Ireland. In view of this, it is highly appropriate that CERME 10 should have taken place in Dublin this year. On a personal level, this experience has coincided with my own formal move from the discipline of mathematics to that of mathematics education, a transition that has brought new and rich perspectives to how I see the subject and its teaching and learning. I am deeply grateful to my new colleagues and, indeed my old ones, for their professionalism and friendship. (I will refrain from mentioning any by name, since I would not know where to stop!)

There are three areas meriting further attention that arise immediately from the discussion above (and from elsewhere too, of course):

1. It is highly desirable to move towards setting up formal (yet simple) structures in Ireland to support mathematics education, analogous to ERME in Europe and ICMI globally.

2. IMTA members are in a good position, due to the country-wide structure of the Association, to engage collaboratively with research in mathematics education.

3. Individual readers should not hesitate to contact researchers (and the latters' email addresses appear on CERME papers) about matters relating research and practice.

Web links

- CERME 10 homepage: http://cerme10.org/
- Call for submissions (for each TWG): http://cerme10.org/scientific-activities/twg-teams/
- Submissions available for all to see: http://cerme10.org/scientific-activities/twg-papers/
- The busy programme: http://cerme10.org/scientific-activities/scientific-programme/
- DCU news item: https://www.dcu.ie/news/2017/feb/s0217c.shtml
- ERME homepage: http://www.mathematik.uni-dortmund.de/ erme/
- CERME history: http://www.mathematik.uni-dortmund.de/ erme/index.php?slab=conferences
- CERME proceedings: http://www.mathematik.uni-dortmund.de/ erme/index.php?slab=proceedings
- ICMI homepage: http://www.mathunion.org/ICMI/
- ICMI history: http://www.icmihistory.unito.it/
- ICME-13, Hamburg, July 2016: http://www.icme13.org/
- Maths Counts, March 2017: http://www.projectmaths.ie/for-teachers/conferences/maths-counts-2017/

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# Linear Regression — an Introduction

Given a scenario where we have scrupulously collected a set of data, is it possible to make statements about some variable not contained within the data? Such a feat has many practical applications and the recent advancement of capabilities in this area has led to inventions that were previously consigned to the pages of science fiction — driverless cars and speech to text recognition to name but two. This is possible through mathematics, where we can express observations of variables in terms of some metric or category of interest that we may not always have information for. What we will talk about in the coming article are the basics of how this is typically achieved, illustrated with a real world example from the field of marketing analytics, with anonymous data kindly provided by Analytic Partners.

We will begin with a description of linear regression, which is an example of a statistical model. A statistical model is a set of assumptions about a variable or a relationship between variables. It provides us with a mathematical description of reality which we hope approximates reality closely enough for our purposes. What this translates to in practical terms is a scenario where we have information about a number of variables (let's call them $x_1$, $x_2$, etc.) but we want to investigate what the relationship is between these $x$ s and one single variable $(y)$. From Junior Cert maths we recall the concept of a function in the equation $y = f(x)$. In a statistical model the goal is to utilise the values of $x$ and $y$ that we have, to infer what is the function $f$. Graphically the idea behind the simplest form of linear regression is to fit a line which describes the linear relationship that is closest to the observed values of two variables. Think of a dataset which contains the number of units of a product that are sold with differing prices per unit. Our task is to produce the line shown in Fig. 1.

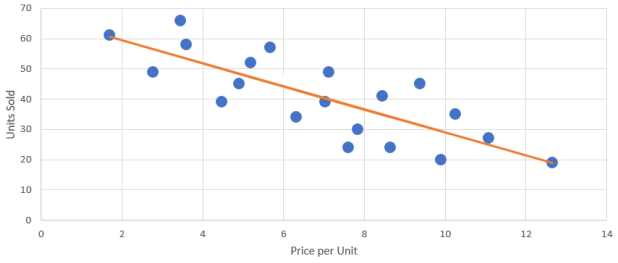


Figure 1: Simple Regression

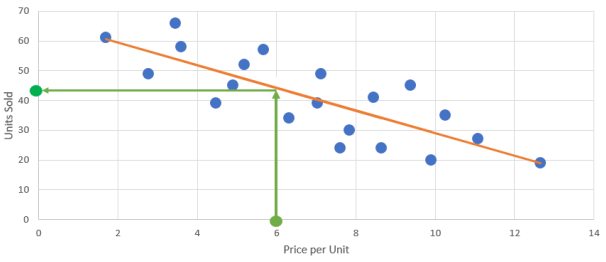


Figure 2: Prediction using Regression

As noted above there are a significant number of applications of this type of thinking, however linear regression itself is primarily used for predictive and descriptive purposes. With regards to prediction and our specific example, once we use linear regression to establish the relationship between price and units sold we are in a position to determine what the likely number of units sold will be, given some new value for price, as graphically shown in Fig. 2. We will describe the method and usage of descriptive models in the upcoming example.

