Cumann Oidí Matamaitice na hÉireann

# NEWSLETTER

Irish Mathematics Teachers' Association

Number 114
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Cumann Oidí Matamaitice na h-Éireann
Irish Mathematics Teachers' Association

Founded in 1964 to promote and assist the teaching of mathematics at all levels. Membership is open to all those interested in mathematics and mathematics education. The IMTA is represented on all NCCA mathematics course committees. Individual membership may be obtained through a Branch organisation.

# Editorial

The year 2014 has been a most significant one in the life-time of the Irish Mathematics Teachers' Association. This is the year in which the Association celebrated the 50th Anniversary of it founding - its Golden Jubilee Celebration. To honour the occasion and maintain a strong link with the teachers of 1964 a special Celebration Banquet was organised where the Guest of Honour was the President of Ireland, Michael D. Higgins. Founding members were invited as Special Guests as were all Past Chairpersons or the family representatives of deceased Past Chairpersons. President Higgins was accompanied at the celebration by his wife, Sabina. The President gave a speech to the guests and brought them to their feet at the conclusion. The text of the prepared speech is contained in this edition. What it does not contain is the 'significant departure from the script', as President Higgins said, which was a true delight to the guests. In this departure, among other remarks about mathematics, the President made reference to the Memorabilia Exhibition which he had viewed earlier and which obviously piqued his interest.

Towards the end of the banquet Professor Elizabeth Oldham delivered the Fr. Ingram Memorial Lecture in the most enthralling manner. The text of this speech is also published in this edition.

Huge congratulations to the IMTA on its 50th Anniversary and to all concerned in enabling it to be such a success. Here's to the success of the next 50 years!

The Celebration Banquet followed on from another successful MathsFest held in UCD. The organisation and production of this event requires the outstanding commitment and ability of some very dedicated persons and, above all, the MathsFest Organiser, Natalie Noone. Her infectious enthusiasm carried through to the spirit of everyone involved on the day. *Míle buíochas!*

Of course there is more in this Newsletter to bring you back to earth or perhaps to soar again. Fancy a construction activity or a game of snooker? Activities of this nature are explored by Michael O'Loughlin and Neil Hallinan. These are followed up by an article on geometry which should, perhaps, be indicated as 'required reading'. Professors Patrick D. Barry and Anthony G. O'Farrell contextualise this document as a precursor to the Geometry section of the Syllabus and, as such, it is most enlightening.

Awards and Competitions now form a major part of the mathematics year. From the 'old reliables' of the Irish Junior Mathematics Competition, Team Math and the Mathematical Olympiad to the newer arrivals of Peter's Problem, the Accenture Awards and the John Hooper Medal for Statistics, students around the country are encouraged and rewarded in some small part for their earnest endeavours. The IMTA is involved in many of these competitions directly but applauds them all. Congratulations to all those involved.

But, just in case there are any who think that their efforts are in vain, a reflection piece from yours truly goes to indicate that the pathway to success is sometimes not clearly visible. Labels of 'success' and 'failure' are often applied too quickly. It is, perhaps, apposite that the amalgamated work of researchers supplied by Aoibhinn Ní Shuilleabháin also reaches a conclusion regarding 'learning through failure'.

Thanks to all contributors.

On a personal note: It is now time for this Editor to move on to pastures new. I have thoroughly enjoyed working on the IMTA Newsletter over the past ten years. I hope that it brought you some enjoyment too. My best wishes to the next incumbent of this office.

I want to pay special tribute to Elizabeth Oldham, the other half of the editorial team. Elizabeth has provided me with her insightful comments and wise counsel over a number of years. Her rigorous and high standards have been applied in a most dedicated manner. Ever generous with her time she has been an unfailing support to me as Editor. My sincere thanks to you, Elizabeth.

*Slán agus beannacht.*

Neil Hallinan hallinann@gmail.com

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# Notes From Council

**Branch changes:** The Limerick Branch of the IMTA was amalgamated into the Clare Branch and the new Branch is now referred to as the Clare/Limerick Branch. The Carlow Branch is no longer in operation due to the lack of a committee.

**Survey on time and mathematics teaching:** As mandated at the AGM in October, 2013, a survey on time allocation for mathematics at second level in Ireland was conducted. Some five hundred and sixty seven responses were received before it was closed at the end of January, 2014, and a subcommittee of the IMTA Council was formed to work on presenting a report on the results.

**Maths-in-Practice Report:** Two members of the IMTA were members of the *Maths-in-Practice* group. Its report was issued in June 2014 and is available at http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Project_Maths/Information/Reports/Maths-in-Practice-Report-.pdf.

**Launch of pilot module for Transition Year students:** Minister of State, Seán Sherlock, launched Accenture's Transition Year module on analytics in March. This was piloted by members of the IMTA. Its purpose is to introduce students to the role of analytics in the business world. The students will be submerged into the world of an analytics consultant through a case study in which they will be given information about a company along with data that must be managed and used for insights. While working on the case study they will be introduced to a business model and will be challenged with analytics puzzles that will have them naturally thinking out-of-the-box. The course is a typical eight-week Transition Year course which the teacher may extend to their own comfort. It is available here: http://www.accenture.com/ie-en/company/sponsorships/Pages/acn-ty-analytics-module.aspx

**Constitution Subcommittee:** A subcommittee was formed in March to consider the IMTA Constitution and come back to Council with appropriate updates/amendments.

Brendan O'Sullivan Hon. Secretary

Fr. Brendan Steen C.M.

# A Special Presentation

At the 50th Celebration Banquet a special presentation of a plaque was made to Fr. Brendan Steen C.M. for his life-long devotion to the cause of mathematics, for his services to the IMTA and for his work as Treasurer of the Dublin Branch of the IMTA over the past 50 years.

# Speech by President Michael D. Higgins at a Gala Dinner to mark the 50th Anniversary of the Irish Mathematics Teachers' Association Stillorgan Park Hotel, Saturday, 11th October 2014

Tá an-áthas orm bheith anseo tráthnóna chun chothrom caoga bliain bhunú Chumann Oidí Matamaitice na hÉireann a chomóradh; ar chumann é a raibh duine de mo réamhtheachtaithe mar Uachtarán na hÉireann, Eamonn de Valera ina chéad phatrún aige.

[I am delighted to be here this evening as you mark the 50th anniversary of the establishment of the Irish Mathematics Teachers' Association; an association which had, as its first patron, one of my predecessors as Uachtarán na hÉireann, Eamonn de Valera].

I would like to thank your Chairperson, Seán MacCormaic, for his invitation to Sabina and me to join you tonight at this gala banquet, and all of you for the very warm welcome we have received.

I was most interested while preparing to attend this event, to read a newsletter prepared by the Association during the year of its foundation – 1964. The paper was old and the typescript quite faint and it was no doubt written on an old Imperial typewriter such as we now see in antique shops, and which I am sure none of you here could possibly remember. But the words resounded with a contemporaneous tone and the aim of the Association remains as relevant today as it did during that first gathering in Newman House fifty years ago – 'to further the study and teaching of Mathematics, and to foster co-operation among teachers at all levels'.

Today that aim sits as pertinently on your website as it did on that sepia toned page of long ago, reminding us that while its aim has remained steadfast, the Association has continued to progress and develop in a rapidly changing environment.

I know that one of the questions which most challenges mathematics teachers, and which I am sure many of you dread hearing is "why do I need to know this"? While the question may not have changed over the years, I am sure that the answer is one which must constantly be adapted and developed and made relevant to the world in which each generation of questioners live.

Indeed there can be no doubt that a central challenge for teachers of mathematics is not only that of inspiring students to understand the intrinsic worth of mathematics as an intellectual pursuit, but also to enable them to comprehend its role as a powerful tool in understanding and appreciating the manner in which the world around them operates.

We are, indeed, living in a very different world from the one inhabited by those who attended that inaugural meeting of the Association in January 1964. While a fundamental grasp of mathematics has always been an essential tool for rational and analytical thinking, and a powerful aid to the development of problem solving skills, today in a global and knowledge based economy it is also an essential competence for many of the disciplines which are fundamental to Ireland's continued economic growth and competitiveness. In addition, mathematics has been identified at EU level as one of the key competencies for personal fulfilment, active citizenship, social inclusion and employability in the knowledge society which we, as global citizens, now inhabit.

I speak often of the task of working to build together an active, inclusive citizenship; where all citizens will be enabled to participate and realise their full possibilities; a citizenship which must define any truly democratic republic. If we are to build such a citizenship we must ensure that our young people are equipped with the necessary fundamental tools on their journey through

life, enabling them to fully engage and interact with society, and to play their part in their communities, making their own important contribution to those communities and to their society as a whole.

We live in an age when important policy issues in areas such as economics, finance, public health or scientific research are presented as being technical concerns, which are beyond the comprehension of the citizens affected. It is critical, therefore, that we develop and encourage confident, resourceful and enthusiastic maths students in both our primary and post primary classrooms, who are empowered with the skills and knowledge to engage critically with the world around them and to see the appropriate use of mathematics in model building and statistical description. This commitment to nurturing life-long skills is something that I know lies at the heart of your work.

It is a commitment that is evident in the theme of your Mathsfest this weekend: *Celebrating the past, embracing the present and preparing for future change',* and also in the vibrant programme that was prepared for this event. It is a programme that demonstrates the real will that exists amongst those who work in the field of mathematical education to innovate, reconsider and indeed question the status quo as they seek to know more and constantly progress the teaching of mathematics in this country.

It is also a programme that reflects a genuine and ongoing desire to work together and forge relationships between schools and mathematics teachers across the country.

Each member of the Irish Mathematics Teachers Association has clearly demonstrated their understanding of the importance of creating a spirit of community between those involved in the policy and practice of primary and post primary mathematical education in Ireland and may I take this opportunity to commend you for that.

Indeed, throughout the years since its foundation the Association has constantly worked to develop partnerships with key stakeholders in the field of education such as the Department of Education and Skills, the State Examinations Commission and the National Council for Curriculum and Assessment, demonstrating a profound understanding of just how much can be achieved when interested groups come together in a spirit of co-operation in order to work towards common goals.

It is this understanding that really lies at the heart of the ethos of the Association and there can be no doubt that, as a society, we owe a strong debt of gratitude and appreciation to the Irish Mathematics Teachers Association. By providing educators with valuable opportunities to share experiences, discuss new ideas, and move those ideas from proposals to reality while providing support to each other as they carry out their day to day roles, the Association is an enormously important resource in ensuring the constant improvement of mathematical education here in Ireland.

I thank you for that and I thank you all once again for welcoming me here this evening.

I know you have been working your way through a varied and highly interesting programme, and I trust much fruitful discussion and thought has grown from this important event.

Mar fhocal scoir, ba mhaith liom gach rath a ghuí ar bhur gcuid oibre ríthábhachtaí agus go mbeidh tráthnóna fíor-thaitneamhach a bheith agaibh chun bhur gcomóradh suntasach a cheiliúradh.

I would like to conclude by wishing you every success with your critical work and a most enjoyable evening as you celebrate your significant anniversary.

Go raibh míle maith agaibh go léir.

# FR. INGRAM MEMORIAL LECTURE 2014

## Celebrating the past, embracing the present, and preparing for future change

Elizabeth Oldham

11/10/2014

Friends... *a chairde:*

Maurice OReilly was wondering if, in starting to speak, I would follow the example of a very distinguished namesake of mine who addressed diners three years ago in Dublin Castle.[1] I might have done so, but the President has gone; so we'll settle for *a chairde*. After all, friendship is one of the features of the Association.

It is a great honour to give the Fr. Ingram Memorial lecture. I remember Fr. Ingram; he used to come to mathematics seminars in Trinity when I was an undergraduate in the early 1960s. I never dreamed that more than fifty years later I would be here and talking under his banner, using the MathsFest theme: "Celebrating the past, embracing the present, and preparing for future change." Traditionally, the "Fr. Ingram" was a solid academic lecture; that would not work as an after-dinner speech, so tonight I am presenting a subset of a paper that may eventually appear somewhere in print. I shall focus on celebrating the past via reminiscence and anecdote, but I hope to incorporate messages for the future. And I shall try to keep the talk to about fifteen minutes from here on.

A few years after those undergraduate days, I started my teaching career in Alexandra College. "Alex" was then in Earlsfort Terrace: opposite UCD where Dublin Branch meetings were held, and round the corner from Newman House where the Association was born. I must pay tribute here to the two senior mathematics teachers, Miss Gillespie and Mrs. Dobbs; they did not quite grab my arms and frog-march me across the road, but they put considerable and (I believe) utterly appropriate pressure on me to join the Association. As a young teacher I somehow found time to keep a diary, and it shows that I attended my first Branch meeting on 15th October 1970. The lecture was on one of the topics newly introduced into the Leaving Certificate syllabus: (can you guess?) *Vectors*.[2] My diary entry states: "Prof. O'Brien spoke, and many people argued." In other words, there was a lively discussion on the very nature of mathematics as reflected in the new programme.

That lecture took place six years after both the foundation of the Association and the beginning of the last really fundamental revision of mathematics education in Ireland: the introduction of "modern mathematics" (well flagged by Neil's archive display next door). Coincidentally, six years have elapsed since the inauguration of the current and likewise fundamental revision involved in Project Maths. Then, as now, the IMTA was engaged in ***supporting mathematics teachers*** in assimilating the changes, as well as ***taking part in passionate discussions*** with regard to mathematics education. I might add that then, as now, there were complaints about insufficient guidance on interpreting the new syllabuses; then, as now, these complaints were

documented in IMTA records and were communicated to the Department of Education. I have an idea for a film about the life of the Association; it could be called "Back to the Future."

However, the context then was different – different in at least three ways. I mention them here because there are parallels in planning for future change.

First: consider the uptake of Mathematics courses. We know that far fewer students took the State examinations than is the case now; but I did not realize until recently that, in 1961, only 56 girls sat for Honours Leaving Certificate mathematics. Through the sixties and seventies the numbers (boys and girls) staying on in school to take the Leaving Certificate were growing very fast, changing the target population for Leaving Certificate education. If the timing had been different, perhaps the syllabus changes of the sixties and early seventies would have been different too.

However, perhaps they would not, for a second contextual issue arises here: that of curricular trends. The "modern mathematics" trend was influential worldwide, and it was said that by 1980 (excuse the awful puns)

- "geometry had been TRANSFORMED" (REFLECT on that!)
- "it was now À LA MODE to teach statistics" (indeed, there was a greater CHANCE than before of finding probability in school curricula)
- "there were INEQUALITIES [across countries] in the treatment of algebra"
- … moreover, in some countries, algebra had been (re)STRUCTURED, as we SET out on a great mathematical adventure.

But by 1980 also, the next big change in mathematics education worldwide was under way. To continue the bad puns, that was the stage at which it became appropriate to talk of PROBLEMS IN MATHEMATICS EDUCATION – though, unlike for modern mathematics, we in Ireland did not take part in that discussion until some two decades later.

Thirdly: we were only discovering that bringing about profound change takes *time*. As an illustration, consider the seminar run by the Association and Dublin Branch for school students in 1971; it was a fascinating session of "Any Questions" type, with students putting questions to a panel including mathematicians and teachers. A few participants are here tonight (hands up please!). One questioner raised the issue of students being "caught between two stools" as the new syllabuses were introduced. (This could be "Back to the Future, part 2.") A third-level panellist expressed surprise that the new syllabuses – some parts of which were still within their first cycle of implementation – were not yet fully bedded in. We in education may now have learnt the lesson that change takes time; it remains for our partners in industry and business to do likewise … but *sin scéal eile*.

Let me return to that young teacher in the 1970s. The demographics were notably different from now; for example, the archive contains photographs of lectures attended by rows and rows of nuns, all in their habits – what great characters there were among them! But in essence Branch life was similar. I encountered many new ideas and much wisdom via the experienced teachers

who gave lectures, or who shared their thoughts over a cup of tea – it really was tea then – after the meetings; again, some of them are here tonight. I made friends. I began to take part in the committee work of the Branch … and so learnt the very important lesson, chiefly from Fr. Steen, that IMTA events should *never* be scheduled to clash with major sporting fixtures. Of course, I was an apt pupil! More seriously, I encountered themes that have recurred over my 44 years in the Association: themes in addition to the focus on ***teacher support*** and the ***debate around curriculum*** apparent on that first evening in UCD, and ***organizing events for school students*** such as that early seminar. I shall mention just one: ***members' dedication to students' success***, illustrated almost annually at the "post-mortems" on examination performance by outbursts of indignation at the difficulty levels of the papers and the vagaries of mark allocation. "Back to the Future, Part 3"… except that the marking schemes then were on a par with official secrets, albeit "leaked" to us with some connivance from those in high places.

The post-mortem meetings were essentially a Branch activity. Over time, I learnt about the work of the Association, initially via the Annual Conferences. Looking over conference programmes and my notes, I find an enormously rich set of contributions, many as relevant now as then. The same is true of the *Newsletter*. It goes to show that *in mathematics education there are no theorems, establishing truth once for all; there are questions – often the same questions, with answers that change according to changing contexts.*

The conferences provided (and provide – like today's) further ***teacher support*** and ***CPD***; the Association, like the Branches, has initiated ***events for school students***; but, as I indicated earlier, the Association has a further role as an ***advocate with powers-that-be***. We have access to people and processes that can allow us to transmit our views. Over the years we became proactive as well as reactive, notably in aiming to initiate syllabus change but also in other ways. (How many people here are aware that we ourselves formulated a policy on calculator use – basically "go for it!" – around 1980?) Our representatives have been among the most active on course committees; in my own time as Education Officer, I could see that we alone systematically involved our members: myself and Peter Tiernan (God rest him) visiting branches and ducking the squashed tomatoes thrown at us when people disagreed with syllabus drafts! Of course we have not always has as much influence as we would like; that can be frustrating and disheartening. However, we have tried, and our efforts are on record. I believe it is important to ***maintain our status as the, or at least a major, go-to body*** as we work for future change in mathematics education.

So what other messages for the future can be gleaned from looking back – or what TRANSFORMATIONS can we look for as we REFLECT on the past? Here is a couple to argue over later! First: as contexts change – and they will – so should curricula. Ireland was near the head of the posse as regards adopting modern mathematics; we were bringing up the rearguard on problem solving; perhaps neither gives the best balance. As an Association, I believe we should be open to changes – prepared to go outside our comfort zone – but also watchful that the changes are feasible and fitting. Secondly, we may need to adjust how we work and even what

we do, though on this one I have fewer concrete ideas. The trajectory that so suited me, and others of my time, may not help the coming generation to embrace Association membership.

*A chairde*: that "young teacher" is no longer a young teacher. However, the friendships I have made through the IMTA remain. Yes, many past members have gone to their reward, but they are much in my thoughts, and I – and we – continue to benefit from their insights and wisdom as well as from those of present colleagues.

We bear this in mind as we think tonight about what we should celebrate most as an Association. I can SUM it up (or even INTEGRATE it?) by saying that we have been "***there***."

- We have been ***there*** for *our students*. We have been ***there*** to fight for them getting the results they have earned. We have fought, not just for their short-term success in examinations, but also for their long-term interests by working with other stakeholders to improve all aspects of curriculum: content, teaching, learning, assessment, resources.
- We have been ***there*** for *each other*. I have indicated how much we absorbed (not just tea, or later beer, but ideas) from each other in the early years, and how much we still do.
- We have been ***there*** to initiate *young teachers* into the profession – though we may need to improve our recent performance on this one.

Of course there are things we should do better or do differently (echoes of Her Majesty again), but that is life. We are still ***there***. And in fifty years' time, I believe that we – not the same set of people, but perhaps a non-null intersection; not maybe in quite the same kind of organization – shall still be ***there***, working for the good of mathematics education in Ireland.

On which note, I think we should raise a glass (or a cup) to that future!

[1]On the occasion of her visit to Ireland in 2011, Queen Elizabeth famously began her address at the State Dinner in Dublin Castle with the words "*A Uachtaráin agus a chairde*."

[2]At the time at which this lecture was delivered, vectors had recently been removed from the Leaving Certificate syllabus.

# A big "thank you" to Neil Hallinan

As indicated in the Editorial, Neil has expressed his intention of stepping down from the post of Editor after ten years at the helm. I am hoping that he will accord me – his assistant (occupying a very minor role) for the last few years – the space in which to try to thank him for the marvellous job he has done. Under his guidance, the *Newsletter* has reflected his passion for mathematics and his unending curiosity about the subject, his vibrant interest in education, his support for so many aspects of the Association's work, his prodigious capacity for work, and much else.

It is even appropriate that the final issue under his editorship is that celebrating our Jubilee year, for he contributed enormously to the organisation of the Jubilee Banquet, and especially to arranging the visit of President Michael D. Higgins and Mrs. Sabina Higgins and to displaying the fascinating archive of early IMTA material that he has done so much to assemble. Hopefully, his role as archivist will continue! We hope also that more of his delightful articles, combining mathematics and quirky humour, will grace future issues. It remains only to say "*Go raibh míle míle maith agat!"*

Elizabeth Oldham

# A Construction Activity

**Michael O'Loughlin**

The following activity can be used as a revision exercise coming up to the Leaving Certificate Higher Level examination. It combines geometrical constructions with some related algebraic activities and it demonstrates that working on a mathematical problem requires the use of more than one branch of mathematics. Its primary purpose, however, is that while working through the activity the student learns some mathematics.

As I no longer work in the classroom I have not had the opportunity to try it out so I would welcome some feedback from any teacher who tries it in class. This activity is best done using A3 or A4 card or stiff paper. If using A4, set the page in landscape orientation and position the starting point C at 6.5 cm from the left edge of the page and at 8.5 cm from the bottom edge so that the construction will just fit on the page.

**Initial Task**

Using a compass and ruler, construct the triangle *CAB*, making [*CA*] the base, given that |CA| = 8cm, |CB| = 6cm and |AB| = 10cm.