Mathematically we describe the regression line as:

$$y_i = \alpha + \beta x_i + \epsilon_i \quad (i = 1 \ldots m) \tag{1}$$

Specifically for our example:

$$Units_i = \alpha + \beta Price_i + \epsilon_i \quad (i = 1 \dots m)$$

The index 1 to $m$ represents the number of rows that are available within our data set, in practice this can vary from a dozen to billions.

There are three main elements to Eq. 1 to consider. The first element is $x$ which represents the variable whose contribution to the value of the variable $y$ we want to express. $\epsilon$ represents the residual error between actual values and predicted; a key assumption of linear regression is that these $\epsilon$ follow a Normal distribution. We then have two parameters $\alpha$ and $\beta$ which are respectively referred to as the *intercept* and *coefficient*. The intercept is a figure that allows the regression line to pass either side of the origin. The coefficient is a constant that represents how the predicted value of $y$ changes for each unit change in $x$. In a scenario where we have $x_1, x_2, .., x_i$, the coefficient for $x_i$ only represents how predicted $y$ changes for a unit change in $x_i$ when all other $x$ s remain unchanged, $i.e$ the additional impact of adding $x_i$ to the model. In this simple example it represents the slope of the regression line. The question then becomes how we define what the intercept and coefficients should be? There are a number of different methodologies for this but one of the more common methodologies is that of least squares.

The objective behind the method of least squares is to minimise the sum of the squares of the differences between the $y$-coordinates and the predicted $y$-coordinates, (the predicted $y$-coordinates are calculated from the $x$ coordinates and the equation of the regression line). Taking our example from above, the squared errors are represented graphically in Fig. 3. The least squares method seeks to choose $\alpha$ and $\beta$ such that the total area of all the squares (the sum of the squared errors) is as small as possible. The height of the squares is determined by the smallest distance between the $y$-coordinates and the predicted $y$-coordinates.

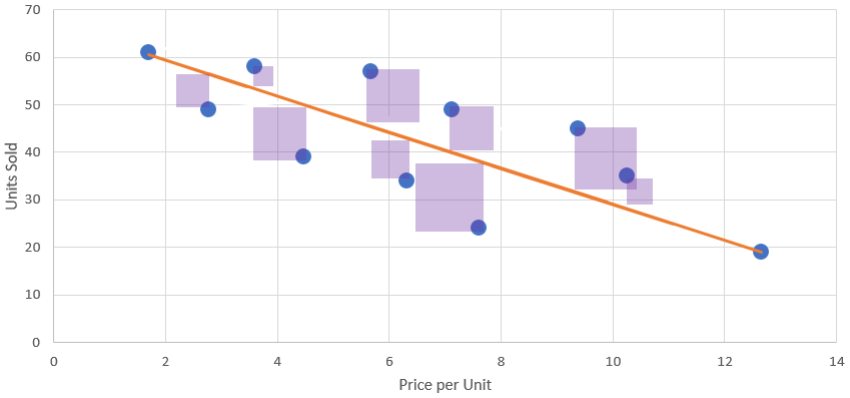


Figure 3: Square errors of least squares method

To derive the solution for least squares, given a series of points $(x_1, y_1)$, $(x_2, y_2)$ $\dots$ $(x_m, y_m)$, we proceed as follows:

First calculate the predicted $y$ values from the line $y = \alpha + \beta x$.

$$\begin{aligned} y_1 &= \alpha + \beta x_1 \\ y_2 &= \alpha + \beta x_2 \\ &\vdots \\ y_m &= \alpha + \beta x_m \end{aligned} \tag{2}$$

Next calculate the differences between the observed $y$ values and the predicted $y$ values.

$$y_1 - (\alpha + \beta x_1),\ y_2 - (\alpha + \beta x_2),\ \ldots\ y_m - (\alpha + \beta x_m)$$

The next step is to calculate the sum of the squares of the differences.

$$L = \sum_{i=1}^{m} (y_i - \alpha - \beta x_i)^2 \tag{3}$$

Notice here that the $y_i$ and $x_i$ are constants and $\alpha$ and $\beta$ can vary. The objective is to find the values of $\alpha$ and $\beta$ that will minimise $L$. We proceed as in single variable calculus by finding derivatives.

$$\begin{aligned} \frac{\partial L}{\partial \beta} &= \sum 2(y_i - \alpha - \beta x_i)(-x_i) = \sum (-2x_i y_i + 2x_i\alpha + 2x_i^2\beta) \\ \frac{\partial L}{\partial \alpha} &= \sum 2(y_i - \alpha - \beta x_i)(-1) = \sum (-2y_i + 2\alpha + 2x_i\beta) \end{aligned}$$

Setting the derivatives equal to zero, dividing by the common factor 2, and rearranging terms we get:

$$\begin{aligned} \left(\sum x_i^2\right)\beta + \left(\sum x_i\right)\alpha &= \sum x_i y_i \\ \left(\sum x_i\right)\beta + \sum \alpha &= \sum y_i \end{aligned} \tag{4}$$

To prevent too much clutter we can define $s_1$, $s_2$, $c_{xy}$, $c_y$ as

$$\begin{aligned} s_1 &= \sum x_i \\ s_2 &= \sum x_i^2 \\ c_{xy} &= \sum x_i y_i \\ c_y &= \sum y_i \end{aligned}$$

Noting that $\sum_{i=1}^{m} \alpha = m\alpha$ our two equations now become:

$$\begin{aligned} s_2\beta + s_1\alpha &= c_{xy} \\ s_1\beta + m\alpha &= c_y \end{aligned}$$

Solving for $\beta$ and $\alpha$ leads to

$$\beta = \frac{mc_{xy} - s_1 c_y}{ms_2 - s_1^2} = \frac{m\left(\sum x_i y_i\right) - \left(\sum x_i\right)\left(\sum y_i\right)}{m\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$\alpha = \frac{s_2 c_y - s_1 c_{xy}}{ms_2 - s_1^2} = \frac{\left(\sum x_i^2\right)\left(\sum y_i\right) - \left(\sum x_i\right)\left(\sum x_i y_i\right)}{m\sum x_i^2 - \left(\sum x_i\right)^2}$$

Thus these values of $\alpha$ and $\beta$ define the line of regression (line of best fit) $y = \alpha + \beta x$. The minimum value of the sum of the squared errors can of course now be calculated also with these values of $\alpha$ and $\beta$. An added benefit of using least squares is that we can use a standard theorem from the field of linear algebra to extend the above process to work with multiple explanatory variables.

The theorem supposes that $A$ is an $m \times n$ matrix, where $m > n$. Suppose $\mathbf{y}$ is an $m \times 1$ column vector. The $n \times 1$ vector $\hat{\mathbf{x}}$ that minimises $\|\mathbf{y} - A\mathbf{x}\|^2$ equals $(A^T A)^{-1} A^T \mathbf{y}$ (as long as $A^T A$ has rank $n$).

To relate this to what has gone before, we can express the system of equations (2) in matrix form as $\mathbf{y} = A\mathbf{x}$ where:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \quad A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_m \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The right hand side of Eq. 3 is then $\|\mathbf{y} - A\mathbf{x}\|^2$.

Now $A^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_m \end{pmatrix}$ and $A^T A = \begin{pmatrix} m & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}$

and $A^T \mathbf{y} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$. Thus the system of equations (4) can be expressed as: $A^T A\hat{\mathbf{x}} = A^T \mathbf{y}$.

In reality statisticians rarely deal with models just consisting of two variables. Instead there are typically multiple factors that influence a variable of interest. Each extra factor adds a column to our matrix $A$ and an extra entry to our vector $\mathbf{x}$. So now instead of the matrix $A$ being $m \times 2$ in our simple example above, it becomes $m \times n$ and the vector $\mathbf{x}$ changes from being $2 \times 1$ to $n \times 1$. One such example (of multiple factors influencing the variable of interest) is a confectionery company who want to better understand what key factors influence their sales on a weekly basis. More specifically their Chief Marketing Officer wants to know the following information:

- Of all sales that the company makes, what percentage is driven by marketing activity?
- Is the brand sensitive to pricing changes — what would be the impact of a 10% price increase?
- Is TV still the best way to advertise or is the investment in online video a more efficient replacement?

- How badly did an out of stock issue affect their sales?

In order to help, we can build a linear regression model to answer these (and many subsequent) questions. The first step of any statistical modelling project begins with relevant data collection. In our example we have to think about what the key factors that drive sales are.

- Non Promoted Price: Does the item represent good value for money?
- Distribution: Is the item available in a store for consumers to buy?
- Trade: Are temporary price promotions or any additional merchandising present in the store?
- Media: Are the company running any advertising campaigns which may encourage consumers to buy?

As well as these key pillars there are a number of other factors that influence sales. Competitors who have price promotions running are likely to steal sales. Significant changes in weather or economic factors can also have an impact. Once data has been collected for each of these key areas, we normally begin by graphing the data to gain a better understanding of what exactly has been happening over the model time period. This is a crucial step that normally uncovers unexplained gaps or erroneous data entries. An example of this is shown in Fig. 4, where tracking software has not been able to distinguish between the promoted price of a product and its non-promoted price. In consultation with the brand this is corrected for and more accurate data is now available to the model.