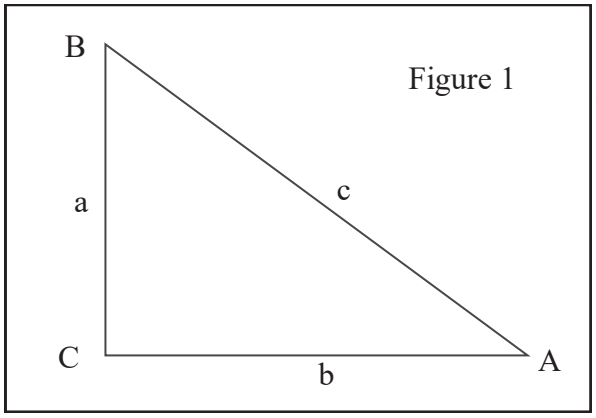


Figure 1

Note that the diagram here is not to scale and is just shown here as a guide to what your construction should look like [Figure 1].

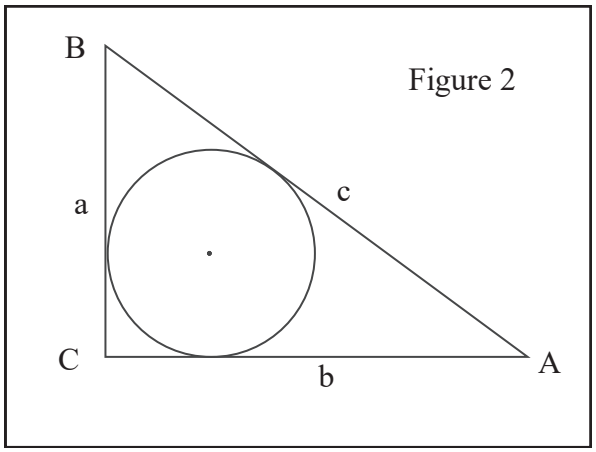


Figure 2

**Subsequent tasks**

1. Verify, using your set square or protractor, that $|\angle ACB| = 90^o$.
2. Verify the same property by calculation using the lengths of the three sides.
3. Construct, using a compass and straight edge, the incentre of ΔCAB [Figure 2].
4. Construct the incircle of ΔCAB [Figure 2].
5. Construct the square on side [CA] [Figure 3].
6. Construct the square on side [CB] [Figure 3].

7. Construct the square on side [BA] [Figure 3].

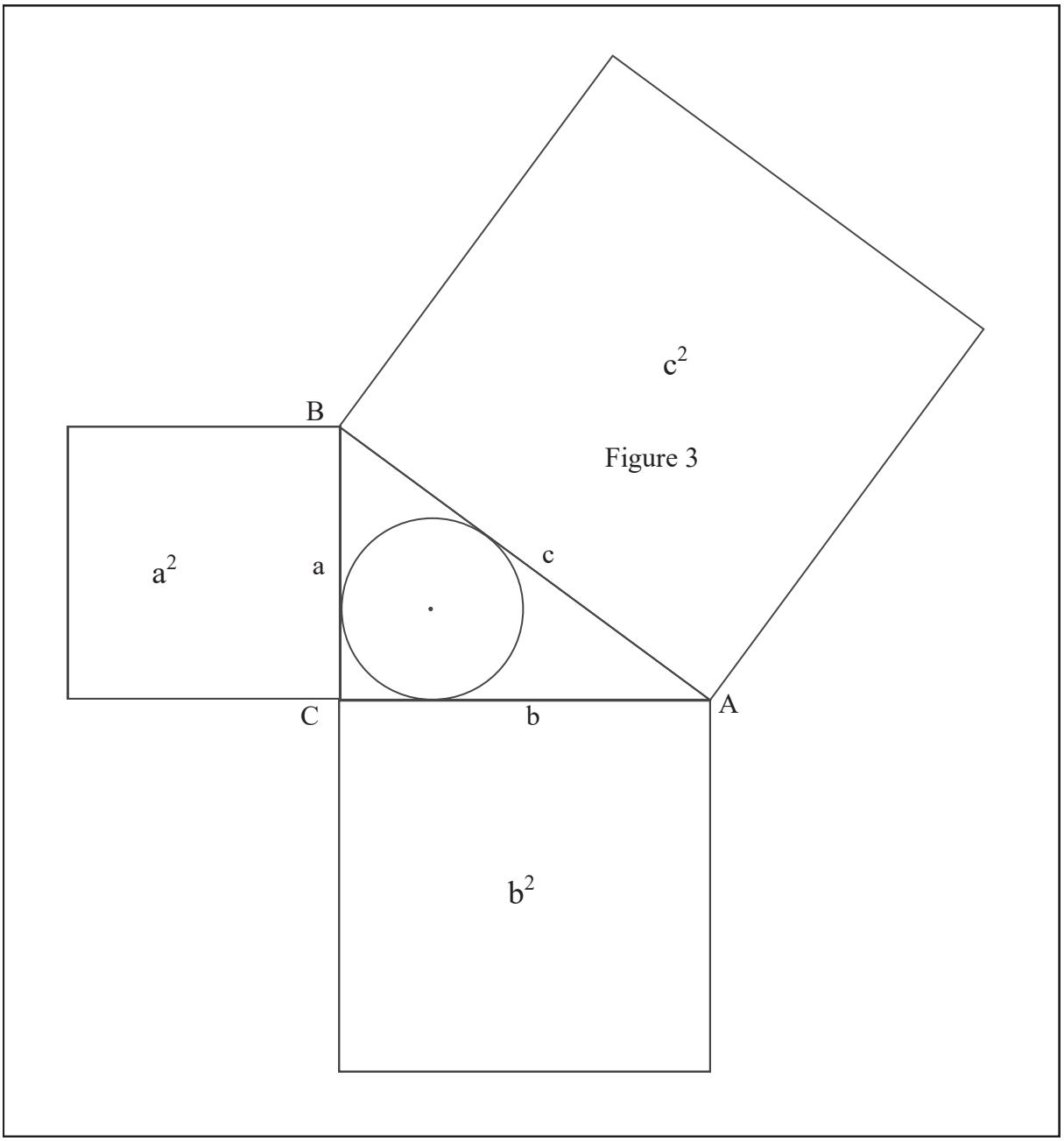



Figure 3

8. Carefully cut out the square on side [CA].
9. Do likewise with the square on side [CB].
10. Carefully cut out the incircle.
11. Rotate the page so that the remaining square has a horizontal base. (This is just to make it easier to work with.)
12. Now place one of the cut out squares (either $a^2$ or $b^2$) on the square remaining on the page ($c^2$) so that its bottom left vertex sits on the bottom left vertex of the larger square and so that their bases align [Figure 4].

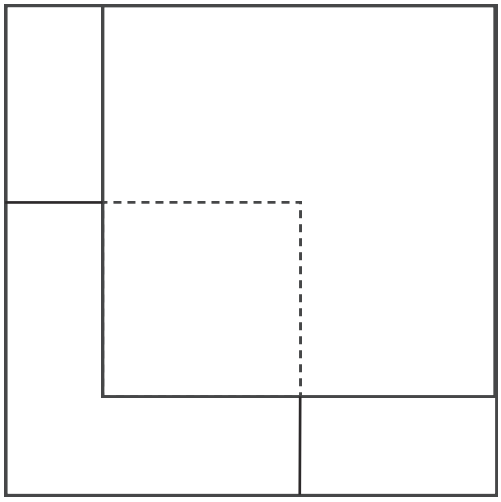


Figure 4

13. Place the other cut out square diagonally opposite so that its top right vertex aligns with the top right vertex of the larger square and their right edges align [Figure 4].
14. Notice that the two smaller squares overlap and that the overlapping part is also a square. Also notice that the two smaller squares leave two rectangular pieces of the larger square uncovered.
15. Carefully place the incircle that you cut out onto the square formed by the overlap.

What do you notice?

What can you say about the diameter of the incircle and the side of this square?

Check by measuring both.

16. Since the triangle is right-angled with $a^2 + b^2 = c^2$ and you have cut out both $a^2$ and $b^2$ and placed them on $c^2$ then the two uncovered rectangular pieces ought to have the same area as the new square formed by the overlapping of $a^2$ and $b^2$.
17. Measure the length and width of one of the uncovered rectangles.
18. Verify that the other rectangle has the same dimensions.
19. Calculate their combined areas.
20. Calculate the area of the square formed by the overlapping of $a^2$ and $b^2$.
21. Write down your findings.

**Generalising**

Instead of working with $a = 6$, $b = 8$ and $c = 10$, let's now work with any $a$, $b$ and $c$ which form a right-angled triangle, i.e. any $a$, $b$ and $c$ for which $a^2 + b^2 = c^2$ but without being concerned with the actual values of $a$, $b$ and $c$. This process is called generalisation and is a very powerful step up from working with specific values of $a$, $b$ and $c$. It allows us to work with any imaginable right-angled triangle of which there is an infinite number.

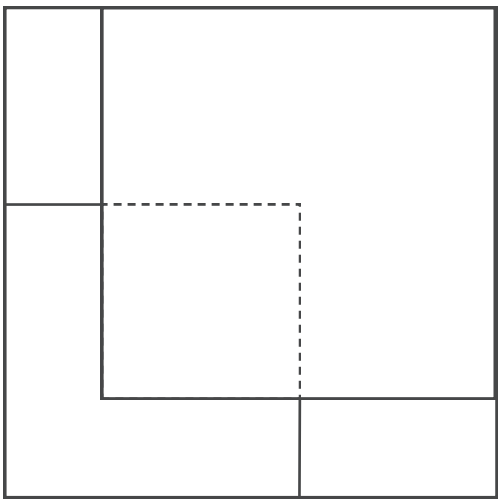


Figure 4

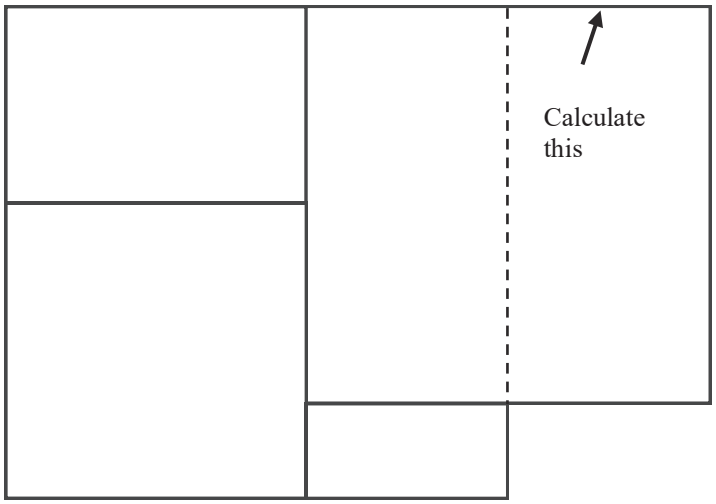



Figure 5

For convenience I am taking $a$ to be less than $b$.

1. Try to work out the length of the side of the square formed by the overlapping of $a^2$ and $b^2$ in terms of $a$, $b$ and $c$. You can calculate this directly from your construction or in a more active way by moving the upper square outwards until its left edge lines up with the right edge of the lower square [See Figure 4 changing to Figure 5]. In this position there is no overlap. Calculate the distance by how much the two squares combined extend past $c^2$. Now move the square back into its original position (as in Figure 4) and the overlap is re-formed. What is the length of the overlap?
2. In terms of two of $a$, $b$ and $c$ calculate the length of one of the uncovered rectangles (as in Figure 4).
3. Calculate the width of this rectangle in terms of two of $a$, $b$ and $c$.
4. Prove the identity: $(a + b – c)^2 = 2(c – a)(c – b)$ given that $a^2 + b^2 = c^2$.
5. In relation to your construction what is this identity saying? Use the word 'area' in your answer.

**Working with integer values**

1. Suppose now that the length of the radius of the incircle is $n$, where $n \in \mathbb{N}$.
2. Write down the length of the side of the square formed by the overlapping of $a^2$ and $b^2$ in terms of $n$ and mark it on your construction.
3. Write down the area of this square in terms of $n$.
4. Suppose that $i$ is the width of one of the uncovered rectangles and that $j$ is its length and mark them on your construction.
5. Write down the area of the rectangle in terms of $i$ and $j$.
6. Use the areas of the square and the two uncovered rectangles to find a relationship between $i$, $j$ and $n$.
7. Is it possible for $i$ and $j$ to be integers too?
8. Suppose $n = 1$ , what are possible integer values for $i$ and $j$?
9. Calculate $a$ in terms of $n$ and one of $i, j$.
10. Calculate $b$ in terms of $n$ and one of $i, j$.
11. Calculate $c$ in terms of $n$ and one of $i, j$.
12. If $n = 1$, calculate possible values for $a$, $b$ and $c$.
13. Try it with $n = 2$. How many sets of values did you get?
14. Now try with $n = 3$.
15. Does this statement appear to be true: *If the length of the radius of the incircle of a right-angled triangle is an integer then there is at least one triangle whose side lengths are integers too*?

**Review**

Write down what mathematical knowledge you have discovered from carrying out this activity.

Michael O'Loughlin Dublin Branch

# The Mathematics Of Snooker

**Neil Hallinan**

The red was on. Not only that, the red was pottable. I knew it. All I had to do was hit it at the correct angle with the white. Easy. Should be easy…

But it wasn't. The red bounced off the cushion tantalisingly close to the middle pocket. I refrained from comment but my puzzled look said it all.

"That was a half-ball and you used near a quarter", the voice of my coach knowledgeably informed me.

I had been playing some stunning shots up to now but this was new.
"What do you mean, Geoff?" I asked.
"Well, I have shown you how to play the stun shot already. That is when the ball is full on. Looking down the line of the cue-stick the cue-ball or white ball appears to cover the object ball – the ball you want to pot – completely.
"Just like a full eclipse of the sun", I chimed in. "That's what you told me when I hit the white ball dead centre with the cue and it stops dead when it hits the red."
"Yes", he agreed. "And you have been hitting them stunningly, if you forgive the pun. Look, there is a picture on the wall-chart of what I am talking about. It should mean something to you now."

I looked at the picture which showed an imagined overhead view of a full ball situation like the following diagram (Figure 1):

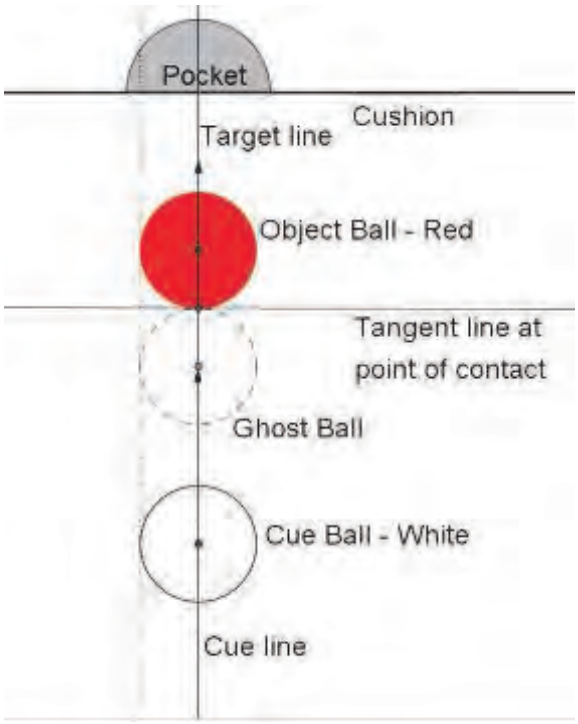



Figure 1. A Full Ball Situation

Geoff explained that the red ball would move along the line drawn through the centres of the object ball (red) and the ghost ball (imagined) through the point of contact. This line determines

the target line. The object of hitting the cue-ball (white) is to enable it to reach the position of the ghost ball. In the case of the full ball the cue-line is directly along the target line. i.e., there is a zero degree deviation of the cue-line from the target line. Looking along the cue-line from directly behind the cue-ball none of the object ball may be seen. It is fully eclipsed. Hence the reference name of 'full ball'.

"Now", said Geoff, "in the next situation I have set up a half-ball for you as I mentioned earlier. This means that half of the object ball appears to be covered when you look from directly behind the cue ball. By hitting the cue ball along this line the object ball will move off at an angle of 30° to the line of the cue."
"Always?" I queried.
"Always!" he emphasised. "Don't ask me why but it always has and always will."

I reflected for a moment as the numbers and shapes began to link up – curve on ball – angle 30° – ½ … Could it be that some trigonometry was at play? I decided to test my theory before my next lesson.

## Doing Some Mathematics

I set up the following diagram in order to analyse the situation where less than half the object ball was covered by the cue ball (Figure 2).

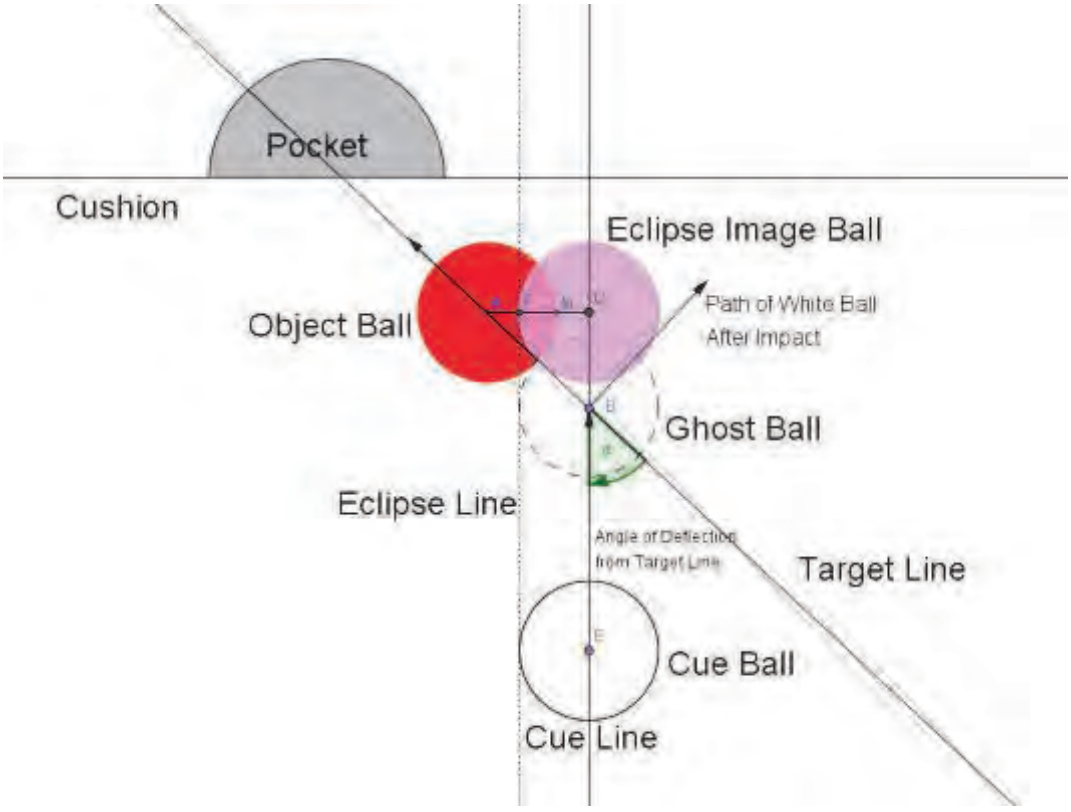



Figure 2. Diagram Showing Less than Half the Object Ball Covered

Let the diameter of a ball be 1 unit so the radius is ½. Let the angle of deflection of the Cue line from the Target line be α (also called the angle of cut).

Consider the triangle ΔBAD where B is the centre of the Ghost ball, A is the centre of the Object ball and D is the centre of the Eclipse Image ball. There is a 90° angle at D since the Eclipse Image Ball is placed so that its line of centre with the Object Ball is at right angles to the Cue Line. The following diagram (Figure 3) is an expanded view of the relevant triangle.

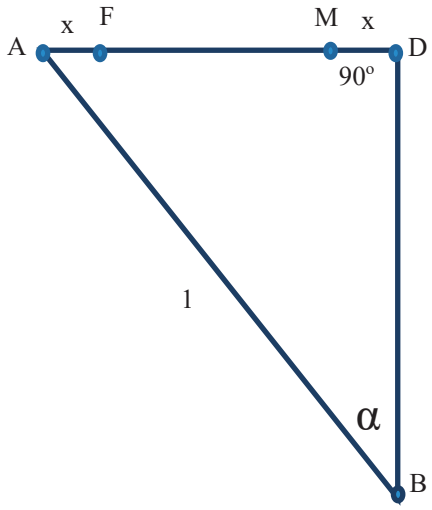



Figure 3. Diagram Showing the Connection of the Lines of Centres

|AB| = 1 since it is the segment joining the centres of the Object Ball and the Ghost Ball which are in contact.
Let |MD| = x. Then |AM| = |DF| = ½ since both lengths are radii.
Hence |AF| = x since |AD| = |AM| + |MD| = ½ + x = |DF| +|FA|.
The angle ∠DBA = α, a vertically opposite angle to the angle of deflection of the Cue Line from the Target Line (the cut angle).

Then $\text{Sin } \alpha = \frac{|AD|}{|AB|} = \frac{|AD|}{1} = |AD|$.

∴ Sin α = |AM| + |MD| = ½ + x.
⇒ x = Sin α - ½

Also |FM| = |FD| - |MD| = ½ - x
∴ |FM| = ½ - (Sin α - ½) = ½ - Sin α + ½
⇒ |FM| = 1 – Sin α
⇒ Sin α = 1 - |FM|

Since |FM| is the fraction of the Object ball which is covered then 1 - |FM| is the fraction of the Object ball which is **not** covered.

Thus we have
Sin α = Fraction of the Object Ball Not Covered
i.e., Sin (angle of deflection) = Fraction of the Object Ball Not Covered
and
The Angle of Deflection (α), i.e., the cut angle = $\text{Sin}^{-1}$(Fraction of the Object Ball Not Covered).

Similarly, we can look at the situation where the amount of the object ball which is covered is greater than half as in the following diagram (Figure 4).