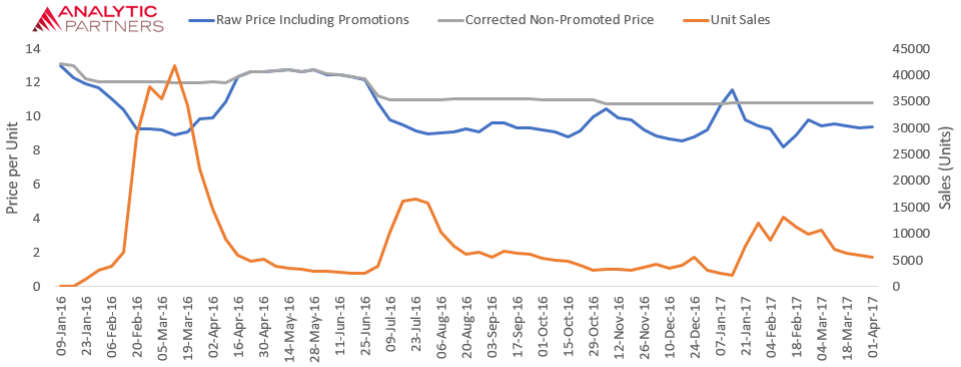


Figure 4: Data Validation Process

Once we are satisified that all relevant data has been collected and is as accurate a representation as possible, we can proceed with modelling. While statistical modelling is possible across a wide number of software applications an industry favourite is the program called 'R'. Feeding our data into R and running a linear regression, we can obtain coefficients for all of our variables. An extract of the formula we feed R for our data is similar to the example above:

$$Units_i = \alpha + \beta 1 Price_i + \beta 2 Distribution_i + \beta 3 TV_i + \ldots + \epsilon_i \tag{5}$$

The dots in Eq. 5 represent the additional data points available, (Out of Stock, Trade, etc.).

Once our model has been run our next task is to assess if the model approximations are indeed close to our actual data points. There are a number of statistical methods of measuring this. One that is typically used for linear regression is the R-Squared metric which is defined as:

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}} \tag{6}$$

Where $SS_{res}$ is the sum of squares of residuals ($i.e$ the difference between predicted and actual dependent points) and $SS_{tot}$ is the total sum of squares ($i.e$ the difference between each actual dependent point and the mean of the dependent). This provides a measure between 0 and 1 which tells us how much of the variation that exists within the dependent variable is captured by the model. We can also use other methods to check how the model is doing, the simplest of which is graphing the actual dependent points against those the model has predicted, such as in Fig. 5.

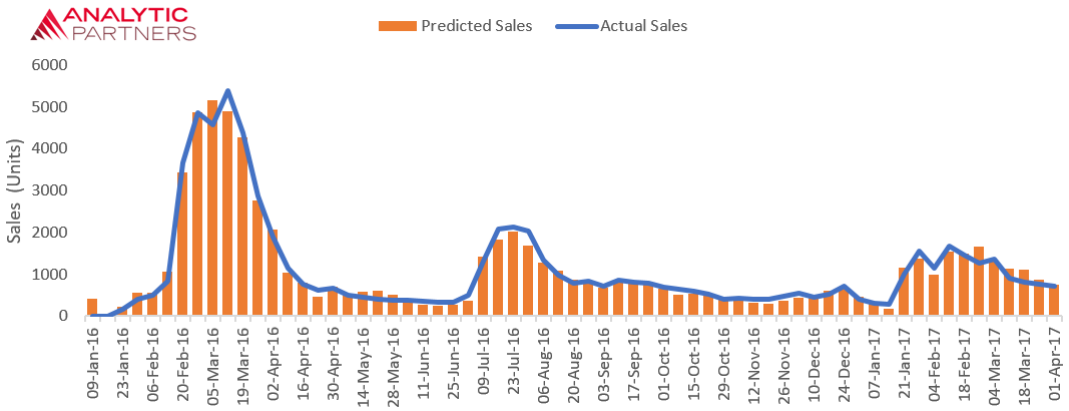


Figure 5: Linear Regression Model Accuracy

Looking at the graph the predicted sales do indeed appear to be following the shape of the actual sales. That, combined with an R-Squared value of 0.86, leads us to believe that our model is doing a reasonable job of describing the actual sales. However in practice the most difficult model test is also that which makes these types of models so useful: Do the coefficients make practical sense for the respective variables?

This is an area where subject-matter expertise and logical thinking are important in determining how valid the model findings are. For instance, from our example were we to find that the $\beta 2$ from Eq. 5 was negative, that would raise suspicion as it would imply that the more stores the brand was stocked in the less sales they would have.

However once we are in a situation where the model is statistically and logically sound there is great value in having a coefficient associated with each of our descriptive variables. To express each of our variables in terms of their contribution to unit sales, we simply multiply the coefficient by the value of the variable. So for instance for TV we can say

$$\text{Units Sold in } ith \text{ Week due to TV} = \beta 3 \times TV_i \tag{7}$$

So where the number of TV GRPs (an activity measure for TV, related to the percentage of a population who see an ad) we have is 101, we observe

$$\text{Units Sold in } ith \text{ Week due to TV} = (0.301) \times 101 = 30 \tag{8}$$

The $0.301$ in Eq. 8 is the coefficient for TV as derived from the model. Using this same methodology we can build up a volumetric picture in terms of unit sales for each variable in our dataset and then begin to answer the questions of the Chief Marketing Officer:

- Q. Of all sales that the company makes, what percentage is driven by marketing activity?
- A. Multiplying all of our coefficients by our respective variables allows us to see the volumetric impact of each. Grouping the variables by those that are marketing activities we can produce the numbers in Fig. 6.

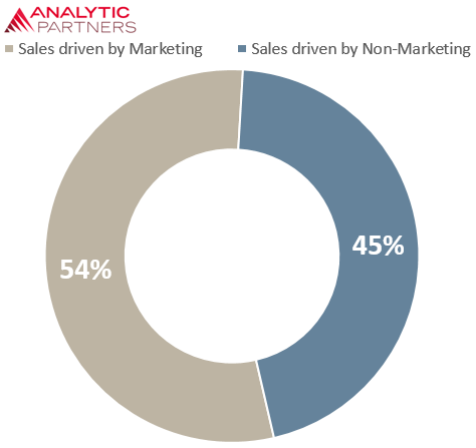


Figure 6: Sales Decomposition

- Q. Is the brand sensitive to pricing changes — what would be the impact of a 10% price increase?
- A. Pricing sensitivity is measured as the change in volume driven by price, relative to the absolute change in price. Applying this to our model yields a sensitivity of $-0.92$. This means that for a 10% price increase, volume will decrease by 9.2%. Additionally revenue will decrease by 0.12%.
- Q. Is TV still the best way to advertise or is the investment in online video a more efficient replacement?
- A. With both TV and online video we can ascertain the volume driven by each by multiplying the coefficients by the activity over the model time period. Here however the absolute volume driven is not the answer to our question, as different amounts were invested in each channel over the model time period. If we divide the volume driven due to each channel by the amount spent we can derive the return on investment per channel. Working through this excercise leads to a return on investment of $0.37$ for TV and $0.41$ for online video. This means that online video can be a more efficient way of driving sales in this example.
- Q. How badly did an out of stock issue affect their sales?

- A. This is a case where we don't have a direct data source to join to our model, but we can create one. The brand can tell us what time period the stock was missing from stores. Subsequently we can create a binary variable for that time period: $1$ indicating the product was out of stock, $0$ indicating otherwise. Adding this to our model leads to a coefficient of $-0.829$, which added over all weeks and regions leads to a total loss of 99 units.

The data provided for this example comes from Analytic Partners who are a premium data driven marketing consultancy. They answer questions similar to those seen in this paper among many other business intelligence capabilities. For more examples of this type of work or for more information on Analytic Partners visit `https://analyticpartners.com/`

Linear regression helps provide us with a mathematical description of reality which we hope approximates that reality closely enough for our purposes. It has been important in a wide number of fields from biology to economics and beyond. In this paper we have touched on the methodology behind linear regression but also seen how it can be used in a business context.

Cian O'Loughlin (Cian works as a data scientist in Toronto, Canada).

# Peter's Problem 2016

A company representative's weekly work schedule requires a visit to customers in each of the following 21 towns:

1. Athlone, Co. Westmeath
2. Balbriggan, Co. Dublin
3. Carrick-on-Shannon, Co. Leitrim
4. Drogheda, Co. Louth
5. Enniscorthy, Co. Wexford
6. Freshford, Co. Kilkenny
7. Greystones, Co. Wicklow
8. Headford, Co. Galway
9. Kilcullen, Co. Kildare
10. Lismore, Co. Waterford
11. Mountrath, Co. Laois
12. Nenagh, Co. Tipperary
13. Oranmore, Co. Galway
14. Portlaoise, Co. Laois
15. Rathkeale, Co. Limerick
16. Swinford, Co. Mayo
17. Tullamore, Co. Offaly
18. Urlingford, Co. Kilkenny
19. Virginia, Co. Cavan
20. Westport, Co. Mayo
21. Youghal, Co. Cork

The attached table of distances gives the Automobile Association preferred routes and distances between each of the towns listed.

NOTE: Identify and list the reference for all details used in your solution.

Table 1: Table of Distances (in km)