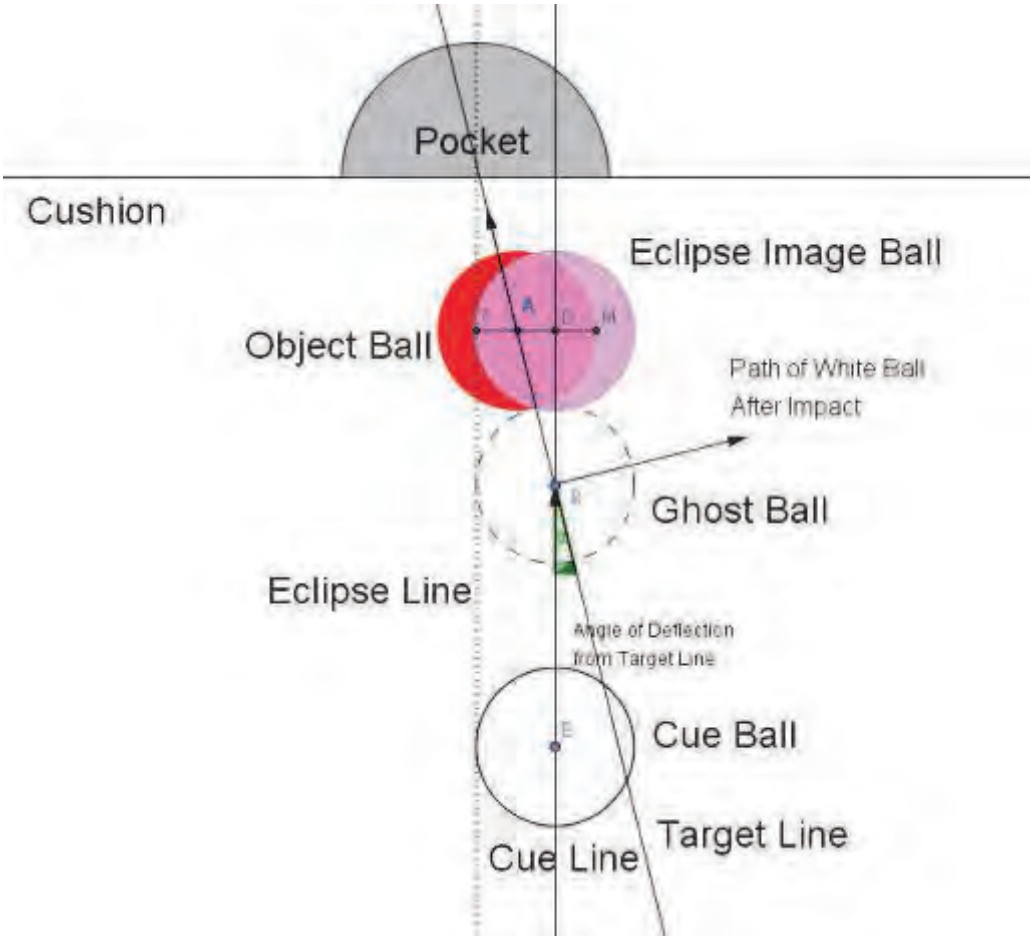



Figure 4. Diagram Showing More than Half the Object Ball Covered

We can now look at the triangle ΔBAD (Figure 5) and make the following observations:

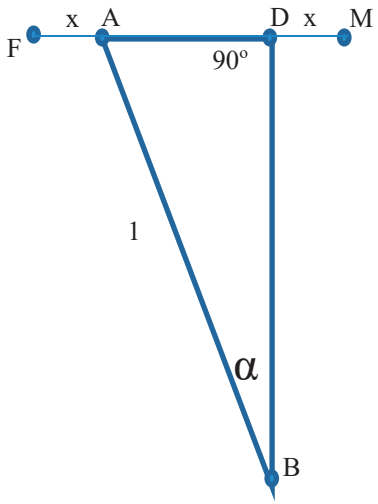



Figure 5. Diagram Showing the Connection of the Lines of Centres

$|\angle ADB| = 90^o$; $|AM| = |DF| = ½$; $|AB| = 1$; $\angle ABD = \alpha$, the angle of deflection.

Let $|DM| = x$. Then $|FA| = x$ since $|AD| = |AM| - |DM| = |DF| - |AF|$.

$$\text{Sin } \alpha = \frac{|AD|}{|AB|} = \frac{|AD|}{1} = |AD| = |AM| - |DM| = ½ - x$$

$\therefore x = ½ - \text{Sin } \alpha$
Also $|FM| = ½ + x = ½ + ½ - \text{Sin } \alpha$
$\therefore |FM| = 1 - \text{Sin } \alpha$ or $\text{Sin } \alpha = 1 - |FM|$

Sin (Angle of Deflection) = 1 – Fraction of Object Ball covered
Which is the same as
Sin α = Fraction of the Object Ball Not Covered.
This is the same result as in the case when less than half the object ball is covered.

## The Half Ball

In the case when half the object ball is covered the value of x in the diagram becomes zero and Sin α = ½.
Again, this is the same as the result Sin α = Fraction of the Object Ball Not Covered.

Thus, in the Half Ball situation where half the object ball is covered the angle of deflection will be $\text{Sin}^{-1}(½) = 30^o$.

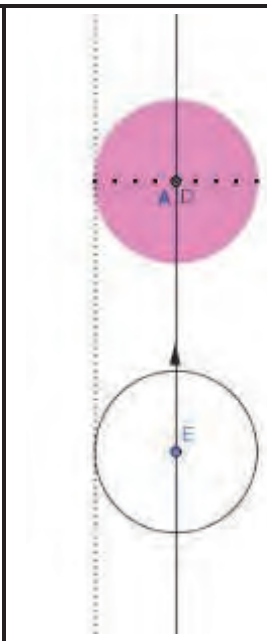
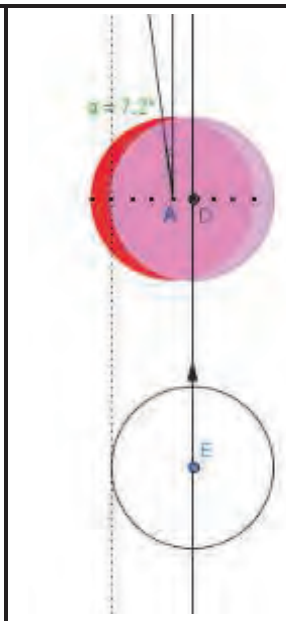
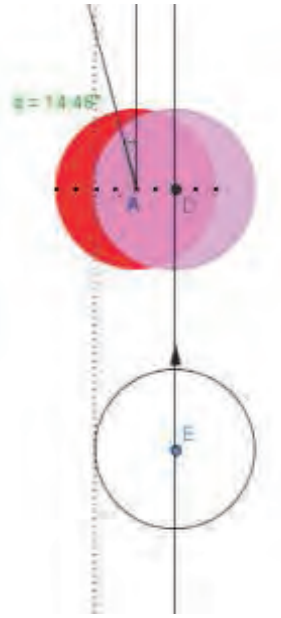
Or, vice versa, if the cut angle is $30^o$ then a Half Ball shot should be used to ensure that the object ball is potted.
The mystery had been solved.

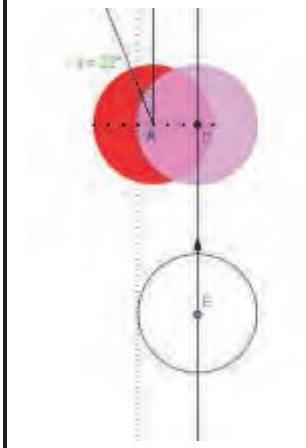
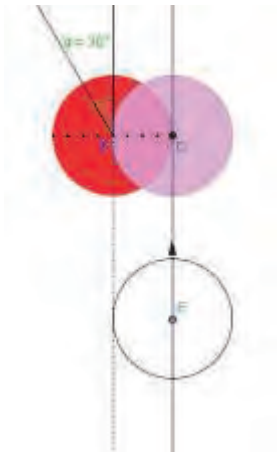
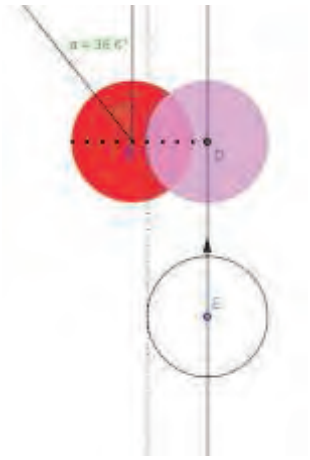
## Further Investigations

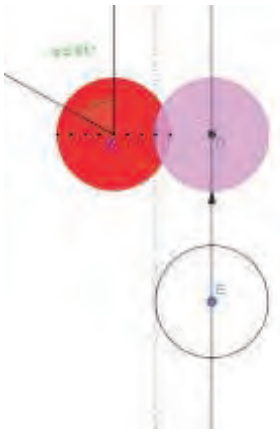
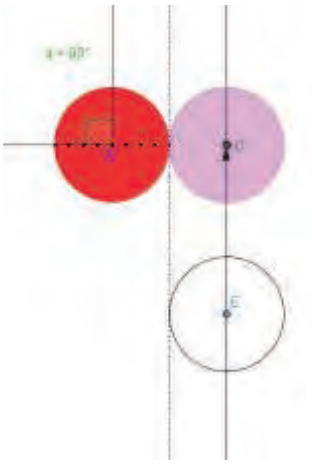
Of course, I began to wonder if there were such things as a Quarter Ball or Three-quarter Ball.
"Oh, yes!" said Geoff, when I mentioned it to him at the next lesson.
"But the angles aren't specified. You begin to know them by experience. A thick Quarter Ball will get you about $45^o$", he said.
To Geoff's amusement I had brought my calculator.
"I can check that for you", I said.
"A Quarter Ball means that three-quarters of the ball is not covered. So, finding Sine Inverse of three-quarters should give us the precise angle, which is exactly $48.59^o$", I told him presently.
"So by hitting it a little 'thicker' I think you mean a slight bit closer towards the Full Ball which will reduce the angle and bring it towards $45^o$."
"That's pretty impressive", said Geoff. "Can you tell me what a Three-quarter Ball angle will be?"
"Easily", I replied. "It's Sine Inverse of a quarter, so it is $14.48^o$. That's almost half of the Half Ball angle. So a very slightly thinner Three-quarter Ball will give $15^o$, presuming that 'thinner' is the opposite of 'thicker'."
"That's it!" exclaimed Geoff. "I can see your calculator has been put to good use. I would be interested to know what you would figure out for the Eighths."
"Me, too!" I exclaimed. "I will make out a chart for you that you can put on the wall."

By the time the next lesson had come around I had drawn up some charts and a graph (Charts 1, 2, 3; Figure 6) illustrating the range of shots from Full Ball to a Miss in steps of One Eighth.

## Charts to Illustrate the Deflection Angle of Snooker Shot

| Illustration of Snooker Shots: Chart 1 | | | |
|---|---|---|---|
| Red Ball: Object Ball<br><br>White Ball: Cue Ball<br><br>Shaded Ball: Image of Cue Ball Imagined to Cover the Object Ball when viewing along the Cue Line from behind the Cue ball |  |  |  |
| Amount of Object Ball Covered | Full Ball | Seven-eighths | Three-quarter Ball |
| Not Covered | Zero | One eighth | One quarter |
| Deflection angle from Target Line (Cut Angle) | 0° | 7.2° | 14.48° |
| Aiming point on Object Ball | Centre of Object Ball | One eighth from centre | Half-way from centre to edge<br>or<br>a quarter from the edge |

| Illustration of Snooker Shots: Chart 2 | | | |
|---|---|---|---|
| Red Ball: Object Ball<br><br>White Ball: Cue Ball<br><br>Shaded Ball: Image of Cue Ball Imagined to Cover the Object Ball when viewing along the Cue Line from behind the Cue ball |  |  |  |
| Amount of Object Ball Covered | Five-eighths | Half Ball | Three-eighths |
| Not Covered | Three-eighths | Half | Five-eighths |
| Deflection angle from Target Line (Cut Angle) | 22° | 30° | 38.6° |
| Aiming point on Object Ball | One eighth inside the edge | On the edge | One eighth outside the edge |

| Illustration of Snooker Shots: Chart 3 | | | |
|---|---|---|---|
| Red Ball: Object Ball<br><br>White Ball: Cue Ball<br><br>Shaded Ball: Image of Cue Ball Imagined to Cover the Object Ball when viewing along the Cue Line from behind the Cue ball | |  |  |
| Amount of Object Ball Covered | Quarter Ball | One-eighth | Zero Ball - Miss<br>Hint: Use a Very Fine Cut |
| Not Covered | Three-quarters | Seven-eighths | Almost a full ball in the case of a Very Fine Cut |
| Deflection angle from Target Line (Cut Angle) | 48.6° | 61° | Slightly less than 90° to ensure contact for the Very Fine Cut |
| Aiming point on Object Ball | Quarter ball outside edge | Three-eighths outside edge | Almost half-ball outside edge |

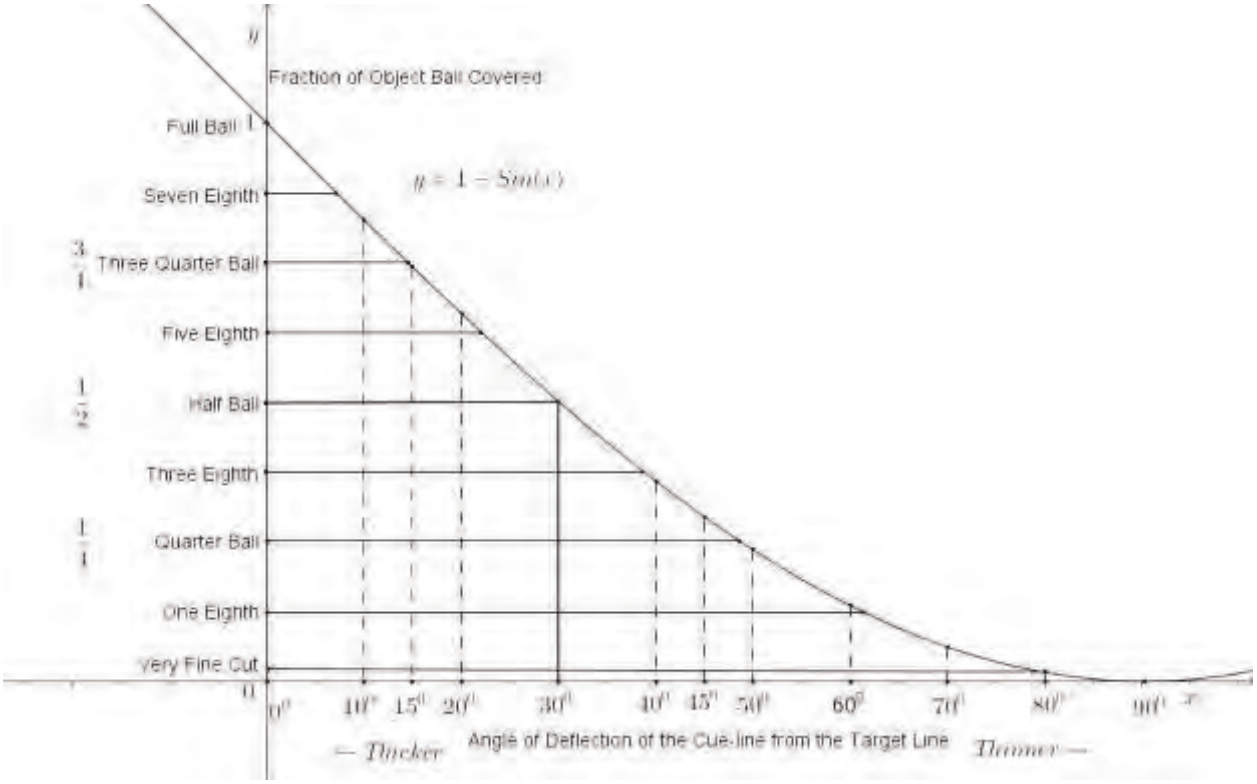



Figure 6. Graph Showing the Connection between the Cut Angle and the Fraction of the Object Ball Covered

Geoff thought that the charts should be very useful to me. However, he had a little warning to give me.

"These charts will give you a good insight for the first few lessons although they do not allow for the small 'throw' angle", he said. "But in the next lesson I will want you to use something invented by a prisoner in France around 1807."

Quickly consulting my convenient iPad I discovered that Geoff was referring to François Mingaud (1771-1847). He had been a political prisoner for some years but was released in 1807. To his credit he had invented the leather tip for the cue while perfecting his billiards skills in prison.

"Up to now you have been hitting the cue-ball in the centre", said Geoff. "When you begin to use the tip of the cue properly you will find how to control the path of the cue-ball following the impact with the object ball. We can investigate the niceties of spin and screw so that the cue-ball will not follow the 90° deflection from the target line as illustrated earlier by you in the path leading away from the ghost ball."

"I'm looking forward to that", I enthused.

"Of course there is nothing to beat experience", concluded Geoff. "The nap of the table, the force of the shot, the elasticity and relative weights of the balls, … I could go on. All these conspire to test the best analysis of mathematicians and physicists. But if you can combine the pleasures of mathematics with the pleasure of learning snooker I will not dissuade you."

"Thank you, Geoff", I replied. "This lesson has been a real pleasure. I wonder now, could we use mathematics to figure out how to play a ball which is snookered?"
"That, my friend, is a lesson for another day", said Geoff.

Neil Hallinan Dublin Branch

P.S. Since snooker is complex, Geoff is not my coach's real name.

# GEOMETRY IN THE TRANSITION FROM PRIMARY TO POST-PRIMARY

PATRICK D. BARRY AND ANTHONY G. O'FARRELL

## 1. Introduction

This article is intended as a kind of precursor to the document *Geometry for Post-primary School Mathematics*, which forms Section B, pp. 37–84 of the Mathematics Syllabus for Junior Certicate issued by the National Council for Curriculum and Assessment [16] in the context of Project Maths.

Our purpose is to place that document in the context of an overview of plane geometry, touching on several important pedagogical and historical aspects, in the hope that this will prove useful for teachers.

The main points we want to emphasize are these:

- Geometry is a key part of mathematics.
- Children must pass through different stages in studying geometry.
- Each stage plays an important rôle.
- Care must be taken in managing the transitions.
- Some knowledge of history is useful for teachers.

## 2. The Main Parts of Mathematics

At present, the NCCA presents the mathematics curriculum in terms of *strands*. For primary level the five strands are labelled Number, Algebra, Shape and Space, Measures, and Data. For secondary level they are (1) data, statistics and probability, (2) geometry and trigonometry, (3) number and measure, (4) algebra and (5) functions. The similarity between the two classifications is part of an attempt to foster continuity across the transition. A further initiative designed to foster this was the publication of a *bridging framework* [15] which provides a dictionary linking terminology used in primary to that used in secondary schools. Initially, the divisions were perhaps intended as much to reflect a more-or-less equal division of teaching and learning effort as much as a division of mathematics into its main areas. It was never intended that the strands would represent watertight divisions of the

2010 *Mathematics Subject Classification.* 97B70.

subject, and was generally recognised that there is necessary interaction between them.

An analysis of the secondary curriculum reveals that the division becomes progressively more forced at the more advanced levels. It would also be completely impossible to impose it on third-level studies in mathematics. It is standard among mainstream professional researchers to say that there are three main branches in the tree of mathematics: Algebra, Analysis and Geometry. Some would add Probability and Computation to these, but many others would regard Probability and Computation as two branches of Analysis. However, is is also standard that the connections between the main fields of mathematics are so many that it is actually possible to say that any one of these embraces the whole. There are fields such as Algebraic Geometry, Algebraic Topology and Geometric Analysis which may be regarded as branches of either one of two main branches, and which use fundamental results from both.

The point we make here is that from a bird's-eye viewpoint, Geometry is about one third of mathematics. At the research level, it accounts for a solid proportion of new PhD theses, as may be seen by examining the tables in reports of the AMS annual surveys of new graduates [1]. Its applications are in active areas of fundamental and applied physics, robotics, coding, graphics and other commercially significant areas. So it is important. Our students deserve a sound formation in geometry.

For various reasons, in the past decades many students have emerged from secondary school with a poor opinion of geometry, the result of unfortunate experience with the subject. PISA assessment results also showed relatively mediocre performance by the Irish 15-year-old cohort on problems requiring geometrical skill. This needs to change, and indeed change is mandated by the Project Maths curriculum.

## 3. Stages

3.1. **Primary stages.** The present Primary Curriculum [19] specifies the study of geometrical shapes in two and three dimensions under the heading of Shape and Space, and of length and area under Measure (with support from Number). This begins right at the start, and is developed further year by year. Students are introduced to simple shapes (triangle, rectangle, circle, semicircle, cube, cuboid, sphere, cylinder, cone) and progressively more complex shapes and properties (isosceles triangle, parallelogram, rhombus, pentagon, hexagon, triangular prism, pyramid, scalene triangle, trapezium, regular hexagon) learn to distinguish them and learn names for them and for their parts

and properties. They make use of suitable materials (blocks, paper and scissors, folded paper, art straws, geoboard, mazes, grids, board games, software, plasticine, prisms, compass, string, tangrams, squared paper) and diagrams and learn to recognize shapes in their environment. Most of the work involves flat, planar shapes, but they also manipulate 3-D shapes and solve problems about them. They learn about measuring lengths, areas, volumes and angles (using a progressively richer number system). They learn how the describe and evaluate spatial relations, give directions, construct and draw 2-D shapes using instruments, subdivide and combine shapes, draw tesselations, construct 3-D polyhedra (by folding nets), and use coordinates. They are encouraged to look for common patterns such as lines of symmetry and the result of counting faces minus edges plus vertices for polyhedral shapes. They learn about parallel lines, and right, acute, obtuse and reflex angles. They explore properties of 2-D shapes, including the angle-sum of a triangle and a quadrilateral, and the ratio of diameter to radius of a circle. The syllabus specifies the linkage of geometry to other areas of the curriculum (motor skills, science, art, physical education and dance, geography) and to aspects of everyday life. This is all very useful, and is appropriate for their ages.