| | Athlone | Balbriggan | Carrick-on-Shannon | Drogheda | Enniscorthy | Freshford | Greystones | Headford | Kilcullen | Lismore | Mountrath | Nenagh | Oranmore | Portlaoise | Rathkeale | Swinford | Tullamore | Urlingford | Virginia | Westport | Youghal |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Athlone | | 153 | 73 | 169 | 167 | 118 | 153 | 102 | 104 | 172 | 82 | 76 | 78 | 74 | 144 | 144 | 40 | 123 | 86 | 106 | 218 |
| Balbriggan | 153 | | 181 | 23 | 163 | 156 | 75 | 252 | 82 | 248 | 131 | 189 | 228 | 122 | 256 | 236 | 128 | 163 | 85 | 279 | 268 |
| Carrick-on-Shannon | 73 | 181 | | 150 | 253 | 176 | 182 | 100 | 169 | 266 | 141 | 150 | 111 | 137 | 218 | 63 | 103 | 181 | 86 | 106 | 312 |
| Drogheda | 169 | 23 | 150 | | 184 | 177 | 96 | 273 | 103 | 270 | 153 | 210 | 249 | 143 | 277 | 205 | 149 | 185 | 63 | 248 | 289 |
| Enniscorthy | 167 | 163 | 253 | 184 | | 97 | 93 | 266 | 87 | 121 | 99 | 164 | 241 | 90 | 231 | 292 | 121 | 112 | 205 | 335 | 129 |
| Freshford | 118 | 156 | 176 | 177 | 97 | | 149 | 187 | 92 | 101 | 39 | 77 | 147 | 40 | 144 | 219 | 73 | 15 | 198 | 238 | 148 |
| Greystones | 153 | 75 | 182 | 96 | 93 | 149 | | 253 | 73 | 211 | 123 | 181 | 228 | 113 | 247 | 237 | 129 | 155 | 117 | 279 | 219 |
| Headford | 102 | 252 | 100 | 273 | 266 | 187 | 253 | | 203 | 216 | 147 | 118 | 180 | 178 | 148 | 66 | 140 | 177 | 189 | 53 | 238 |
| Kilcullen | 104 | 82 | 169 | 103 | 87 | 92 | 73 | 203 | | 181 | 58 | 116 | 32 | 49 | 183 | 207 | 60 | 90 | 123 | 250 | 188 |
| Lismore | 172 | 248 | 266 | 270 | 121 | 101 | 211 | 216 | 181 | | 126 | 113 | 185 | 131 | 98 | 274 | 163 | 90 | 290 | 266 | 29 |
| Mountrath | 82 | 131 | 141 | 153 | 99 | 39 | 123 | 147 | 58 | 126 | | 57 | 122 | 14 | 124 | 184 | 38 | 41 | 129 | 227 | 172 |
| Nenagh | 76 | 189 | 150 | 210 | 164 | 77 | 181 | 118 | 116 | 113 | 57 | | 88 | 71 | 68 | 167 | 73 | 65 | 230 | 169 | 160 |
| Oranmore | 78 | 228 | 111 | 249 | 241 | 147 | 228 | 32 | 180 | 185 | 122 | 88 | | 155 | 117 | 89 | 116 | 138 | 163 | 81 | 207 |
| Portlaoise | 74 | 122 | 137 | 143 | 90 | 40 | 113 | 178 | 49 | 131 | 14 | 71 | 155 | | 139 | 180 | 34 | 132 | 163 | 223 | 178 |
| Rathkeale | 144 | 256 | 218 | 277 | 231 | 144 | 247 | 148 | 183 | 98 | 124 | 68 | 117 | 139 | | 204 | 142 | 132 | 300 | 198 | 115 |
| Swinford | 144 | 236 | 63 | 205 | 292 | 219 | 237 | 66 | 207 | 274 | 184 | 167 | 89 | 180 | 204 | | 141 | 145 | 300 | 198 | 295 |
| Tullamore | 40 | 128 | 103 | 149 | 121 | 73 | 129 | 140 | 60 | 163 | 38 | 73 | 116 | 34 | 142 | 141 | | 224 | 145 | 44 | 209 |
| Urlingford | 123 | 163 | 181 | 185 | 112 | 15 | 155 | 177 | 90 | 90 | 41 | 65 | 138 | 132 | 132 | 78 | 224 | | 86 | 184 | 136 |
| Virginia | 86 | 85 | 86 | 63 | 205 | 198 | 117 | 189 | 123 | 290 | 129 | 230 | 163 | 163 | 300 | 145 | 86 | 205 | | 228 | 309 |
| Westport | 106 | 279 | 106 | 86 | 335 | 238 | 279 | 53 | 250 | 266 | 227 | 169 | 81 | 223 | 198 | 44 | 184 | 198 | 228 | | 181 |
| Youghal | 218 | 268 | 312 | 289 | 129 | 148 | 219 | 238 | 188 | 29 | 172 | 160 | 207 | 178 | 115 | 295 | 209 | 136 | 309 | 288 | |

Questions

1. Using the **table of distances**, plan and write down the shortest route that should be taken to minimise the distance travelled to visit all of the towns in one complete circuit, starting in Athlone as home-base and returning to that base following the final visit.

2. Using a suitable chart, table or otherwise, illustrate the route you have chosen, itemising the distance between each town in turn.

3. Using the Statistics section of the SIMI (Society of the Irish Motor Industry) website, find the most popular 152-registered car – car and model – sold in the Republic of Ireland.

4. Using the 'combined driving' performance claimed by the manufacturer in their Product Guide for the car model, find the amount of petrol, in litres, used by the car on those trips (the rep is provided with a 1.4 litre, 150HP, manual, 5-door car of the make and model identified in Q3).

5. If the rep was driving the 1.6 litre manual, 90HP, 5 door diesel version of the **same** car model, calculate the amount of diesel, in litres, used by this car, again using the 'combined driving' performance claimed by the manufacturer.

6. If diesel is 10.2 cent per litre cheaper than petrol, calculate the difference between the cost of fuelling the two cars for all 21 journeys, rounded to the nearest 5 cent.

7. Again, using the manufacturer's statistics, calculate which car is cleaner for the environment in terms of $CO_2$ emissions by using the $CO_2$ emissions figures provided in the Product Guide for each vehicle and state the difference between these emissions for the total journey, correct to the nearest gram.

Neil Hallinan and Dominic Guinan

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# Peter's Problem 2016 — Sample Solution

**Q1.** Preferred route:

| | | | |
|---|---|---|---|
| Athlone | → Tullamore | → Portlaoise | → Mountrath |
| Mountrath | → Freshford | → Urlingford | → Nenagh |
| Nenagh | → Rathkeale | → Lismore | → Youghal |
| Youghal | → Enniscorthy | → Kilcullen | → Greystones |
| Greystones | → Balbriggan | → Drogheda | → Virginia |
| Virginia | → Carrick-on-Shannon | → Swinford | → Westport |
| Westport | → Headford | → Oranmore | → Athlone |

Or the reverse journey may also be outlined.

The distances (in km) are taken from the Distance Chart supplied:

| | | |
|---|---|---|
| Athlone | $\xrightarrow{40}$ Tullamore | $\xrightarrow{34}$ Portlaoise |
| Portlaoise | $\xrightarrow{14}$ Mountrath | $\xrightarrow{39}$ Freshford |
| Freshford | $\xrightarrow{15}$ Urlingford | $\xrightarrow{65}$ Nenagh |
| Nenagh | $\xrightarrow{68}$ Rathkeale | $\xrightarrow{98}$ Lismore |
| Lismore | $\xrightarrow{29}$ Youghal | $\xrightarrow{129}$ Enniscorthy |
| Enniscorthy | $\xrightarrow{87}$ Kilcullen | $\xrightarrow{73}$ Greystones |
| Greystones | $\xrightarrow{75}$ Balbriggan | $\xrightarrow{23}$ Drogheda |
| Drogheda | $\xrightarrow{63}$ Virginia | $\xrightarrow{86}$ Carrick-on-Shannon |
| Carrick-on-Shannon | $\xrightarrow{63}$ Swinford | $\xrightarrow{44}$ Westport |
| Westport | $\xrightarrow{53}$ Headford | $\xrightarrow{32}$ Oranmore |
| Oranmore | $\xrightarrow{78}$ Athlone | |

**Q2.** Total Distance travelled is **1208** km.

**Q3.** Most popular 152-registered car:
From the site `www.simi.ie` the tab BeepBeep Stats brings the search to `www.beepbeep.ie/stats`. Using the year 2015 and Months from July to December the relevant result may be found. The most popular 152-registered car was the **VW Golf**.

**From the site** `http://www.volkswagen.ie/en.html`
**select Models/Golf/Brochures and Prices to find where the Golf Product & Price Guide may be accessed or downloaded.**

**Specifications for Petrol car: 1.4 litre, 150hp engine, 5-door, Manual:**
From the Listing '*Effective from 28 October, 2015*' or the listing '*Effective from 1 February, 2016*' this is item No. 36 on the list:
5G14PX Golf HL BMT ACT 1.4 TSI 5DR 150 HP M6F Highline Petrol 115 ($CO_2$)

**Specifications for Diesel car: 1.6 litre, 90hp engine; 5-door, Manual:**
This is item No. 7 on the same list:
5G122V Golf TL BMT 1.6 TDI 5DR 90HP M5F Trendline Diesel 101 ($CO_2$)

**Q4.** Petrol usage:
The figures for petrol usage are given on the Technical Data (pages 19–20 of the specifications). Under Environmental Information for Manual gearbox Combined driving, 1.4 TSI 150HP: 4.8 litres per 100km
Petrol usage: 1208km @ 4.8 litres/100km = **57.984 litres**

**Q5.** Diesel usage: The figures for diesel usage are given on the Technical Data (pages 21–22 of the specifications)
Under Environmental Information for Manual gearbox Combined driving, 1.6 TDI 90HP: 3.8 litres per 100km
Diesel usage: 1208km @ 3.8 litres/100km = **45.904 litres**

**Q6.** Difference in costs:
Take an example:
Let the price of the diesel be €1. This implies the petrol costs €1.102.
Using these values the cost of the trip for the diesel car is €45.904.
The cost of the petrol is €$57.984 \times 1.102 =$ €63.898368. The difference is €$(63.898368 - 45.904) =$ €17.994368 = €18.00 rounded to the nearest $5c$. We note that the difference is not constant since it depends on the price of fuel.