At the end of primary school, children should have acquired most basic geometrical concepts and the language that goes with them. They should be in a position to use their understanding to solve many practical problems.

3.2. **Secondary stages.** When they start post-primary school, students should not be allowed to abandon all this geometrical experience, but should continue to draw on it, solidify and develop their understanding of it, and stay in touch with geometrical ideas on a continual basis.

They have a lot more to learn about geometry. There are in fact two further components needed beyond primary level, corresponding to the two main reasons that further geometrical study is needed: the practical utility of more advanced material and skills, and the role of geometry in developing and honing the student's reasoning power.

The case for exposure to rigorous mathematical thought as a preparation for life and for any further studies was well made by John Stuart Mill (quoted in [20]):

> *The value of Mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, &c.) consists in the applicability not of*

> *its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general ...*
>
> *These grounds are quite sufficient for deeming mathematical training an indispensible basis of real scientific education, and regarding, with Plato, one who is ἀγεωμέτρητος*[1] *as wanting in one of the most essential qualifications for the successful cultivation of the highest branches of philosophy.*

Geometry is not the only branch of Mathematics that may serve to develop reasoning power, but it has long been used for that purpose, and many consider it well-suited. The geometrical theory expounded in the Elements of Euclid (cf. [8]), rediscovered in the West at the end of the Middle Ages and adopted as the preferred text by the first European universities has been the most popular. It is important for teachers to understand some key points about it:

- It is an abstract theory about space (without matter).
- It was not written to be studied by children.
- It has some logical flaws.
- Mathematicians have figured out various ways to fix these flaws so that the main propositions can be proven from a set of axioms.
- Each such amended theory is called Euclidean Geometry.
- Each such theory is even less suited for children.
- Euclidean geometry is very useful.
- There are other geometrical theories, in which some of the propostions of Euclidean Geometry are false.
- We do not actually know which of these is the best approximation to "real empty space"[2].
- We do know that actual space, with matter, does not fit well with Euclidean Geometry[3], although there is a close correspondence at everyday length scales.
- Abstract geometry has to be simplified, if it is to be used in school to develop reasoning power.

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[1]– *ageometretos* , i.e. ignorant of geometry (or, perhaps, unskilled in geometry, or indifferent to geometry). The motto said to have been carved above the entrance to Plato's Academy was: Οὐδεις ἀγεωμέτρητος εἰσίτω – *Let no-one ignorant of geometry enter.*

[2]– another abstraction.

[3]when lines are interpreted as light rays.

- Even when simplified, it is not feasible just to fling children into the abstractions without a careful preparatory stage.

We shall elaborate on some of these points, and comment on the pedagogical implications.

We start with some history.

## 4. Historical development of geometrical theory

4.1. **The arc of history.** Euclidean Geometry has had a long history. Following on practical studies of shapes, lengths, areas and volumes in the Sumerian and Egyptian civilizations *inter alia*, it started to evolve into a logically-organised science as a result of the efforts of philosophers in Greece c.700-600 BC, who wanted to base knowledge on solid foundations. The basic idea was to identify and define purely geometrical (i.e. non-material) abstractions (point, line, etc.) and also identify uncontroversial starting principles about them, and then to use logic to work out the consequences. Ideally, the building blocks of the theory should be as simple as possible. This simple idea proved extremely effective in practical applications (such as tunnel construction), and gave encouragement. Understanding of geometrical theory evolved gradually ever since, although there were many fallow centuries. Euclid's synthesis of the geometry of his day (about 300 BC) was a major landmark, but after his time many further theorems unknown to him have been discovered, and our understanding of the basic plan of his work has also evolved. René Descartes, in his Discourse on the Method (1637 AD) showed how to link numbers to geometry – in Euclid's books, and up to that point geometrical magnitudes and numerical magnitudes had been considered different species. This created the field of algebraic geometry, and in a sense reduced geometry to arithmetic. However it may come as a surprise that the system of real numbers has been fully understood only since about 1860 AD (thanks to Richard Dedekind). Non-euclidean geometries were discovered early in the 19th century, rubbishing Kant's view that our knowledge of Euclidean geometry is "synthetic a priori", and raising the question whether the real world is Euclidean or not.

4.2. **Deductive reasoning.** A proper understanding of logical deductive systems was only arrived at in the late 1800's, and this prompted Hilbert to produce the first fully-rigorous account of Euclidean geometry, i.e. an account in which all the theorems of Euclid can be proved rigorously from first principles. What is now understood as a mathematical theory, or deductive system, has five components [10]:

(1) Undefined terms
(2) Definitions
(3) Axioms
(4) A system of logic (rules for valid deductions)
(5) Theorems (a term that embraces also propositions, lemmas, and corollaries).

In other words, in a logical system we list up-front the terms and assumptions that we start with, and thereafter proceed by way of definitions and proofs.

4.3. **Definitions.** Definitions are about specifying what we are dealing with. A *definition* identifies a new concept in terms of accepted or known concepts. In practice a definition of a word, symbol or phrase $E$ is a statement that $E$ is to be used as a substitute for $F$, the latter being a phrase consisting of words and possibly symbols or a compound symbol. We accept ordinary words of the English language in definitions and what is at issue is the meaning of technical mathematical words or phrases. In attempting a definition, there is no progress if the technical words or symbols in $F$ are not all understood at the time of the definition.

The disconcerting feature of this situation is that in any one presentation of a topic there must be a first definition and of its nature that must be in terms of accepted concepts. Thus we must have terms which are accepted without definition, that is there must be *undefined* or *primitive* terms. This might seem to leave us in a hopeless position but it does not, as we are able to, and must, assume properties of the primitive terms and work with those.

There is nothing absolute about this process, as a term which is taken as primitive in one presentation of a topic can very well be a defined term in another presentation of that topic, and vice versa. We need *some* primitive terms to get an approach under way.

4.4. **Proof.** Proof is the way to establish the properties of the concepts that we are dealing with. A *proof* is a finite sequence of statements the first of which is called the *hypothesis*, and the last of which is called the *conclusion*. In this sequence, each statement after the hypothesis must follow logically from one or more statements that have been previously accepted. Logically there would be a vicious circle if the conclusion were used to help establish any statement in the proof.

There is also a disconcerting feature of this, as in any presentation of a topic there must be a first proof. That first proof must be based on some statements which are not proved (at least the hypothesis),

which are in fact properties that are accepted without proof. Thus any presentation of a topic must contain unproved statements; these are called *axioms* or *postulates* and these names are used interchangeably.

Again there is nothing absolute about this, as properties which are taken as axiomatic in one presentation of a topic may be proved in another presentation, and vice versa. But we must have *some* axioms to get an approach under way.

4.5. **Hilbert's system.** In Hilbert's system [11] there are undefined terms such as *point, line, plane, between, congruent*, seven axioms of connection, five axioms of order, an axiom of parallels, six axioms of congruence, and an axiom of continuity, and definitions of terms such as *segment, vertex, side of a line*. The logic used is standard Aristotelian logic.

Notice that this leaves aside completely the question of any relation between this theory and the real world. There are equally satisfactory and equally-consistent[4] theories of various geometries in which some of Euclid's theorems are false.

Incidentally, the main aspects of Euclid's work that needed to be "cleaned up" were (1) the attempt to prove the SAS congruence criterion, Prop. I:4, using superposition, instead of just assuming it; (2) the absence of any postulates about line separation or plane separation, and "betweenness", needed for instance in Prop. I:16; and (3) the absence of any continuity or completeness assumption, already an issue in Prop I:1. The notion gained currency in the 1960's that "Euclid is all wrong" and should just be dumped. The truth is that with a little careful tweaking early on, almost everything stands and the proofs can still be used.

4.6. **Alternative Versions of Euclidean Geometry.** Over the period from c.500 BC to the present quite a few different approaches to Euclid's theorems have been published. The superabundance of these is one of the major problems that we face now. Hilbert's was the first that was logically watertight and categorical[5]. Before his time, and since, many variants were invented by teachers who wanted to make Euclid more accessible to children. After him, other professional research mathematicians produced complete versions involving different

[4]The consistency of Euclidean geometry cannot be proven. It can be shown that it is consistent if elementary arithmetic is consistent.

[5] in the sense that any two interpretations (models) of it are essentially the same

undefined terms, definitions and axioms, but of course the same theorems. They were motivated by the desire to have an equivalent system with simpler axioms. For instance, Hilbert's system does not include or use the real number system, and Birkhoff [3] proposed a system that extended the theory of the real numbers by adding only four axioms and gave all of Euclidean geometry.

## 5. Geometry for secondary school

**5.1.** It should be stated honestly, and faced now that a fully rigorous account of plane Euclidean geometry is too difficult for secondary school. This has been widely understood in academic circles for a very long time. In Mathematics Education circles, it was made explicit in the work of the van Hieles (cf. [5]) in the nineteen-fifties, when they identified five levels at which a person might understand geometry, ranging from level zero ("visualization") up to level four ("rigour"). The top level is only appropriate for university-level work. This does not mean that logical work in geometry is not feasible in school – the van Hiele levels appropriate for school were labelled "analysis", "informal deduction" and "formal deduction". Moreover, competence at the top level is not really needed for working with the manifold applications of geometry[6].

A key point is that you cannot train someone in logical deductive thinking by using an illogical system. So what professional mathematicians urge and press for is that school geometry should be in the the context of some fully-rigorous scholarly background approach. The school version should broadly have the same sequence of topics and the same type of proofs, but leave out some very difficult parts, the latter to be guaranteed by those at a higher level who choose to immerse themselves in a study of this material.

The present situation in Ireland is that the prescribed school geometry has for its scholarly background approach the one laid out in [2]. That system, like Birkhoff's, uses the real number system, but employs a few more axioms, including Playfair's version of the parallel axiom [2, Axiom A7, p.57]. The school system is deliberately simplified, as explained in [16, pp.40-43]. As a result, the proofs are not fully watertight, relying in places on unstated "commonsense notions", and a teacher or student who notices this is encouraged to refer to [2] to satisfy themselves that the gaps can be bridged.

[6]The van Hiele model has had a lot of influence. Early on, it formed the basis for a radical reform in the geometry curriculum in the Soviet Union in the nineteen-sixties, and it has gradually been taken on board in the USA.

An important aspect of Birkhoff-like systems is that one can treat the real number system informally (instead of formally, axiomatically) in school. This avoids explicitly mentioning the topics of continuity and completeness, which are too sophisticated for school. An important reason for basing the system on the book [2] is that there the complete scholarly treatment is fully laid out, *with complete detailed proofs.* A number of good alternatives are backed by complete theories for which the full proofs can be generated easily by any competent professional mathematical researcher, but are in fact only sketched in published sources.

**5.2. Approaching abstraction when teaching geometry.** Euclidean geometry employs abstractions. Right at the start, we have point and line, for instance.

Students have to be prepared carefully for this abstraction. The geometrical concepts must be motivated from the real world around us. Education in geometry (as in everything else) must proceed in stages, as the child's mind develops. These stages have long been recognised, and were explicitly catered for in popular textbooks such as Durell [7]. (Clement Durell's texts were in widespread use wherever British influence acted from 1919 for over forty years.) There has to be a preliminary stage before the stage of formal logical work with the abstract ideas and it is essential that these stages not be confused with each other. The preliminary stage should not be rushed, and time allowed for the abstract concept to sink in. It is not appropriate to plunge into "Theorem 1" immediately after explaining about points.

Later, when abstract results are applied, we should make it clear that we are now assuming they apply to reality.

A point is not a real thing. It has no size. Durell [7] says that teachers should *never* allow points to be drawn as blobs, and instead indicated by a cross made with two very fine lines. He insists that compass punctures should be as small as possible, and straight lines be as fine as possible. This is extreme, but you see his point! There may well be students who think points are little black round things, as drawn by Geogebra, and it is a good idea to make sure that they are disabused of this before they get started on formal work.

Diagrams are vital in teaching geometry, and should always be used. It is precisely because such visual aids are there to support and guide reason that geometry is considered the best way to practise logic.

There should be considerable physical motivation to start with, and diagrams always used to provide insight, but details of motivation

should not be confused with the careful logical presentation of the mathematical model that follows later on.

Every opportunity should be taken to get students to engage with problems that they can tackle using their current understanding of geometry.

As with any mathematics teaching, one proceeds in a cycle [7, 5]: oral discussion of examples, exercises in numerical and non-numerical examples, informal proof ideas, formal proofs, exercises involving "riders", or "cuts" (extra propositions to be proven by the student – the "Propositions" given without proof in the syllabus document are intended to be used in this way, and it is expected that the assessment process will examine skill in creating such proofs), and one provides exercises graduated by difficulty, extra exercises of one kind for students who struggle, and challenging extra exercises for those pupils "who run ahead of the class". Regarding the latter, although the main focus of the school programme is on plane geometry, one should look out for applications to solid geometry.

## 6. Lines and Non-Euclidean Geometry

The modern mathematical concept of *line* is infinite, without ends, and is straight. The English word line is derived from the Latin *linea*, which originally referred to flaxen thread, as is the name of the material *linen*, also made from flax. Similarly, in Irish we have the pair of words *líne* and *línéadach*. In Greek, the word for line was γραμμή (gramme), the stroke of a pen, derived from γραφω (grapho), I write, or draw. In contrast to modern usage the Greeks spoke of a *straight line* (literally εὐθεῖα γραμμή , "right line") and *curved line*. Moreover, by straight lines the Greeks mainly meant what we call *line-segments* which would be *produced* (i.e. extended) as required. It is helpful to bear this in mind when reading older texts.

One motivation of a line-segment was a *linen thread held taut*[7]. The notion of being straight was extended to lines, as a segment was unendingly produced, and at each stage there had to be a segment which contained the starting segment. Of course a taut thread could be copied onto a wax, papyrus or wooden tablet, and tablets with straight edges could be cut from the latter. The use of compasses then enabled them to cut out shapes of triangles and various types of quadrilaterals, as well of course as circles. Nowadays we have rulers, protractors, set

[7]Heath[8] records the classical phrase ἐπ ἄκρον τεταμένη γραμμή – a line stretched to the utmost.

squares and computer software packages to help us draw these figures and make constructions.

Euclid defined a straight line as "a line that lies evenly with the points on itself". This suggests that the concept is an abstraction of the idea of a "line of sight", or if you prefer, of a light ray, although Euclid in the Elements is careful not to refer to any physical thing, perhaps because as a good Platonist he is treating geometry as the form of space, a purer thing than space (cf. [8, pp. 165-169]).

If straight lines are the paths of light rays from an object to our eye, then we now know, from observational Astronomy, that real space is not Euclidean: there are distant objects that can be seen in two different directions ( – usually explained in terms of the theory of General Relativity as the result of "gravitational lensing"). So the most that might be true is that real space is approximately Euclidean at some length-scales and where the density of matter is low. Gauss realized that space might be non-Euclidean before most people, and checked measurements taken of the angle-sum in large terrestrial triangles (with vertices on mountain-tops in the Harz) as part of his geodetic survey of the kingdom of Hanover in the early 1820's. These measurements did not show a deviation from 180°, within experimental error. In certain non-Euclidean geometries ("hyperbolic") the angle-sum is always less than 180°, and the defect grows larger and larger as the triangle gets larger – in extremely large triangles the sum may be arbitrarily small! The defect is small in small triangles[8]. It may be that if we could measure the angles in a triangle with vertices in three different galaxies ( – hard to see how to do this without visiting them), we would find it substantially less than 180°. We don't know. So we are not justified in instructing our students to "discover that the angle-sum is 180°". They can and should discover that it appears to be very close to this, but that is all. In a class of students, measuring hand-drawn triangles to half a degree, one might expect a range of measurements clustering around 180°, and that is fine.

The proof of the angle-sum theorem, Euclid I:32, [16, Theorem 4], just shows that if you accept the Axioms, then the theorem holds. It is probably not advisable to disturb the faith of the young, but it is as well for teachers to understand this. It hinges on the Axiom of Parallels, and it is amusing to list some of the alternative assumptions that could be used instead (along with the other axioms[9]) , and that

[8] In other non-Euclidean geometries ("elliptic", or "spherical") the angle-sum is consistently greater than 180°

[9] – strictly, the other axioms of "neutral geometry" [10]

are false if it fails (the names in brackets after each are associated with them):

- There are two similar (equiangular) triangles in which each side of one is twice the corresponding side of the other (Wallis, 1663)
- There is at least one rectangle (Saccheri, 1773; Omar Khayyam, 11th Century).
- There is a triangle having area as great as you please (Gauss, 1799).
- A line perpendicular to the bisector of an acute angle at a point inside the angle must meet both arms of the angle.
- A line that cuts one of two parallel lines must cut the other (Proclus, 5th Century AD).

An internet search for the parallel postulate will throw up many more such oddities. The fact that there are so many very plausible statements that imply the Parallel Postulate explains why so many eminent mathematicians were deceived into thinking they had proved it without assuming anything[10].

This explains why there is no such thing as a "physical proof" of a mathematical theorem, and why no theorem is "visually obvious". Any attempt to deduce a mathematical fact from a real-world observation involves a logical gap: the implicit assumption that some mathematical theory accurately reflects some physical reality. If such an assumption is made explicit, it becomes a scientific hypothesis, and can only be disproved by observation, never confirmed [18].

It does not seem to be widely appreciated that, from the logical point of view, the abstract results are also needed in order to lay a firm ground for trigonometry and for coordinate geometry. If the parallel postulate

[10]Courses in non-Euclidean geometry are quite usual in the preparation of mathematics teachers these days. To quote Wolfe [21]: "For teachers and prospective teachers of geometry in the secondary schools the study of Non-Euclidean Geometry is invaluable. Without it there is strong likelihood that they will not understand the real nature of the subject they are teaching and the import of its applications to the interpretation of physical space." He in turn quotes Chrystal, who published a small book about what he called pan-geometry in 1880, aimed at teachers. He wrote: "It will not be supposed that I advocate the introduction of pan-geometry as a school subject; it is for the teacher that I advocate such a study. It is a great mistake to suppose that it is sufficient for the teacher of an elementary subject to be just ahead of his pupils. No one can be a good elementary teacher who cannot handle his subject with the grasp of a master. Geometrical insight and wealth of geometrical ideas, either natural or acquired, are essential to a good teacher of geometry; and I know of no better way of cultivating them than by studying pan-geometry."

fails, we have no rectangles, hence no rectangular cartesian coordinates, and triangles are not similar unless congruent, so we have no standard trigonometry. The theorems about ratios have recently been restored to the Leaving Certificate program, in part because they provide this foundation (retrospectively) for Junior Certificate trigonometry and coordinate geometry, and in part because it was considered important that senior students continue to engage with formal proofs in synthetic geometry.

There may be nothing in nature that corresponds exactly to Euclidean geometry, but it cannot be denied that it has been extraordinarily useful in practical matters for over two and a half millennia. It should also be noted that even if it does not fit exactly the shape of the real universe, Euclidean space, *as an ideal mental construct invented by us*, is immensely useful in other areas of pure and applied mathematics[11] and will always be used. By the way, the useful software system Geogebra is a realization of this ideal mental construct. In it the angle sum is always *exactly* $180^0$!

## 7. More History: a fork in the road

We now say a bit about the history of education in geometry.

**7.1.** Euclid's *Elements* had a virtual monopoly as a textbook for geometry for a very long time. A substantial splinter-group was started in France in the 16th century when Pierre de la Ramée (in Latin Petrus Ramus)(1515-1572), among his other publications, attacked the logical approach of the *Elements*. His views attracted considerable support in French educational circles for many generations and led to widely-held views that what is visually obvious should be accepted without proof. All this led to quite a different approach to geometry with many innovations. Some of these were reflex angles and rotations. Many of these ideas accumulated eventually to efficient new approaches to geometry, in modern times e.g to treatments based on transformations on the one hand and to vector spaces on the other. Thus there are now several possible approaches to Euclidean geometry available but, for present purposes, consideration of them should not be confined to their abstract merits but to which are the most suitable for our school students to obtain their grounding in geometry.

[11]Examples are the theory of equations, numerical computation, much of real and complex analysis, and even non-Euclidean geometry, which is studied using coordinates ("charts") that map pieces of the space to Euclidean space.

**7.2.** The lead of France was generally followed on the continent of Europe and the notable large country standing out against this and adhering to Euclid's approach was Great Britain. There there were efforts from 1860 onwards to assimilate elements then current on the Continent. The final bulwark to modifying Euclid's *Elements* fell in 1903 with recommendations of the Mathematical Association, and the 'Cambridge Schedule' proposed and adopted at the University of Cambridge about that time. Subsequently in textbooks there, and used in Ireland, a variety of approaches and concepts were mingled from different strands [12, 13, 14, 17]

**7.3.** This intermingling leads to some severe technical problems. All the textbooks started roughly the same way, focusing on the concepts in the world about us, becoming familiar with shapes and sizes and being led to properties of classes of them. On the whole they were clinging to a bad old habit from Euclid of trying to define everything.

From what we have said above it should be clear that they should have been motivating concepts from the real world, but that in formal geometry undefined but named items are needed. From Euclid they retained the concepts of assumed axioms, or postulates.

The practical difficulty in this is that for some concepts which are obvious and readily understood visually, it is quite difficult to lay down assumed properties for them which lead to their being singled out uniquely. In the next section we deal with two concepts which are at the heart of this problem.

By the way, it is a mistake to think that our forebears were not aware of the need for undefined terms. In books written for undergraduates, as opposed to school children, they express themselves frankly on the point. When Maynooth College was set up in 1795, it was initially staffed by French clerical academics, refugees from the Revolution. Mathematical instruction was compulsory for all students, lay and clerical, and it appears that geometrical teaching was based on French models so that practice at Maynooth was in step with the continental, rather than the British norm. André Darré, first Professor of Mathematics and Natural Philosophy, and formerly of Toulouse, prepared a text in English [6] on plane and spherical geometry for use in the College. He first gives the following definitions:

> *A straight line is that of which the elementary parts run in the same direction. A line is curved, the elements of which change continually their direction.*

Then he says:

> *Such is the most accurate notion Geometry can give of its object; and it is adequate to its object, though not perhaps a logical definition.*
>
> *Sciences mostly begin by such simple ideas sufficiently clear, independently of a definition; and they are no less reasonable than self-evident.*
>
> *For want of such simple notions and self-evident principles, an interminable series of definitions and demonstrations should be required; our mind could find no ground whereon to rest in analysis, or wherefrom to step in synthesis; nothing could be accurately understood, nothing rigorously demonstrated; and a full conviction never be obtained in the pursuit of sciences.*

The organisation of Darré's text leaves something to be desired, even apart from the quality of his English, for which he frankly begs indulgence. He does not give explicit postulates, and his "proof" of the angle-sum theorem for triangles employs a couple of hidden assumptions and a previous result with a useless proof involving motion. Nicholas Callan later wrote what he describes as a revised and improved version of Darré's text, in which he assumes explicit Postulates including a form of the parallel postulate[12]. His definition of parallel[13] is not terribly useful, and is possibly influenced by Legendre [12], two editions of which are in the College Library. But it is clear that he broke with Darré and Legendre in making no attempt to do without an axiom of parallels. Also, Legendre tries to *prove* that all right angles are equal. instead of just assuming it. We suppose that Callan "went back to Euclid", to a large extent.

## 8. Modern difficulties

**8.1. Orientation.** Suppose that we draw a triangle and mark a small arrow-head on its boundary to indicate the sense in which we consider a moving point makes one complete circuit of the boundary. Visually it is very clear that there are two possibilities, one of which we name *clockwise* and the other of which we name *anticlockwise*. But how are we to put a definition or properties of that into our mathematical model? If you have drawn your triangle on a sheet of grease-proof paper you will see that the situation is reversed when looked at through the back of the paper, and that is one complication. We cannot put an

[12] "Postulate 4: A straight line that meets one of two parallels, may be produced until it meets the other."

[13] "having no divergency"

arrow-head on each triangle boundary in the plane, so what can we do? Mathematicians have worked out a way to handle this problem by placing an arrow-head on one boundary and improvising a method of transferring that to the boundary of every other triangle in the plane, so that, for example, we can say what is clockwise on every boundary. This concept is named *orientation* of a circuit on a triangle-boundary.

**8.2. Rotation.** A second awkward topic is the very familiar one of a *rotation* in the real world, or to put it more precisely, 'rotation about a given point, through a given angle'. This involves the difficulty of orientation too. Mathematics has long had a formula for this in coordinate geometry, which uses trigonometry, but how can it be handled if it is introduced early on in pure geometry?

**8.3. How should these difficulties be handled?** Our position is not that these are very difficult concepts but that they should wait until it is much easier to introduce them. The concepts should be made clear visually by diagrams.

For example orientation, which is rarely dealt with formally, can easily be handled (if someone wants to do that) by using the concept of sensed or signed area in coordinate geometry.

The syllabus documents as they stand do not include a formal treatment of rotations, but do mention them. If a formal treatment were to be added within the existing framework of five strands, a rotation would be a type of function $P \to R(P)$. It would take points $P$ of the plane as inputs, and give other points $Q = R(P)$ as outputs. The relationship between the input point $P$, the centre of rotation $C$, and the output point $Q$ would be described in terms of angles and congruence: the angles $POQ$ would all be congruent, and $|PC| = |QC|$.
(A document in which a treatment of this kind is presented as material for a group project may be downloaded from either of the sites `http://www.ucc.ie/en/euclid/edu_and_careers/projectmaths/` or `http://archive.maths.nuim.ie/staff/aof/school.html`. These sites also have a few other resources related to school geometry, which have been submitted for approval to the NCCA Project Maths coordinators.)

We note that the text-book of Hall and Stevens [14], very commonly used in Ireland in the past, contained alternative proofs by rotation of some theorems, starting as early as Theorem 1. Teachers may be familiar with these. It may seem attractive to use such proofs, however, the point of training in deductive thinking is lost if proofs can pull in extra axioms out of the blue, and there are no axioms about rotations in the present system. Formal proofs studied should remain within whatever logical framework is laid down in the syllabus, and this in its

turn must be based upon a scholarly back-ground published treatment which provides a context for it.

**8.4. Angles and Rotations.** This is not to suggest that students should not be made very familiar with the visual concepts of clockwise and anticlockwise rotations in the real world. Of course they should. According to the Primary Curriculum, students are to 'learn to recognise an angle in terms of a rotation' [19, p.75]. This is a bit ambiguous, but perhaps ok. In the formal material on geometry, angle is an undefined term, an abstraction like point and line, so the question is: how is the student to be prepared for this, in the preliminary stage? From this stage, the student will bring some intuitive idea of what an angle is.

It says in the syllabus document that to each angle is associated a unique point called its vertex, two rays starting at the vertex, called its arms, and a piece of the plane called its inside.

It is not going to work very well if the student thinks that an angle is "a rotation". This carries with it some idea of motion, and this is not helpful in studying the angles of, for example, a given triangle. So it would be better, when talking informally about rotations, to say that a rotation is something that can be described in terms of an angle, rather than saying that an angle is a rotation. An angle is a specific "static" object, with vertex, arms and inside.

There are two things: the angle, and the number of degrees in the angle (also known as the measure of the angle [2]). One is a geometrical object, the other is a real number. These can be confused. We suggest that it is a good idea, in the preliminary informal work, to draw various angles, point out the vertex, arms, inside of each, and say that the number of degrees tells us "how big the opening is", "how rapidly the arms diverge", "how much we have to **rotate** one arm about the vertex, in order to reach the other arm", "the amount of turning involved, if we first face along one arm, and then turn and face along the other", and go onto discuss how much of a circle about the vertex is inside the angle, the concepts of degree (and radian, if desired) and the use of the protractor.

**Note.** The references below include some old books, long out of print and probably not accessible to most teachers. Happily, most such out-of-copyright books may now be accessed and read online, thanks to the Google books initiative and other archiving efforts such as archive.org. We recommend that teachers, when time allows, take advantage of these resources.

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# Awards and Competitions

## Accenture Analytics Mathematics Excellence Awards 2013

The Accenture Analytics Mathematics Excellence Awards are organised by the IMTA in conjunction with Accenture Analytics. They are awards given to three students who achieve the most meritorious results in the national Leaving Certificate Mathematics examination. Each student receives a scholarship and an Analytics Masterclass with a Management Scientist from the Analytics Innovation Centre in Dublin.

In March 2014 Seán Sherlock (Minister of State, Department of Enterprise, Jobs & Innovation and Department of Education & Skills) and Graham Healy (Managing Director of Accenture) presented the annual Accenture Mathematics Awards in University College Cork (UCC) to the three chosen students for outstanding performance in the 2013 examination.

Photos:
Brendan O'Brien, graduate of Douglas Community School, Cork, now enrolled at UCC (Physics and Astrophysics) with Graham Healy.
Conor Moloney, graduate of Christian Brothers College, Cork, now enrolled at UCC (Medicine) with Minister Seán Sherlock and Seán MacCormaic (IMTA Chairperson).
Darragh Rice, graduate of St Brendan's College, Killarney, Kerry, now enrolled at TCD (Medicine) with Minister Seán Sherlock.

## The IMO in Cape Town, 2014

Luke Gardiner, Gonzaga College, Ranelagh, Dublin 6;
Oisín Faust, The High School, Rathgar, Dublin 6;
Seoirse Murray, Maynooth Post Primary School, Maynooth, Co. Kildare;
Oisín Flynn-Connolly, Home-schooled, Ballyjamesduff, Co. Cavan;
Karen Briscoe, Kinsale Community School, Kinsale, Co. Cork;
Ivan Lobaskin, St. Benildus College, Kilmacud, Dublin 14.

Team Leader: Bernd Kreussler (MIC Limerick); Deputy Leader: Gordon Lessells (UL)

This is the first time that all of the participants received their own individual Honourable Mention. This means that each of the six contestants achieved a complete solution to at least one problem.

In 2015 the IMO will be in Thailand.
See http://www.irmo.ie/IMO2014report.pdf for a full report.

# Team Math Competition

The Team Math competition has been a long-running activity of the Irish Maths Teachers' Association, with local rounds for many years and local and National Finals since 2005. Team Math is a table quiz for teams of up to 4 students from the Leaving Certificate year, with questions based on the Leaving Certificate Higher Level syllabus. Regional Rounds were held on the evening of Friday January 31, 2014. Winning teams qualified for the National Final in NUI Galway on Saturday March 8, 2014. Congratulations to the winners from High School, Rathgar, Dublin 6.

# John Hooper Medal for Statistics 2014

**First prize and winners of the John Hooper Medal for Statistics**

**Students:** Liam O' Driscoll, David O' Callaghan **Teacher:** John O' Sullivan Hamilton High School, Bandon, Co. Cork. Project Title: *Should the voting age be lowered to 16?*

**Second prize Students**: Rachael Ni Dhonnachadha, Shannon Ni Dhonnachadha **Teacher:** John White St. Vincent's Secondary School, Seatown Place, Dundalk, Co Louth. Project Title: Karate - *The Science of Punching*

**Third prize Students**: Meaghan Dinsmore, Eimear McCausland **Teacher:** Niamh McNally Loreto Secondary School Balbriggan, Co. Dublin. Project Title: *Does music and sport play a role in the development of hand eye coordination?*

See http://www.cso.ie/en/newsandevents/johnhoopermedalforstatistics2014competition/johnhoopermedalforstatisticsresults2014/

# Peter's Problem, 2014

Winners: The students from Coláiste Cholmcille, Ballyshannon, Co. Donegal.

Congratulations to Jonathan, Dara, David and Matthew and to their teacher, Mr. Eoin O'Moore.

(See a report and solution on Page 69 this Newsletter.)

# The PI Quiz

The Pi Quiz is a team competition for Junior Cert. Students organised by the Cork Branch of the IMTA. It began in 2009. There is no National contest. Congratulations to the 2014 winners:

Cork: Christian Brothers College

Kildare/Dublin: Confey College, Leixlip

Midlands: Marist College, Athlone

Donegal: St. Eunan's College, Letterkenny

Tipperary: St. Joseph's College, Borrisoleigh, Tipperary

# *En Passant* - Look! No Calculator!

Find, without using a calculator, which is greater, $9^{15}$ or $15^9$.

Hint: Compare $(2^4 - 1)$ and $(2^3 + 1)$ together with their respective powers.

# Comórtas Sóisearach Matamaitice Éireann 2014

Organised on behalf of
*The Irish Mathematics Teachers Association (IMTA)*

## First Round

***Time : 40 minutes***

**Instructions**

1. Do not open the examination until you are told to do so.

2. **You are permitted to use a calculator but not graph (squared) paper.** You may use rulers, compasses and paper for rough work.

3. Be certain that you understand the coding system for your answer sheet. If you are not sure, ask your teacher to explain it.

4. This is a multiple choice test. Each question is followed by five possible answers marked **A,B,C,D** and **E.** Only one of these is correct. When you have decided on your choice, enter the appropriate letter on your answer sheet for that question.

5. **Scoring:**
   Each answer is worth 5 marks in Section A, 6 marks in Section B, and 8 marks in Section C.
   There is ***no penalty*** for an incorrect answer.
   Each unanswered question is worth 2 marks to a maximum of **6 marks.**

6. Diagrams are not drawn to scale. They are intended as aids only.

7. When your teacher instructs you to begin, you will have 40 minutes of working time.

# Irish Junior Mathematics Competition 2014

**Section A (5 Marks)**

1) How many seconds are there in sixty two minutes?

(A) 62 (B) 3602 (C) 3620 (D) 3680 (E) 3720

2) Sea water consists of a mixture of salt and fresh water.
The masses of salt and fresh water in a certain area are in the ratio 7 : 193.
How many kilograms of salt are there in 1000 kg of sea water?

(A) 35 (B) 186 (C) 193 (D) 200 (E) 350

3) The sum of the ages of Amy, Brian and Conor is 31 years.
What will be the sum of their ages in three years time?

(A) 32 (B) 34 (C) 35 (D) 37 (E) 40

4) The sum of ten numbers is 2624. If one of the numbers is changed from 456 to 654, the new sum is

(A) 2168 (B) 2426 (C) 2812 (D) 2822 (E) 3278

5) How many even whole numbers are between $\frac{17}{4}$ and $\frac{47}{2}$ ?

(A) 4 (B) 5 (C) 6 (D) 8 (E) 9

**Section B (6 Marks)**

6) On the way to a match, the team bus travelled at 60 km/h for 45 minutes, 30 km/h for 30 minutes and 50 km/h for one hour.
What was the average speed per km (to nearest km) travelled by the bus?

(A) 47 (B) 49 (C) 50 (D) 52 (E) 53

7) Jim bought a shirt for €13.98 and gave the cashier €20. By mistake the cashier gave Jim back €13.98 in change. In order to correct the error, the amount, in euros, that Jim should return to the cashier is

(A) 6.02 (B) 6.96 (C) 7.96 (D) 12.04 (E) 13.98

8) One plant is now 21 cm tall and will grow at 2 cm per week. A second plant is now 3 cm tall and will grow at 5 cm per week. The time, in weeks, for the two plants to reach the same height is

(A) 3 (B) 4 (C) 6 (D) 9 (E) 18

9) A €100 article is reduced by 20% for a sale. In order to restore the price to €100, the percentage by which the sale price must be increased is

(A) 20 (B) 22.5 (C) 25 (D) 30 (E) 125

10) The value of $(3\frac{1}{3})^2$ is

**(A)** $11\frac{1}{9}$ (B) $9\frac{1}{9}$ (C) $6\frac{2}{3}$ (D) $\frac{100}{3}$ (E) $\frac{101}{9}$

**Section C (8 Marks)**

11) At a match the stadium was 95% full. There were 19950 people at the match.
How many empty seats were there?

(A) 210 (B) 1050 (C) 19740 (D) 19950 (E) 18953

12) A bag contains balls of five different colours. Two are red, three are blue, ten are white, four are green and three are black. Balls are taken from the bag without looking and not returned. What is the smallest number of balls that should be taken from the bag to be sure that two balls of the same colour have been taken?

(A) 2 (B) 12 (C) 10 (D) 5 (E) 6

13) A driver stopped for petrol when her car was 1/8 full. She bought 40litres of petrol for €60. She then noticed that her tank was ¾ full, so she filled it completely.
How much did she pay for the final ¼ to fill the tank?

(A) €12 (B) €24 (C) €30 (D) €36 (E) €48

14) A 10m by 15m house is built on a 20m by 40m piece of land. The remainder of the land consists of a driveway and lawn. If the lawn occupies 80% of the remainder of the land, then the area of the lawn, in square metres is

(A) 130 (B) 280 (C) 520 (D) 650 (E) 800

15) All the faces of 64 identical small cubes are first painted white.
Then one big cube is made by combining all of these small squares.
All six faces of the big cube are then painted black.
How many of the 64 small cubes have exactly two black faces?

(A) 12 (B) 24 (C) 28 (D) 32 (E) 48

# Comórtas Sóisearach Matamaitice Éireann
# (Irish Junior Mathematics Competition) 2014
## Round 1
## ANSWER KEY

SECTION A

1. E
2. A
3. E
4. D
5 E

SECTION B

6. B
7. C
8. C
9. C
10. A

SECTION C

11. B
12. E
13. B
14. C
15. B

# Comórtas Sóisearach Matamaitice Éireann 2014

Organised by

# The Irish Mathematics Teachers Association

FINAL

*Time : 1 HOUR*

Instructions

1. Do not open the examination until you are told to do so.

2. You are permitted to use a calculator The use of graph (squared) paper is not allowed. **You may** use rulers, compasses and paper for rough work.

3. Be certain that you understand the coding system for your answer sheet. If you are not sure, ask the supervisor to explain it.

4. This is a multiple choice test. Each question is followed by five possible answers marked A,B,C,D **and** E. **Only one of these is correct. When you have** decided on your choice, enter the appropriate letter on your answer sheet for that question.

5. Scoring:
**Each answer is worth 5 marks in** Section A, 6 marks in Section B, and 8 marks in Section C.

Each unanswered or incorrect answer receives no mark.

6. Diagrams are not drawn to scale. They are intended as aids only.

7. Please do not begin until you are instructed, you will have *1 HOUR* of working time.

Section A (5 Marks)

1) If $\frac{5}{9}$ of the children in a choir are boys and the rest are girls, the ratio of boys to girls is

A) 4 : 9 (B) 4 : 5 (C) 5 :4 (D) 9 : 4 (E) 5 : 9

2) The number of pairs of parallel faces on a cube is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

3) One and twenty four hundreds as a decimal is

(A) 0.124 (B) 1.024 (C) 1.24 (D) 124.00 (E) 12400

4) What is 25% of 6% of 1000?

(A) 15 (B) 60 (C) 100 (D) 150 (E) 250

5) What is the product of ten minus seven and seven minus ten?

(A) - 9 (B) -6 (C) 0 (D) 6 (E) 9

6) The average of two numbers is 7. When a third number is included the average becomes 6. What is the value of the third number?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

7) Eight red boxes, six blue boxes and one green box weigh 67 kg.
Three red boxes, five blue boxes and ten green boxes weigh 43 kg.
What is the **total weight** of one red box, one blue box and one green box?

(A) 10 kg (B) 11 kg (C) 12 kg (D) 13 kg (E) 14 kg

8) Jim keeps canaries and cats as pets. Together they have 15 heads and 50 legs.
How many cats does Jim have?

(A) 4 (B) 5 (C) 9 (D) 10 (E) 12

9) A camera takes a picture of a metre stick beside a brick wall. In the developed photograph the metre stick is 2 cm long and the brick wall is 4.5 cm high.
What is the actual height of the wall in cms?

(A) 4.5 (B) 22.5 (C) 45 (D) 225 (E) 450

10) The value of $9 - 6 \times 2 + 5$ is

(A) 0 (B) 3 (C) 7 (D) 10 (E) 13

Section B (6 Marks)

11) Pat wants to sell his house for €150 000. After two months, the house was not sold and he decides to reduce the price by 20%. A new potential buyer likes the house but will only buy if Pat reduces the price by a further 10%. Pat agrees.
What price should the new buyer pay for the house?

(A) €105 000 (B) €108 000 (C) €110 000 (D) €112 000 (E) €115 000

12) Three buses leave the terminal together at 8 am. The buses return again to the terminal after 20 minutes, 30 minutes and 24 minutes respectively.
At what time will all three buses be at the terminal together again for the first time?

(A) 9.30 am (B) 10 am (C) 12 noon (D) 10.30 am (E) 8 am next day

13) On a standard die the numbers on the opposite sides add up to 7.
Three dice are shown. What is the sum of the numbers that are hidden from sight?

(A) 31 (B) 32 (C) 33 (D) 34 (E) 35

14) A monthly bus pass costs €30. Without a pass, each bus ride would cost €0.50.
In a month, the fewest number of times a person with a bus pass must go on a bus ride in order to save money is

(A) 15 (B) 30 (C) 31 (D) 59 (E) 61

15) If December contains five Sundays in a particular year, then December 25$^{th}$ may occur on
(A) Monday (B) Thursday (C) Friday (D) Saturday (E) Sunday

16) On a test consisting of 30 questions Aoife had 50% more right answers than she had wrong answers. She answered all the questions and each answer was either right or wrong. The number of answers Aoife got right was

(A) 10 (B) 12 (C) 15 (D) 18 (E) 20

17) A 1000 seat cinema multiplex is divided into three theatres. There are 470 seats in the first theatre, and the third theatre has 150 seats less than the second.
How many seats are in the second theatre?

(A) 190 (B) 340 (C) 380 (D) 470 (E) 530

18) At the Post office Jill spent a total of €2 to buy some 52 cent stamps and 11 cent stamp. If she received no change how many stamps did Jill buy?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 7

19) Sales of pairs of leather gloves at a shop totalled €473. Each pair of gloves sold for the same price and the price is an exact number of euros. The number of pairs of gloves sold could have been is

(A) 7 (B) 9 (C) 11 (D) 13 (E) 15

20) Sam's car uses 5.3 litres of petrol per 100 km when driving in the city and 4.3 litres of petrol per 100 km when driving on the motorway. Sam drove 60 km in the city and 40 km on the motorway.
What was her average consumption of petrol, in litres per 100 km, correct to 1 decimal place?

(A) 1.92 (B) 3.18 (C) 4.3 (D) 4.8 (E) 4.9

Section C (8 Marks)

21) Joe and Pat both cycled a distance of 27 km, starting at the same time.
Joe completed his journey 18 minutes earlier than Pat. Joe's average speed was 18 km/h. What was Pat's average speed in km/hr?

(A) 1.8 (B) 15 (C) 18 (D) 22.5 (E) 27

22) How many even whole numbers between 1 and 100 are not divisible by 5?

(A) 40 (B) 41 (C) 42 (D) 43 (E) 44

23) Starting with a list of three numbers, the "changesum" procedure creates a new list by replacing each number by the sum of the other two. For example, from {3,4,6} "changesum" gives {10,9,7} and a new "changesum" leads to {16.17.19}.
If we begin with {20,1,3}, what is the maximum difference between two numbers of the list after 2014 consecutive "changesums"?

(A) 1 (B) 2 (C) 17 (D) 19 (E) 2013

24) Sarah lights a candle every ten minutes. Each candle burns for 40 minutes and then goes out. How many candles are lighting 55 minutes after Sarah lit the first candle?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

25) The digits used to number all the pages of a book were counted. If the number of digits used was 216, how many pages are in the book?