A mathematical model is required.
Let the price of diesel be €$x$ per litre ⇒ the price of petrol is €$(x + 0.102)$ per litre.
The cost of diesel is €$45.904x$ and the cost of petrol is €$57.984(x + 0.102)$.

$$\begin{aligned}\text{Difference} &= €(57.984x + 5.914368 - 45.904x)\\ &= €(12.08x + 5.914368)\end{aligned}$$

where €$x$ is the cost of diesel per litre.

**OR**
Let the price of petrol be €$y$ per litre ⇒ the price of diesel is €$(y - 0.102)$
The cost of diesel is €$(45.904(y - 0.102))$ and the cost of petrol is €$57.984y$.

$$\begin{aligned}\text{Difference} &= €(57.984y - 45.904(y - 0.102))\\ &= €(57.984y - 45.904y + 4.682208)\\ &= €(12.08y + 4.682208)\end{aligned}$$

where €$y$ is the cost of petrol per litre.

**Q7.** $CO_2$ Emissions:
The specifications for the models identified in Section 3 above are 101 g/km for diesel and 115 g/km for petrol. The difference is 14 g/km in favour of diesel.
**NOTE:** On the Technical Data pages the $CO_2$ emissions figure for diesel is given as 98 g/km while the figure for petrol is 112 g/km. (These are also the figures listed for the 3-door model of the same specification.) However, the difference is also 14 g/km in favour of diesel.

Difference in emissions is 14 g/km ⇒ for the journey the difference is $1208 \times 14 = 16912$ g in favour of the diesel model.

Dominic Guinan and Neil Hallinan

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Contributions to the Newsletter are welcome from anybody who has an interest in mathematics. So whether you are a student, a primary schoolteacher, secondary, third-level, retired, or your work makes use of mathematics or you have an interest in the history of mathematics then send your article or idea to:

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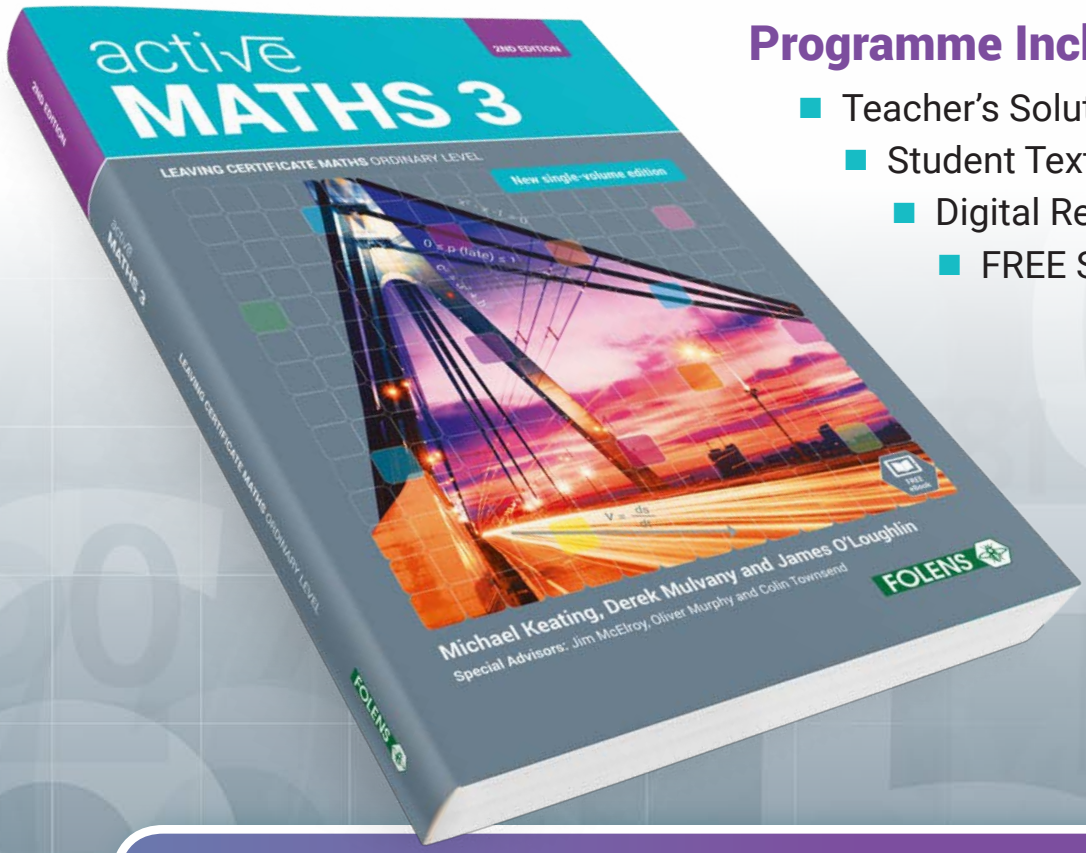


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