(A) 100 (B) 101 (C) 104 (D) 108 (E) 109

# I.M.T.A. (Cork Branch)

## Comórtas Sóisearach Matamaitice Éireann
## (Irish Junior Mathematics Competition) 2014
## ANSWER KEY (Final)

SECTION A

1. C
2. D
3. C
4. A
5. A
6. B
7. A
8. D
9. D
10. D

SECTION B

11. B
12. B
13. D
14. E
15. A
16. D
17. B
18. E
19. C
20. E

SECTION C

21. *B**
22. *A**
23. D
24. C
25. D

# TOP SCORES IN THE IRISH JUNIOR MATHEMATICS COMPETITION, 2014

| PLACE | MARK | NAME | SURNAME | SCHOOL | CONTACT | CENTRE |
|---|---|---|---|---|---|---|
| 1 | 150 | Aindrias | Lyons | Col. Chríost Rí | Susan O'Connell | Cork |
| 1 | 150 | Pádraig | Ó Treasaigh | Gaelcholáiste | Deborah Ní Cheallaigh | Limerick |
| 3 | 145 | David | Troy | Castleknock College | Colin Townsend | Castleknock |
| 3 | 145 | Celine | Schmid | Loreto, Balbriggan | Elma Keane | Castleknock |
| 3 | 145 | Rory | Tummon | St Joseph's | PJ Folan | Galway |
| 3 | 145 | Matthew | Kiely | Ardscoil Rís | Paula Reidy | Limerick |
| 3 | 145 | Fergus | Stafford | Good Counsel | Seán Mac Cormaic | Wexford |
| 8 | 144 | Michael | O'Grady | CBC | Ann Barry Murphy | Cork |
| 8 | 144 | Danny | Kim | Wesley College | Quentin Heaney | Rathdown |
| 8 | 144 | Eoin | Costello | Garbally College | John Deely | Galway |
| 11 | 142 | Tom | Prior | Ballinamore CS | Robert O'Reilly | Ck-on-Shannon |
| 11 | 142 | Cian | Jones | St. Columcille's | Iseult Cleere | Rathdown |
| 13 | 140 | Amy | Dodrill | Our Lady's | Victoria Heaphy | Castleknock |
| 14 | 139 | Dara | O'Boyle | Castleknock College | Colin Townsend | Castleknock |
| 14 | 139 | Holly | Ramsay | Portmarnock CS | Shirley O'Brien | Castleknock |
| 14 | 139 | Matthew | Quigley | Col. Chríost Rí | Susan O'Connell | Cork |
| 14 | 139 | Jack | Meenan | St. Columba's | Cathy Conneely | Donegal |
| 14 | 139 | Anna | Young | Col. Iognáid | Doreen Hynes | Galway |
| 14 | 139 | Sadhbh | Fitzpatrick | St Louis,Monaghan | Joanne Myers | Monaghan |
| 14 | 139 | Cormac | O'Mahony | De La Salle | Peadar Hanratty | Waterford |
| 21 | 137 | Cian | Quinn | Ballymote | Jill Connor | Ck-on-Shannon |
| 21 | 137 | Ciarán | O'Flynn | Castleknock College | Colin Townsend | Castleknock |
| 21 | 137 | Anthony | O'Halloran | Col. An Sp. Naoimh | Peter Fagan | Cork |
| 21 | 137 | Conor | Mc Carthy | Pres, Cork | Siobhán Mc Grath | Cork |
| 21 | 137 | Carol | Martin | St. Mary's, Charleville | Claire O'Donovan | Cork |
| 26 | 136 | Killian | Twomey | Col. Na hInse | Corinne O'Toole | Castleknock |
| 26 | 136 | David | Olowookere | King's Hospital | Dean Maguire | Castleknock |
| 26 | 136 | Antons | Archipovs | St Fintan's HS | Patrick Costello | Castleknock |
| 26 | 136 | Eimhin | Campbell-Carroll | St Fintan's HS | Patrick Costello | Castleknock |
| 26 | 136 | Emily | Mulcahy | Col. Muire, Cobh | Mel Layton | Cork |
| 26 | 136 | Conor | Nolan | Gonzaga College | Paul Kieran | Rathdown |
| 26 | 136 | Shamik | Giri | Villiers School | Máiréad Fitzgibbon | Limerick |
| 33 | 135 | Liam | Mc Grath | Cashel CS | Margaret Cully | Clonmel |
| 34 | 134 | Ben | Lyons | Col. Na Mí | Tracey Ní Chatháin | Castleknock |
| 34 | 134 | Isabelle | Bryans | Holy Faith, Clontarf | Miriam Kearney | Castleknock |
| 34 | 134 | Devin | O'Keefe | Sandford Park | Leah Ketterick | Castleknock |
| 34 | 134 | Adam | Graham | Loreto, Milford | Nicola Mc Laughlin | Donegal |
| 34 | 134 | Fergal | Lonergan | St Eunan's | Emma Farrell | Donegal |
| 34 | 134 | Edward | Collins | Gonzaga College | Paul Kieran | Rathdown |
| 34 | 134 | Rachel | O'Malley | Col. Iognáid | Doreen Hynes | Galway |
| 34 | 134 | Stephen | Kenny | St Joseph's | PJ Folan | Galway |
| 34 | 134 | Ronan | Randles | Ardscoil Rís | Paula Reidy | Limerick |
| 34 | 134 | Séamus | Hurley | CBS,Ennistymon | Ann O'Brien | Limerick |
| 34 | 134 | Kevin | Shine | Athlone CC | Margaret Convey | Tullamore |

| PLACE | MARK | NAME | SURNAME | SCHOOL | CONTACT | CENTRE |
|---|---|---|---|---|---|---|
| 45 | 133 | Jack | Hennessy | Midleton CBS | Eamonn O'Neill | Cork |
| 45 | 133 | Síofra | Collins | Salerno | Jane Keenan | Galway |
| 47 | 132 | Ciara | Lynch | Col. Na hInse | Corinne O'Toole | Castleknock |
| 47 | 132 | Donncha | Mc Donagh | Gonzaga College | Paul Kieran | Rathdown |
| 47 | 132 | Aben | Benny | De La Salle | Peadar Hanratty | Waterford |
| 47 | 132 | Alex | Phelan | De La Salle | Peadar Hanratty | Waterford |
| 51 | 131 | Jamie | Hayes | Glanmire CC | Liam Young | Cork |
| 51 | 131 | David | Curtin | Hamilton HS | Declan Hayes | Cork |
| 51 | 131 | Caoimhe | O'Sullivan | Midleton College | Margaret Rumley | Cork |
| 51 | 131 | Eoin | Higgins | St Peter's | Karyn Lehane | Cork |
| 51 | 131 | Conor | O'Sullivan | Col. Iognáid | Doreen Hynes | Galway |
| 56 | 130 | Ellen | Moore | Mount Anville | Louise Gilmore | Rathdown |
| 56 | 130 | Neda | Pasakinskaite | St Louis,Monaghan | Joanne Myers | Monaghan |
| 58 | 129 | Michael | O'Neill | St Fintan's HS | Patrick Costello | Castleknock |
| 58 | 129 | Neasa | O'Connor | Salerno | Jane Keenan | Galway |
| 58 | 129 | Aaron | Clifford | Gaelcholáiste | Deborah Ní Cheallaigh | Limerick |
| 61 | 128 | Colmán | Randle | Achill | Joanne Mellett | Claremorris |
| 61 | 128 | Faizan | Gul | Col. An Sp. Naoimh | Peter Fagan | Cork |
| 61 | 128 | Keegan | Gallagher | Deele College | Fiona Friel | Donegal |
| 61 | 128 | Harry | Wall | CBC, Monkstown | Ken Walsh | Rathdown |
| 61 | 128 | Máire | Ní Bhuachalla | Col. Íosagáin | Siobhán Uí Sheasnáin | Rathdown |
| 61 | 128 | Jessica | Egbulam | Villiers School | Máiréad Fitzgibbon | Limerick |
| 61 | 128 | Colin | Smyth | Patrician | Breda Disney | Tullamore |
| 68 | 127 | Tim | Palmer | St Mogue's | Martha Brady | Castleknock |
| 68 | 127 | Sam | O'Neill | Carrigaline CS | Nichola Ní Bhriain | Cork |
| 68 | 127 | Brandon | Doran-O'Connor | Bunclody VC | Lisa Foley | Wexford |
| 71 | 126 | Vladimir | Dainovski | Castleknock College | Colin Townsend | Castleknock |
| 71 | 126 | Brian | Hayes | Glanmire CC | Liam Young | Cork |
| 71 | 126 | Mícheál | Ó Críodáin | Scoil Mhuire, Ballingeary | Siobhán Nic Ghearailt | Cork |
| 71 | 126 | Freddy | Searson | Gonzaga College | Paul Kieran | Rathdown |
| 71 | 126 | Triona | Mc Carthy | St Raphaela's | Shane Molloy | Rathdown |
| 71 | 126 | Jack | Maher | Clifden CS | Linda Keaveney | Galway |
| 77 | 125 | Cormac | O'Flynn | Pres, Cork | Siobhán Mc Grath | Cork |
| 77 | 125 | Lee | Carey | Crana College | Susan Gallagher | Donegal |
| 77 | 125 | Finn | Morrissey | Ardscoil Rís | Paula Reidy | Limerick |
| 77 | 125 | Aoife | Buckley | Banagher | Eve Clarke | Tullamore |
| 77 | 125 | Billy | Mc Donald | Marist | Maureen Dooley | Tullamore |
| 77 | 125 | Ryan | Moen | Patrician | Breda Disney | Tullamore |
| 83 | 124 | Bryan | Johnston | Coola PPS | Fiona Mc Niff | Ck-on-Shannon |
| 83 | 124 | Daniel | Windle | Sandford Park | Leah Ketterick | Castleknock |
| 83 | 124 | Liam | Cotter | Col. Pobail Bheanntraí | Séán Pól Ó Síocháin | Cork |
| 83 | 124 | Jack | Ryan | Gorey cs | Frank Duke | Wexford |
| 87 | 123 | James | Lennon | Ballinamore CS | Robert O'Reilly | Ck-on-Shannon |
| 87 | 123 | Louis | Hemeryck | Belvedere | Sheron Sweeney | Castleknock |
| 87 | 123 | Cormac | Slattery | Confey College | Susan Ryan | Castleknock |
| 87 | 123 | Alex | Wall | Sandford Park | Leah Ketterick | Castleknock |
| 87 | 123 | Shane | Elphick | Beara CS | Conor Moore | Cork |
| 87 | 123 | Conor | Mc Carthy | CBC | Ann Barry Murphy | Cork |
| 87 | 123 | Joseph | Cotter | Col. Chríost Rí | Susan O'Connell | Cork |
| 87 | 123 | Fergal | Buckley | Glanmire CC | Liam Young | Cork |
| 87 | 123 | Nicole | Clinch | Holy Child, Killiney | Lisa Burton | Rathdown |
| 87 | 123 | Tommy | Mc Carthy | St. Columcille's | Iseult Cleere | Rathdown |
| 87 | 123 | Rian | Dempsey | Good Counsel | Seán Mac Cormaic | Wexford |
| 87 | 123 | Declan | Quinn | St. Peter's | Jim Mc Carthy | Wexford |

# Comórtas Sóisearach Matamaitice Éireann 2015
# (Irish Junior Mathematics Competition 2015)

***Eligibility*** First Year Students 2014/15

***Format*** One set of question papers and answer key will be posted to each participating school some days before the competition date.

Each school will be responsible for photocopying the question paper and administering the First Round.

***First Round*** Wednesday, March 4th, 2015 Time : 40 minutes

***Final*** April/ May 2015

The top students from the First Round may be invited to compete in the Final at venues to be arranged, ***provided a certain standard is reached.***

***Entrance fee*** €35 **per school** (cheques payable to ***MICHAEL MOYNIHAN)***

If you wish your school to participate please return the completed Registration Form with the fee [~~**no later than November 29th 2014**~~ ] **[Due to the later time of publication of this Newsletter please check re late entries by using the contacts provided below. Apologies. Ed.]**

Applications received after this date may not be accepted.

Applications should be sent to: **[Please see the note re late entries above. Ed.]**

Michael D. Moynihan (Mícheál D. Ó Muimhneacháin)
"Drumacoo",
Leemount,
Carrigrohane,
Co. Cork.

Phone : 087-2860666
**email** mmoynihan@eircom.net

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**Registration Form 2015 [Please check re late entries.]**

Name of Teacher:_____

School:_____

School address_____

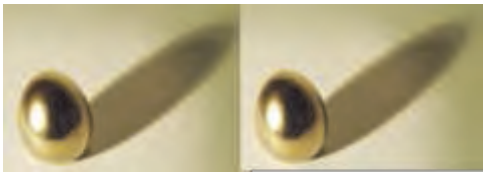
Phone number_____(Home)_____(School)

School Fax number_____ email address_____

Approximate number of students participating_____

# A CRACKING PUZZLE

A firm has invented a super-strong egg. For publicity purposes, it wants to determine the highest floor in a 100-story building from which such an egg can fall without breaking. The firm has given a tester two identical eggs to experiment with. Of course, the same egg can be dropped multiple times unless it breaks. What is the minimum number of droppings that is guaranteed to determine the highest safe floor in all cases?
[From *Algorithmic Puzzles*, Anany Levitin & Maria Levitin, Oxford University Press, 2011.]



This problem was published in Newsletter 113. A solution to the problem is presented here.

## Solution

We are searching for a solution to the 'worst-case scenario'. In this scenario the egg may or may not break on any drop. Our algorithm has to ensure that when the test is complete we have identified the floor which would take the most egg-drops to find within that particular test while the number of egg-drops is lower than that of any other possible test.

### A Possible Test

By dropping the first egg on sequential floors, starting with Floor 1, we can be sure to establish the floor where the egg breaks. In the worst-case scenario this will be Floor 100 which means that 100 trials must be performed. Indeed, the egg does not need to break on Floor 100. One hundred trials must still be done. Floor 100 is the floor which will take the most egg-drops to identify within this particular test. We will consider if there is an algorithm which requires fewer drops. We will use the two eggs in the next test.

### A Second Possible Test

By dropping the first egg at various stages on the way up the building we can establish upper and lower limits to the where the egg breaks. Then we can utilise the second egg to sequentially search between those limits.

For instance, by dropping the first egg at every tenth floor we can see that in the worst case the

egg finally breaks on the drop from Floor 100 at drop number ten. The upper limit is Floor 100 and the lower limit is Floor 90 (ten floors below) where it was last dropped without breaking.
The second egg is then used to test the floors from Floor 91 to Floor 99. In the worst case this will take nine trials for the second egg.

The total number of trials is then 19 in the worst case for this algorithm.

## A Third Possible Test

By looking at other constant intervals between drops of the first egg we find that 19 trials is the best that can be done. For instance, if we drop the first egg on every twelfth floor until it breaks on Floor 96 and then drop the second egg sequentially from Floor 85 we will have a worst-case of 8 + 11 or 19 trials to find that it breaks on Floor 95. We will consider another algorithm by improving the process used in this one.

## A Fourth Possible Test – The First 'Telescopic' Test

The first egg is used to partition the 100 floors. By making the partitions of smaller size as we go up the building we can ensure that the additional number of drops required from the second egg does not make the total progressively larger. This may be called a ‘telescopic search’ or ‘triangle search’ because the partition sizes follow the pattern of triangular numbers. Following an egg-break on the first search, a linear search (testing successive floors in turn) may be done using the second egg within the limits found by the telescopic search. Our aim is to ensure that the worst case for each partition does not get progressively worse as we go further upwards.

For instance, let us drop the first egg at Floor 18. If it breaks then the worst case for the second egg is 17 further drops making a total of 18 drops. If the first egg does not break on Floor 18 then the next drop can be done at Floor (18 + 17) which is Floor 35. Again, if it breaks, the worst case for the second egg is 16 further drops. This ensures that the total for either worst case is still 18 (2 for the first egg and 16 for the second). This process may be continued until Floor 100 is reached. We need to know how many partitions of 100 are created by starting at 18 and going up in decremented intervals so that the total of the interval number and the interval difference does not increase above 18. We have 18 + 17 + 16 + 15 + 14 + 13 + 7 = 100 which is 7 partitions. The worst case here will involve 18 drops – i.e., the 1$^{st}$ egg breaks on either of the first six drops. Already, we have a better worst-case scenario than in the case of the constant partitions.

## A Fifth Possible Test – The Second 'Telescopic' Test

By reducing the first term of the series and extending the number of terms we can improve the overall result as long as the number of terms does not outgrow an established minimum.

A minimum will occur when the last term of the series of partitions is 1. Or, in reverse, we require a series 1 + 2 + 3 + … to reach 100. This is the sum of the first n natural numbers. Thus

we have to solve the equation $\frac{n}{2}(n+1) = 100$.
That is, solve $n^2 + n - 200 = 0$.

The solution is: $n = \frac{-1 \pm \sqrt{1+800}}{2} = 13.65$ correct to 2 decimal places. This tells us that we require more than 13 terms in our series.

Observe that 1+2+3+4+5+6+7+8+9+10+11+12+13 = 91 but because we have to use a reducing series we will write 13+12+11+10+9+8+7+6+5+4+3+2+1= 91. This series has 13 terms and falls short by 9. The series may be rectified by using an extra 9 at the start to get 9+13+12+11+10+9+8+7+6+5+4+3+2+1=100. This series has 14 terms. In the worst-case scenario the total number of drops of both eggs is 14. [Notice that if 9 is used in a position other than at the start of the series or a number less than 9 is used at the start then the combination of 1st and 2nd egg-drops will exceed 14 in the worst cases since the number of terms will have to increase for the 1st egg or the intervals will increase beyond the minimum achieved for the 2nd egg. For instance, **8**+13+12+11+10+9+8+7+6+5+4+3+2+1**+1**=100 has 15 terms
or, **8**+13**+13**+11+10+9+8+7+6+5+4+3+2+1=100 requires 15 drops if the lowest egg-break is between Floor 21 and Floor 34 (3 drops of the 1st egg) with 12 drops of the 2nd egg required in the worst case.

## A Sixth Possible Test – The 'Trumpet' Test

Alternatively, we can observe that the series 14+13+12+11+10+9+8+7+6+5+4+3+2+1= 105. This has 14 terms and exceeds the target by 5. By removing 5 we can again get the required series. If the 5 is removed from the first term we get the same series, (**9**+13+12+11+10+9+8+7+6+5+4+3+2+1), as above. However, the 5 may be removed from any element of the series or from a combination of elements. For example, the series
**10**+13+12+11+10+9+8+7+6+5+4+3+**2**=100 may be used. The combination of egg-drops will not exceed 14 in the worst cases. This series may be called a 'trumpet series' or, more correctly, a 'reverse-trumpet series'. It begins large and telescopes to a small ending while individual elements along the way may be 'pushed in'.

### Conclusion

Our conclusion is that the minimum number of drops required for any worst-case is 14.

### The number of solutions

Other partitions of 100 are possible. In total, 8391 different solutions are possible. For instance 14+**12+11+10+9+8**+8+7+6+5+4+3+2+1 = 100 (1 has been removed from each of the five terms shown in bold).
The total number of solutions may be established by considering in how many ways 5 may be removed from the series 14+13+12+11+10+9+8+7+6+5+4+3+2+1= 105.

## Showing the pattern of the series in a table: Table 1

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Start with Floor 14. The intervals '+2' and '+3' are omitted.** | | | | | | | | | | | | | | |
| Interval between floors | | +13 | +12 | +11 | +10 | +9 | +8 | +7 | +6 | +5 | **+4** | +1 | If Breaks on Floor 99 | |
| Floor | 14 | 27 | 39 | 50 | 60 | 69 | 77 | 84 | 90 | 95 | 99 | **100** | | |
| No. drops: 1st Egg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | **12** | | |
| No. drops: 2nd Egg | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | **0** | Travel with egg: 995 floors | |
| Worst | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | **12** | | |
| **Start with Floor 13. The intervals '+3' and '+1' are omitted.** | | | | | | | | | | | | | | |
| | | +13 | +12 | +11 | +10 | +9 | +8 | +7 | +6 | +5 | **+4** | +2 | If breaks on Floor 98 | |
| Floor | 13 | 26 | 38 | 49 | 59 | 68 | 76 | 83 | 89 | 94 | 98 | **100** | | |
| No. drops: 1st Egg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | **12** | | |
| No. drops: 2nd Egg | 12 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | **1** | Travel with egg: 981 | |
| Worst | 13 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | **13** | | |
| **Start with Floor 12. The intervals '+2' and '+1' are omitted.** | | | | | | | | | | | | | | |
| | | +13 | +12 | +11 | +10 | +9 | +8 | +7 | +6 | +5 | +4 | **+3** | If breaks on Floor 100 | |
| Floor | 12 | 25 | 37 | 48 | 58 | 67 | 75 | 82 | 88 | 93 | 97 | **100** | | |
| No. drops: 1st Egg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | **12** | | |
| No. drops: 2nd Egg | 11 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | **2** | Travel with egg: 979 | |
| Worst | 12 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | **14** | | |
| **Start with Floor 11. The interval '+2' is omitted.** | | | | | | | | | | | | | | |
| | | +13 | +12 | +11 | +10 | +9 | +8 | +7 | +6 | +5 | +4 | **+3** | +1 | If breaks on Fl. 99 |
| Floor | 11 | 24 | 36 | 47 | 57 | 66 | 74 | 81 | 87 | 92 | 96 | 99 | **100** | |
| No. drops: 1st Egg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | **13** | |
| No. drops: 2nd Egg | 10 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | **0** | Travel: 965 |
| Worst | 11 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | **13** | |
| **Start with Floor 10. The interval '+1' is omitted.** | | | | | | | | | | | | | | |
| | | +13 | +12 | +11 | +10 | +9 | +8 | +7 | +6 | +5 | +4 | +3 | **+2** | If breaks on Fl.100 |
| Floor | 10 | 23 | 35 | 46 | 56 | 65 | 73 | 80 | 86 | 91 | 95 | 98 | **100** | |
| No. drops: 1st Egg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | **13** | |
| No. drops: 2nd Egg | 9 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | **1** | Travel : 957 |
| Worst | 10 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | **14** | |
| **Start with Floor 9. No interval is omitted.** | | | | | | | | | | | | | | |
| | | +13 | +12 | +11 | +10 | +9 | +8 | +7 | +6 | +5 | +4 | +3 | +2 | **+1** | B: 100 |
| Floor | 9 | 22 | 34 | 45 | 55 | 64 | 72 | 79 | 85 | 90 | 94 | 97 | 99 | **100** | |
| No. drops: 1st Egg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | **14** | |
| No. drops: 2nd Egg | 8 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | **0** | T: 945 |
| Worst | 9 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | **14** | |

Table 1 sets out possible series which may be used for various starting floors. The highest starting floor is Floor 14 while the lowest starting floor is Floor 9. The table also shows the greatest number of floors which have to be travelled with an egg (see the extension problem below).

## An extension problem: Egg-carrying

In order to carry out the experiment the first egg has to be carried up a number of floors and dropped. If it does not break it has to be retrieved and again carried up to the next required floor. If it breaks, the second egg has to be carried to the required floor and repeatedly carried up until the floor immediately below where the first egg broke is reached. (Note: This order for testing the second egg may be reversed.)

If we calculate how many floors, cumulatively, both of the eggs have to be carried in the worst-case scenario then the series which begins with 9 has the least grand total which is 945 floors. The series beginning with 14 entails carrying an egg up through 995 floors which is 50 extra floors. Thus, the series 9+13+12+11+10+9+8+7+6+5+4+3+2+1 is the most efficient plan for our test. [Note that the totals do not count the number of floors travelled without any egg in hand.]

## Improved Egg-carrying

We may consider if the egg-carrying algorithm may be improved. If both eggs are carried together then if the first does not break when dropped the second may be carried further to the next dropping point. This procedure will reduce the number of flights through which the eggs must be carried. In the most efficient series already identified (starting with Floor 9) two eggs may be carried for a total of 448 floors and a single egg for a further total of 49 floors to give a grand total of 497 floors of egg-carrying. (9 + 13 + 34 + 11 + 55 + 9 + 72 + 7 + 85 + 5 + 94 + 3 + 99 + 1 = 497).

Alternative methods can be used to find the minimum number of drops required.

**Alternative A**

Here the number of floors which can be tested for a given number of drops is calculated.

For 1 drop only 1 floor can be tested. 1st egg: 1.

For 2 drops, up to 3 floors can be tested. 1st egg: 2, 3.

For 3 drops, up to 6 floors can be tested: 1st egg: 3, 5, 6.

For 4 drops, up to 10 floors: 4, 7, 9, 10.

For 5 drops, up to 15 floors: 5, 9, 12, 14, 15.

For 6 drops, up to 21 floors: 6, 11, 15, 18, 20, 21.

For 7 drops, up to 28 floors: 7, 13, 18, 22, 25, 27, 28.

For 8 drops, up to 36 floors: 8, 15, 21, 26, 30, 33, 35, 36.

For 9 drops, up to 45 floors: 9, 17, 24, 30, 35, 39, 42, 44, 45.

For 10 drops, up to 55 floors: 10, 19, 27, 34, 40, 45, 49, 52, 54, 55.

For 11 drops, 66 floors: 11, 21, 30, 38, 45, 51, 56, 60, 63, 65, 66.

For 12 drops, 78 floors: 12, 23, 33, 42, 50, 57, 63, 68, 72, 75, 77, 78.

For 13 drops, 91 floors: 13, 25, 36, 46, 55, 63, 70, 76, 81, 85, 88, 90, 91.
For 14 drops, up to 105 floors: 14, 27, 39, 50, 60, 69, 77, 84, 90, 95, 99, 102, 104, 105.
Therefore, the number of drops required to test 100 floors is 14.

**Alternative B:**

The set-up may be viewed as triangular numbers. We have the usual triangular numbers, starting at 1, arranged as follows in Table 2. This is a tabulated view of the solution provided above.

| Floor number | | | | | | | | | | | | | | Row no. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | 1 |
| 2 | 3 | | | | | | | | | | | | | 2 |
| 4 | 5 | 6 | | | | | | | | | | | | 3 |
| 7 | 8 | 9 | 10 | | | | | | | | | | | 4 |
| 11 | 12 | 13 | 14 | 15 | | | | | | | | | | 5 |
| 16 | 17 | 18 | 19 | 20 | 21 | | | | | | | | | 6 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 | | | | | | | | 7 |
| 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | | | | | | | 8 |
| 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | | | | | | 9 |
| 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | | | | | 10 |
| 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | | | | 11 |
| 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | | | 12 |
| 79 | 80 | 81 | 82 | 83 | 84 | 85 | 96 | 97 | 88 | 89 | 90 | 91 | | 13 |
| 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | **100** | 101 | 102 | 103 | 104 | 105 | 14 |

Table 2

To see the intervals involved in the actual process of testing the eggs we can rearrange the rows. For instance, we can let the first step go up 9 floors and the following steps go up as shown in the table (Table 3) using the pattern of the ‘reverse-trumpet series’:

| | | | | | | | | | | | | Use floor for 1st test | 1st egg: No. of drops | 2nd egg: No. of drops (worst case) | Total no. of egg-drops (worst case) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 2 | 12 | 14 |
| | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 3 | 11 | 14 |
| | | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 4 | 10 | 14 |
| | | | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 5 | 9 | 14 |
| | | | | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 6 | 8 | 14 |
| | | | | | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 7 | 7 | 14 |
| | | | | | | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 8 | 6 | 14 |
| | | | | | | | 80 | 81 | 82 | 83 | 84 | 85 | 9 | 5 | 14 |
| | | | | | | | | 86 | 87 | 88 | 89 | 90 | 10 | 4 | 14 |
| | | | | | | | | | 91 | 92 | 93 | 94 | 11 | 3 | 14 |
| | | | | | | | | | | 95 | 96 | 97 | 12 | 2 | 14 |
| | | | | | | | | | | | 98 | 99 | 13 | 1 | 14 |
| | | | | | | | | | | | | 100 | 14 | 0 | 14 |

Table 3

Another configuration is that as many rows as possible have as large a step as possible. This is illustrated in the following table (Table 4):

| | | | | | | | | | | | | | Use floor for 1st test | 1st egg: No. of drops | 2nd egg: No. of drops (worst case) | Total No. of egg-drops (worst case) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 13 | 14 |
| | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 2 | 12 | 14 |
| | | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 3 | 11 | 14 |
| | | | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 4 | 10 | 14 |
| | | | | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 5 | 9 | 14 |
| | | | | | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 6 | 8 | 14 |
| | | | | | | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 7 | 7 | 14 |
| | | | | | | | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 8 | 6 | 14 |
| | | | | | | | | 85 | 86 | 87 | 88 | 89 | 90 | 9 | 5 | 14 |
| | | | | | | | | | 91 | 92 | 93 | 94 | 95 | 10 | 4 | 14 |
| | | | | | | | | | | 96 | 97 | 98 | 99 | 11 | 3 | 14 |
| | | | | | | | | | | | | | 100 | 12 | 0 | 12 |

Table 4

In this case the number of steps is reduced to 12 which counts the drops for the 1st egg. The drops for the second egg will bring the total for the worst case up to 14.

In general, floor numbers may be redistributed among the rows provided that no row becomes any bigger than (15 – drop number).

| | | | | | | | | | | | | | Use floor for 1st test | 1st egg: No. of drops | 2nd egg: No. of drops (worst case) | Total No. of egg-drops (worst case) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 13 | 14 |
| | | | | | | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 2 | 7 | 9 |
| | | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 3 | 11 | 14 |
| | | | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 4 | 10 | 14 |
| | | | | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 5 | 9 | 14 |
| | | | | | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 6 | 8 | 14 |
| | | | | | | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 7 | 7 | 14 |
| | | | | | | | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 8 | 6 | 14 |
| | | | | | | | | 80 | 81 | 82 | 83 | 84 | 85 | 9 | 5 | 14 |
| | | | | | | | | | 86 | 87 | 88 | 89 | 90 | 10 | 4 | 14 |
| | | | | | | | | | | 91 | 92 | 93 | 94 | 11 | 3 | 14 |
| | | | | | | | | | | | 95 | 96 | 97 | 12 | 2 | 14 |
| | | | | | | | | | | | | 98 | 99 | 13 | 1 | 14 |
| | | | | | | | | | | | | | 100 | 14 | 0 | 14 |

Table 5

There are 8391 possible lists for the column showing the floor number to use for the 1st test.

A formal treatment of the mathematics of the decision-making in the case of the egg-dropping problem is carried out in http://archive.ite.journal.informs.org/Vol4No1/Sniedovich/#wuegg

The problem first appeared in *Which Way Did the Bicycle Go?* (page 53); Konhauser J.D.E., D. Velleman, and S. Wagon (1996), Dolciani Mathematical Expositions - No. 18, The Mathematical Association of America.

Neil Hallinan

Dublin Branch

# Fr. Brendan Kearney S.J. (1936 – 2014)
# An Appreciation

Rev. Brendan Kearney S.J., who passed away in February 2014, aged 78, was a staunch supporter of the IMTA Dublin Branch and a prodigious contributor to the Association's Newsletter over many years. He was a Member of the Editorial Board of the Newsletter from 1971 to 1978.

Ordained by Archbishop John Charles McQuaid in 1968, he taught for many years in Belvedere College, Dublin. In later years he moved to California where he was curate in St. James Parish, Redondo Beach, California, from September, 1995 to July, 2010.

His sometimes quirky, witty and always erudite articles appeared in the Newsletters spanning a period of 40 years. To quote from the Appreciation piece in the Irish Jesuit News (Vol. 3, Issue 4, Feb., 2014) *'His strong sense of the ridiculous must reflect the moods of his father, who contributed the hilarious "Dear Sean and Mammy" pieces to Dublin Opinion'.* (His father was the late A. P. Kearney who was the first editor of the IMTA Newsletter. He was also the author of the 'Look Likely Law', oft-quoted by his son, as in the Newsletter 108 article referenced below.)

From Fr. Brendan's first article in No. 7 (Nov., 1966) on '*A Companion to the Foregoing'* (accompanying an article by James Hyslop titled '*The Remainder Theorem*') to his final one in No. 108 (2008) entitled *'$x^n - a^n$ and some ssociated horror stories'*, he contributed almost 40 papers, as well as providing solutions and comments to Leaving Certificate Higher Level Mathematics papers for many years. He provided background material on many topics of the Leaving Certificate mathematics course: calculus, algebra, trigonometry, number theory, probability and functions among them.

Fr. Brendan embraced new technologies from their inception into education. In the early 1970s he took a course in TCD (Diploma in Computers in Education) - the brainchild of Fr Cyril Byrne CSSP - and, in 1974, he was using the computer in his study of Euclidean nunbers. In 1975 he outlined a case in support of the use of calculators in State Examinations. By 1991, in his article *'Humachine*' he was delighting in the possibilities afforded by the programmable calculator.

The Booklet on Integration for teachers (1976) was written by Fr. Kearney and he kindly donated the proceeds from its sale to IMTA.

His trademark duck cartoons first appeared in mathematics articles sometime around 1976. Ducks were used for illustration purposes in various expressions of puzzlement, questioning, disinterest and, finally, delight. Instead of the traditional 'QED.' he sometimes ended a proof with the word 'Quack'.

Fr. Brendan will be fondly remembered by his many friends in the IMTA. We extend sincere sympathy to his sister, Ann, brother-in-law, Michael, nieces and nephews, friends and Jesuit confrères.

# Streetmath

$$5(3x - 1) = 25$$

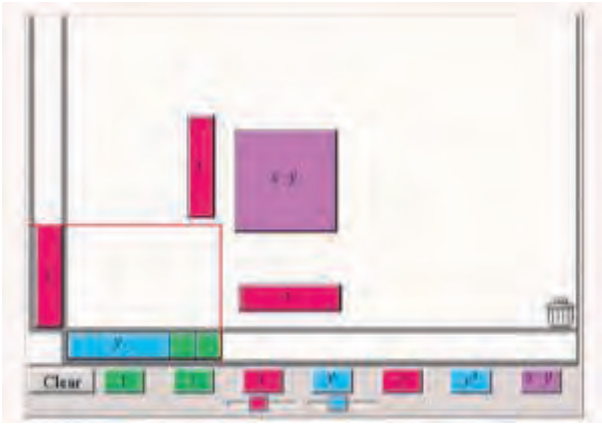
“He's in there! Nobody's ever seen him, he causes trouble all the time, we're definitely going to get him now...”
“x? Nobody knows anything about x.”
“We do know something about one of his aunts, more a sort of cousin, 3x - 1.”
“What do we know about her?”
“Five of her are 25”
“So what?”
“Five of her are 25, which makes her ...”
“5?”
“You're beginning to get the idea ... if 5 of his cousin are 25 then his cousin has to be 5.”
“OK, there's five 5's in 25, we know that.”
“Used to be in the 5 times tables, now you only have to press the old buttons in your calculator ...”
“Calculators are so yesterday, you can do it on your phone.”
“Have you got that froggy new ring tone ?”
“No, and do you know how much you pay every time you use it?”
“Get back to his cousin ... 3x - 1 = 5 ?”
“So, x = ?”
“Well, three of him would be six?”
“Ya, 3x = 5 + 1 = 6, which makes x = ?”
“x = 2, — we got him !”
Robin Harte, TCD

## The Fields Medal, 2014

An Iranian mathematician working in the US has become the first ever female winner of the celebrated Fields Medal.
In a landmark hailed as "long overdue", Professor Maryam Mirzakhani was recognised for her work on complex geometry.
Awarded by a committee from the International Mathematical Union (IMU), the Fields Medal is regarded as something akin to a Nobel Prize for maths. It was established by Canadian mathematician John Fields and comes with a 15,000 Canadian dollar (over €10,000) cash prize.
First awarded in 1936 and then every four years since 1950, the medal is awarded to between two and four researchers, who must be no older than 40, because Fields wanted to encourage the winners to strive for "further achievement" as well as recognise their success.
http://www.bbc.com/news/science-environment-28739373

# Have you tried Virtual Manipulatives?

Here is a site which will encourage experimentation with those physical objects in a virtual environment:




**Multiplying Binomials**

Illustrate the question, "What is $x$ groups of $(y + 2)$ things?" by placing an $x$ piece along the vertical side and a $y$ piece combined with 2 green cubes along the horizontal side. Fill in the rectangle with pieces that represent an $xy$ value, and $2x$ values. What is your answer? Change the value of $x$ or $y$ by dragging the sliders.

Question and image taken from: http://nlvm.usu.edu/en/nav/frames_asid_189_g_1_t_2.html?open=activities&from=topic_t_2.html

The three pieces required to fill in the answer are shown here although one of the x-pieces must be rotated.

This is part of the National Library of Virtual Manipulatives at http://nlvm.usu.edu/en/nav/topic_t_2.html

# This Puzzle is a Killer

An evil king is informed that one of his 1000 wine barrels has been poisoned. The poison is so potent that a miniscule amount of it, no matter how diluted, kills a person in exactly 30 days. The king is prepared to sacrifice 10 of his slaves to determine the poisoned barrel.

A) Can this be done before a feast scheduled in 5 weeks?

B) Can the king achieve his goal with just eight slaves?

[From *Algorithmic Puzzles*, Anany Levitin & Maria Levitin, Oxford University Press, 2011.]

The Solution is on page 83. But first you may like to read the hint and attempt it yourself.

Neil Hallinan

Hint: Answer A) Yes; B) Yes. The entire 5 weeks are not necessary.

# Cumann Oidí Matamaitice na hÉireann
# Irish Mathematics Teachers' Association
# 50th Golden Jubilee Celebration

**President Michael D. Higgins and Sabina Higgins with the Past Chairpersons, Representatives of deceased Past Chairpersons, Members from the beginnings of the IMTA and their guests**

Back (l. to r.): Elizabeth Caird, Dominic Guinan, Goretti O'Dea, Liam Sayers, Sheila Sayers, John Fahey, Marie Fahey, Seán Ashe, Catherine Ashe, Peter Forde, Carolanne Forde, Olivia Bree, Daragh (Buckley) Murphy

Front (l. to r.): Jane Williams, Anne McNamara, President Michael D. Higgins, Sabina Higgins, Fr. Brendan Steen C.M., Patricia Buckley

The Guest of Honour at the Gala Celebration Banquet was President of Ireland, Michael D. Higgins. The President was accompanied by his wife, Sabina.

Members who attended the first meeting of the IMTA 50 years ago were Special Guests at the banquet. They were: Elizabeth Caird, Jane Williams, Fr. Brendan Steen C.M. and Con O'Keefe. Regrettably, Con was unable to attend on the night.

## President Michael D. Higgins and Sabina Higgins with the Council Officers, Newsletter Editorial Team, Top Table Guests and their guests.

Back (l. to r.): Neil Hallinan, Carmel Hallinan, Michael Brennan, Michael Moynihan, Máire Moynihan, Donal Coughlan, Valerie Coughlan, Elizabeth Oldham

Front (l. to r.): Mark Comiskey, Natalie Noone, Brendan O'Sullivan, Sabina Higgins, President Michael D. Higgins, Seán MacCormaic, Thérèse MacCormaic

Back (l. to r.): Neil Hallinan, Michael Moynihan, Donal Coughlan, Michael Brennan

Front (l. to r.): Natalie Noone, Brendan O'Sullivan, Sabina Higgins, President Michael D. Higgins, Seán MacCormaic, Elizabeth Oldham

# President Michael D. Higgins, Viewing, Speaking and Chatting

# MathsFest14, UCD, Belfield, 11 October

## MathsFest Opening, Speakers and More

1. Volunteers at Registration, Pamela O'Brien, Dr. Maria Meehan, The Opening Ceremony
2. Aoibhinn Ní Shuilleabháin, Louise Boylan, Paco Gomez, Aoife Kelly, Dr. Steve Humble, Catherine Kierans, Andrew McKimm
3. Michelle Kelly, Aibhin Bray, Michael Keating, Paudie Scanlon, Dr. Seán Delaney, Jacinta Burke
4. Owen McConway, Chris Green, Liz O'Neill, Fionnuala Croke, Maria Broderick
5. The Project Maths Team, Aidan Roantree, Pádraig Kirk, Rachel Dool (Midleton College) receiving Engineers Ireland's Kindle prize from Michael Moynihan

# PETER'S PROBLEM 2014

A local authority plans to build a pedestrian bridge across a canal. The canal is 8.2 metres wide. The ground on both sides of the canal is flat and at the same level. The banks of the canal are parallel to each other.

The bridge will have 3 sections. The span will cross the canal horizontally at $90^0$ to the canal banks. The approaches to the span will both be parallel to the canal banks.

The foundations of all supports will be at least 1.5 metres from the water's edge on either side. The level of water in the canal is 75 cm below ground level of the bank.

The span of the bridge must be 2.3 metres above the level of the water to facilitate traffic on the canal. Allow 25cm for the thickness of the frame of the bridge below the surface of the pedestrian walkway in this section of the bridge.

The pedestrian walkway on all sections of the bridge must be wheelchair accessible and all construction must conform to the building guidelines in Part M of the Department of the Environment, Community and Local Government Building Regulations 2010, under the section 'Approach to buildings other than dwellings' which also governs the building of pedestrian bridges. ( NOTE: A stepped access route or lift as referred to on page 26 does not form part of this problem).

The width of the construction should be uniform for its entire length.
Find the minimum area of the pedestrian surface of the construction, including turning areas at access points at either end of the construction.
Give the answer in $m^2$, correct to two places of decimals.

If the condition that "The span will cross the canal horizontally" is withdrawn from the specification for the bridge, can you suggest the most efficient amendments that could be made to reduce the length of the walkway.

Calculate the reduction resulting from your suggestion.

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In conjunction with Engineers Ireland Peter's Problem is designed for Transition Year students. It is a commemoration of the late Peter Tiernan.

Peter's Problem Sub-committee: Dominic Guinan, Michael Moynihan, Michael Brennan, Neil Hallinan

# Peter's Problem 2014: Report

The overall impression is of groups of students who engaged with the problem and worked to see how its instructions could be implemented. While the detail of development was not always shown it is obvious that as time went on the problem developed in an interesting way for all. Some students brought the real-life dimension of the problem to life by referencing their family situation or by observing details of local constructions. Others widened their searches on the internet although this could prove misleading as the regulations for Ireland are sometimes slightly different from other countries.

The writing-up part of the exercise also showed how the students implemented a variety of forms. Drawings were done as sketches, design drawing and Computer Aided Drawing packages. Reports were presented through Word packages, PowerPoint and handwriting which is probably most useful for speed of communication of mathematics.

The problem itself entailed building a ramp which had to rise a height of at least 1.8m. This necessitated an appreciation that the ramp length had to be quite long and that rest areas should be built in to the construction. Searching for a minimum length (and hence area) could only be done by looking at the different grades of slope allowable under Guidelines Part M. Different slopes required rests at different stages and thus a comparison of possibilities could be made.

In general, a comparison of possibilities was not made by the students. However, while most groups utilised a slope of 1:20 which is the gentlest slope, a smaller surface area is found by using a slope of 1:15. Furthermore, a careful reading of Part M shows that the requirement of ‘level’ may be translated as a slope of 1:50 or less. Using this slope for ‘level’ rest areas significantly reduces the target height for the remaining parts of the ramp.

The Peter's Problem Committee

# SOLUTION TO PETER'S PROBLEM 2014

1. The specifications.
Canal: 8.2m wide. Foundations of supports: 1.5m from water's edge. Span 11.2m.
Frame of bridge: 2.3m above water level and 0.25m thick. Water level: 0.75m below ground level.
Walkway surface is 2.3 + 0.25 – 0.75 = 1.8m above ground level.
The ramp sections must rise 1.8m to meet this walkway.
The figures for the width and slope are found in Part M (Building Regulations).

2. Information from the Department of the Environment, Community and Local Government Building Regulations 2010, Part M. This may be found at: http://www.environ.ie/en/Publications/DevelopmentandHousing/BuildingStandards/FileDownLoad,24773,en.pdf
We consider *'Section 1.1: Approach to buildings other than dwellings'* as specified.
A gradient of 1:50 or less steep is considered to be level.
Landings should be at least 1800mm wide and 1800mm long when the slope has three flights or more.
This gives the uniform width of the bridge.
The ramp gradients are limited to those in Table 1 and Diagram 3 (from Part M) as shown below.

**Table 1 Limits for ramp gradients and lengths**

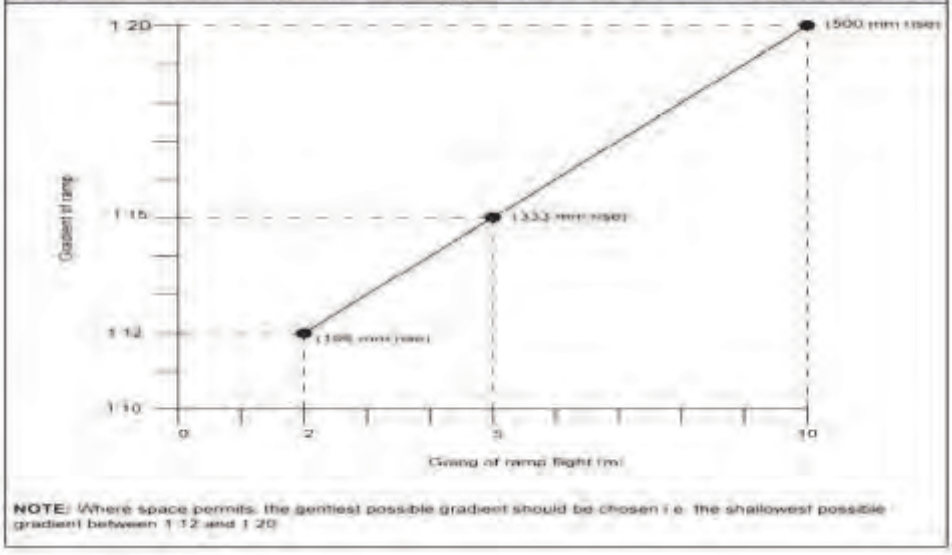
| Maximum going of a flight | Maximum gradient | Maximum rise |
|---|---|---|
| Not exceeding 10 m | 1:20 | 500 mm |
| Not exceeding 5 m | 1:15 | 333 mm |
| Not exceeding 2 m | 1:12 | 166 mm |

NOTE: For goings between 2 m and 10 m, it is acceptable to interpolate between the maximum gradients (refer to Diagram 3).

Note: 'Going' is the same as 'Run' in

$$\text{Gradient} = \frac{\text{Rise}}{\text{Run}}$$

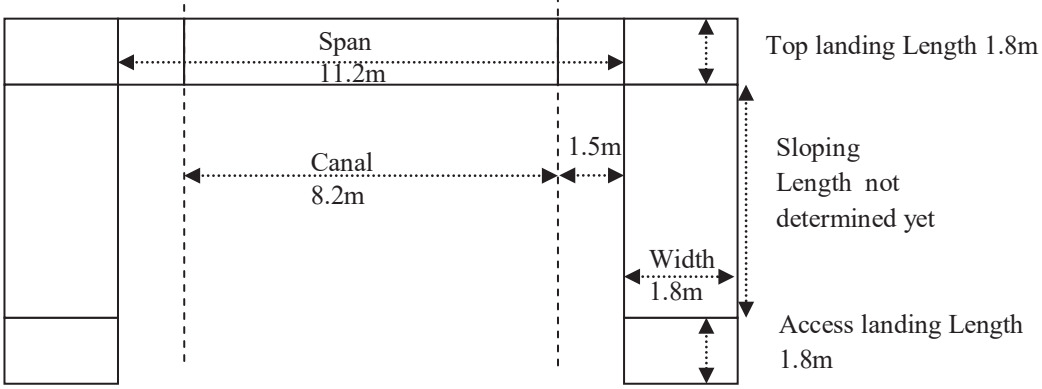
**Diagram 3 Relationship between ramp gradient and the maximum going of a flight**




NOTE: Where space permits, the gentlest possible gradient should be chosen i.e. the shallowest possible gradient between 1:12 and 1:20

3. Outline drawing:

Plan view of the bridge dimensions at this stage shown below

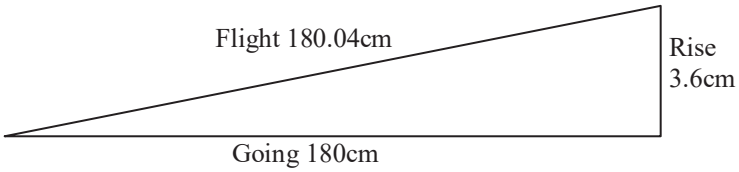


**Part I of the problem:**

4. Determining the slopes section of the ramp: Looking at landings.

To simplify the entrance to the ramp a flat or zero gradient landing is used. Also a zero gradient top landing is used. Intermediate landings or rest areas are level with a gradient of 1:50. Use Going = 180cm.

$$\frac{Rise}{Going} = \frac{1}{50}$$
$$\Rightarrow Rise = \frac{Going}{50}$$
$$= \frac{180}{50} = 3.6cm$$




$$Flight = \sqrt{Going^2 + Rise^2}$$
$$= \sqrt{180^2 + 3.6^2}$$
$$= \sqrt{32412.96} = 180.0359964cm$$

Alternatively, if the Flight length is 180cm then

$$\frac{Rise}{Flight} = Sin\theta \text{ and } \theta = Tan^{-1}(\frac{1}{50})$$
$$\Rightarrow Rise = Flight \times Sin(Tan^{-1}(\frac{1}{50}))$$
$$= 180 \times Sin(Tan^{-1}(\frac{1}{50})) = 3.599280216cm$$

= 3.6cm Rise correct to 3 decimal places.

Therefore, the rise for each rest area is 3.6cm.

5. Determining the slopes section of the ramp: Using 1:20, 1:15, 1:12 only.

The overall rise required is 1.8m. Since this is greater than the maximum 500mm or 0.5m rise per flight from Table 1 we cannot use one flight (and no rests) only.

The same holds true for either 1 or 2 rest areas. Therefore we start with 3 rest areas.

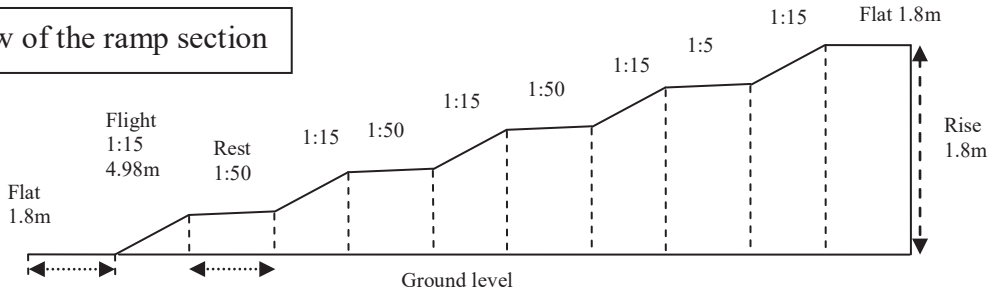
| **Part I** Calculation of Area for various ramp designs using gradients of 1:20, 1:15 and 1:12 only | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Rests | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Flights | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| Rise required = 180 – (3.6)x(No. Rests) Rise per flight = Rise required/No. Flights | | | | | | | | |
| Rise (cm) | 169.2 | 165.6 | 162 | 158.4 | 154.8 | 151.2 | 147.6 | 144 |
| Rise per flight (cm) | 42.3 | 33.12 | 27 | 22.63 | 19.35 | 16.8 | 14.76 | 13.09 |
| Max gradient (1/m) | 1:20 | 1:15 | 1:15 | 1:15 | 1:15 | 1:15 | 1:12 | 1:12 |
| Going cm (Rise x m) | 3384 | 2484 | 2430 | 2376 | 2322 | 2268 | 1771.2 | 1728 |
| Flight (cm) (by Pythagoras) | 3388.23 | 2489.51 | 2435.39 | 2381.27 | 2327.15 | 2273.03 | 1777.34 | 1733.99 |
| Length of the ramp inclined section = Flight length + (No. Rests)x(180.04) | | | | | | | | |
| Incline (cm) | 3928.35 | **3209.67** | 3335.59 | 3461.51 | 3587.43 | 3713.35 | 3397.70 | 3534.39 |
| Total length of the bridge = 2(Incline length) + 1840<br>Bridge Area = (Bridge length)x(180) | | | | | | | | |
| Bridge (cm) | 9696.70 | 8259.34 | 8511.18 | 8763.02 | 9014.86 | 9266.70 | 8635.40 | 8908.78 |
| Bridge Area ($cm^2$) | 1745406 | 1486681 | 153201 | 1577343 | 1622674 | 1668006 | 1554372 | 1603580 |
| Bridge Area ($m^2$ (2d.p.)) | 174.54 | **148.67** | 153.20 | 157.73 | 162.27 | 166.80 | 155.44 | 160.36 |

Table A

We observe that as the number of rest areas increases from 9 upwards the gradient is confined to 1:12 and that the overall length of the incline will continue to increase.

We see that the minimum area occurs when there are 4 rest areas and 5 flights of sloped sections at a gradient of 1:15.

The side-view of the ramp section

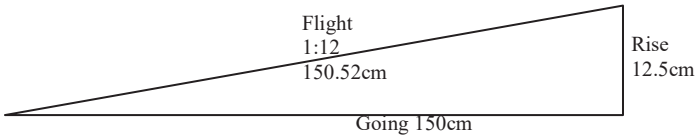



**The minimum area of the surface is 148.67m$^2$ under these conditions.**
An investigation of combinations of gradients does not show any reduction of area.

## Part II of the problem:

6. The span need not cross horizontally. We may convert the sections of the span which are not over water into sloped sections. Since this 1.5m section is shorter than 2m a gradient of 1:12 may be used.

By Pythagoras Theorem the flight length is 150.52cm.




$$\frac{Rise}{Going} = \frac{1}{12}$$
$$\Rightarrow Rise = \frac{Going}{12}$$
$$= \frac{150}{12} = 12.5cm$$

7. Re-calculate sloped section (see Table B). Rise =180- 12.5 = 167.5cm.

| **Part II** Calculation of Area for various ramp designs using gradients of 1:20, 1:15 and 1:12 only | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Rests | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Flights | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| Rise required = 167.5 – (3.6)x(No. Rests) Rise per flight = Rise required/No. Flights | | | | | | | | |
| Rise (cm) | 156.7 | 153.1 | 149.5 | 145.9 | 142.3 | 138.7 | 135.1 | 131.5 |
| Rise/flight (cm) | 39.18 | 30.62 | 24.92 | 20.84 | 17.79 | 15.41 | 13.51 | 11.95 |
| Max gradient (1/m) | 1:20 | 1:15 | 1:15 | 1:15 | 1:15 | 1:12 | 1:12 | 1:12 |
| Going cm (Rise x m) | 3134 | 2296.5 | 2242.5 | 2188.5 | 2134.5 | 1664.4 | 1621.2 | 1578 |
| Flight (cm) (Pythagoras) | 3137.92 | 2301.6 | 2247.48 | 2193.36 | 2139.24 | 1670.17 | 1626.82 | 1583.47 |
| Length of the ramp inclined section = Flight length + (No. Rests)x(180.04) | | | | | | | | |
| Incline (cm) | 3678.04 | 3021.76 | 3147.68 | 3273.6 | 3399.52 | 3110.49 | 3247.18 | 3383.87 |
| Total length of the bridge = 2(Incline length) + 4(180) + 2(150.52) + 820 = 2(Incline) + 1841.04<br>Bridge Area = (Bridge length)x(180) | | | | | | | | |
| Bridge (cm) | 9197.12 | 7884.56 | 8136.4 | 8388.24 | 8640.08 | 8062.02 | 8535.4 | 8608.78 |
| Bridge Area (cm$^2$) | 1655482 | 1419221 | 1464552 | 1509883 | 1555214 | 1451164 | 1536372 | 1549580 |
| Bridge Area (m$^2$ (2d.p.)) | 165.55 | **141.92** | 146.46 | 150.99 | 155.52 | 145.12 | 153.64 | 154.96 |

Table B

**The minimum area is 141.92m$^2$. The previous minimum was 148.67m$^2$.**
**The reduction in area is 6.75m$^2$.**

8. A combination of gradients may be used on Table B. For 5 rest areas and 6 flights we can use 3 flights of gradient 1:12 and 3 gradients of 1:15. The last of these flights at ratio 1:15 requires a rise of 32.83cm (rather than 33.33cm) and is therefore shorter. **The total bridge surface area in this case is 141.07m$^2$** which is smaller than the area previously calculated on Table B. (A combination gives no benefit on Table A.)

**The reduction then is: 148.67m$^2$ (Table A) – 141.07m$^2$ = 7.6m$^2$.**

## Addendum: Interpolated Gradients

9. The option of interpolating gradients could be used (from Diagram 1 of Part M) although this was not expected in submissions from students.

An interpolation formula may be established to give **Gradient = (100-Rise)/1000 where Rise is in cm.**

**Interpolated gradients for Part I**: The results are shown on Table C.

| Table C: For Part I: Interpolated gradients using Rise from Table A<br>Calculation of area for various ramp designs using interpolated gradients | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Rests | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Flights | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| Rise required = 180 – (3.6)x(No. Rests) Rise per flight = Rise required/No. Flights | | | | | | | | |
| Rise (cm) | 169.2 | 165.6 | 162 | 158.4 | 154.8 | 151.2 | 147.6 | 144 |
| Rise per flight (cm) | 42.3 | 33.12 | 27 | 22.63 | 19.35 | 16.8 | 14.76 | 13.09 |
| Max gradient (1/m) | 1:17.34 | 1:14.96 | 1:13.7 | 1:12.93 | 1:12.4 | 1:12.02 | 1:11.74 | 1:11.51 |
| Going cm (Rise x m) | 2933.93 | 2477.38 | 2219.4 | 2048.11 | 1919.52 | 1817.42 | 1732.82 | 1657.44 |
| Flight (cm) (Pythagoras) | 2938.8 | 2482.91 | 2225.3 | 2054.23 | 1925.75 | 1823.7 | 1739.09 | 1663.68 |
| Length of the ramp inclined section = Flight length + (No. Rests)x(180.04) | | | | | | | | |
| Incline (cm) | 3478.92 | 3203.07 | **3125.5** | 3134.47 | 3186.03 | 3264.02 | 3359.45 | 3464.08 |
| Total length of the bridge = 2(Incline length) + 1840<br>Bridge Area = (Bridge length)x(180) | | | | | | | | |
| Bridge (cm) | | 8246.14 | 8091 | 8108.94 | | | | |
| Bridge Area (cm$^2$) | | 1484305 | 1456380 | 1459609 | | | | |
| Bridge Area (m$^2$ (2d.p.)) | | 148.43 | **145.64** | 145.96 | | | | |

Table C

We do not need to calculate the other areas since the minimum length of incline gives the minimum area. In this case there are 5 rest areas and 6 flights with a gradient of 1:13.7.

**The minimum area of the surface is 145.64m$^2$ under these conditions.** (This is less than the minimum 148.67m$^2$ calculated on Table A.)

10. **Interpolated gradients for Part II**: Shown on Table D.

| Table D: For Part II: Interpolated gradients using Rise from Table B<br>Calculation of area for various ramp designs using interpolated gradients | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Rests | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Flights | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| Rise required = 167.5 – (3.6)x(No. Rests) Rise per flight = Rise required/No. Flights | | | | | | | | |
| Rise (cm) | 156.7 | 153.1 | 149.5 | 145.9 | 142.3 | 138.7 | 135.1 | 131.5 |
| Rise per flight (cm) | 39.18 | 30.62 | 24.92 | 20.84 | 17.79 | 15.41 | 13.51 | 11.95 |
| Max gradient (1/m) | 1:16.5 | 1:14.42 | 1:13.32 | 1:12.64 | 1:12.17 | 1:12 | 1:12 | 1:12 |
| Going cm (Rise x m) | 2585.55 | 2207.7 | 1991.34 | 1844.18 | 1731.79 | 1664.4 | 1621.2 | 1578 |
| Flight (cm) (Pythagoras) | 2590.29 | 2213 | 1996.94 | 1849.94 | 1737.63 | 1670.17 | 1626.82 | 1583.47 |
| Length of the ramp inclined section = Flight length + (No. Rests)x(180.04) | | | | | | | | |
| Incline (cm) | 3130.41 | 2933.16 | **2897.14** | 2930.18 | 2997.91 | 3110.49 | 3247.18 | 3383.87 |
| Total length of the bridge = 2(Incline length) + 4(180) + 2(150.52) + 820 = 2(Incline) + 1841.04<br>Bridge Area = (Bridge length)x(180) | | | | | | | | |
| Bridge (cm) | | | 7635.32 | | | | | |
| Bridge Area ($cm^2$) | | | 1374358 | | | | | |
| Bridge Area ($m^2$ (2d.p.)) | | | **137.44** | | | | | |

Table D

Using 5 rest areas and 6 flights the rise required is 149.5cm (see Table D). The interpolated gradient is 1:13.32. **This gives a final area of 137.44$m^2$.**
The reduction then is 145.64 (from Table C) – 137.44 = **8.24$m^2$** for interpolated gradients.

This gives a value for area less than any found previously.

## Summary:

From Table A: Minimum surface area: **148.67$m^2$.**
From Table B (non-horizontal span): Minimum surface area: **141.92$m^2$.** Reduction: **6.75$m^2$.**
From Table B with gradient combinations: Minimum surface area: **141.07$m^2$.** Reduction: **7.6$m^2$.**

From Addendum (interpolated gradients)
From Table C: Minimum surface area: **145.64$m^2$.**
From Table D (non-horizontal span): Minimum surface area: **137.44$m^2$.** Reduction: **8.24$m^2$.**

# Ten Reflections following the 'Fail Better' Exhibition

Neil Hallinan

## 1. Hippasus - on the Failure of Pythagoras

Pythagoras failed to measure a particular distance using the unit which was the spirit of existence. Pythagoras and his followers believed that rational numbers underlay everything. He realised that some measures were difficult to find. Of course, he did not believe that they were impossible. However, his disciple, Hippasus (c500 BC), saw that the failure was an opportunity to discover a new type of number to be known as an irrational number. His realisation came through his efforts of work on the dodecahedron (which has 12 regular pentagonal faces). We associate the simplest example of an irrational number with the square root of 2 or the length of the diagonal of a square with unit length sides.

## 2. A Failure of Space

It is said that the Greeks at that time could only establish the irrationality of whole numbers up to square root of 17. Perhaps they ran into limitations of margins for their constructions! That is an idle thought, but later, much later, c.1650, a French mathematician, Fermat, stated that the margin of his book was too small to hold a proof that he had found – a failure of space. This started a search for a proof which led to a multitude of other mathematical ideas. Then, about 350 years following Fermat's declaration, a proof was found for the theorem known as Fermat's Last Theorem. The solution was found by Wiles in 1994 using a particular type of failure as its keystone idea. Even as he produced his proof, Andrew Wiles produced a massive failure – his first public presentation of the proof was shown to contain a flaw.

## 3. A Failure by Laplace

In 1843, an Irish mathematician, Hamilton, eventually saw that it was the failure of a particular mathematical rule that would allow him to invent a new type of number known as quaternions. These were not really useful until recently when they became the answer to 3D animation problems. Hamilton, himself, first came to fame by spotting an error in a publication of work by Laplace. Laplace's failure was Hamilton's stepping-stone to success.

## 4. A Personal Failure

One of the greatest failures in mathematical history was a young man called Évariste Galois. He was a revolutionary in spirit against the king of France. But he failed to take part in the real revolution as he was locked up in school. Later, he failed in his romantic affairs and paid the penalty by dying at age 20, in 1832, following a duel with the chief suitor of the lady receiving his attentions. However, during the night before he died he wrote out many of his ideas on the theory of groups which were to become a foundation of later important mathematics. The theory of groups was used to conclude another famous failure – the impossibility of constructing the trisector of an angle using a straight edge and compass only — proved by Wantzel in 1837.

## 5. Number System Failures

It may be observed that it is failure in mathematics which drives it on to new discoveries. No new number system is required unless the previous one fails at some level. The closure of sets under a particular operation is really the failure of that operation to deal with other numbers. The failure of addition to work backwards leads to subtraction. The failure of integers to represent parts leads to rational numbers. The failure of rational numbers to express every measure leads to irrational numbers and the concept of real numbers. The failure of real numbers to supply answers to all quadratic equations led to complex numbers. Where does it end? Well it has been proved that there is another type of number beyond quaternions. It is octonions discovered by Graves in 1843 following the discovery of quaternions by his friend, Hamilton. But there it ends. Octonions are non-commutative and non-associative. The final failure is that there are no other types of numbers which constitute an algebra over the real numbers. Or the final success is that no other numbers are required.

## 6. Failure as a Method of Proof

The idea of failure is logically built into much of mathematics. One of the favourite methods of proof is the method of reductio ad absurdum – proof by contradiction where an accepted statement is shown to produce absurd results and therefore the statement must be rejected. Fermat was particularly adept at using this method of proof and used it in the Method of Infinite Descent. An example of this type of proof may be illustrated by proving the irrationality of the square root of 2. This is based on one of the earliest of mathematical concepts, that of odd and even numbers. The crux of the proof relies on the definition of a rational number as a number which is expressed as a ratio of two numbers which have no common divisor, ie especially they can't both be even numbers. The proof then goes on to show in turn that both of the numbers must be even when the square root of 2 is involved. This flatly contradicts the initial definition and the only conclusion is that the square root of 2 cannot be set up as being a rational number and must be a different type of number which can be called non-rational, or irrational.

## 7. An Urban Failure

One of the greatest cities of the world, for almost 1000 years, was Constantinople, now called Istanbul. For many centuries it held out against all assaults. It was impervious to siege. One emperor let his people know it was time for war by warning them that only people who had at least three years supply of food would be allowed to stay in the city. It was only when its defences eventually failed that the fleeing scholars brought ancient learning to medieval Europe and started the Renaissance which, of course, included the expansion of ideas in mathematics as well as a host of other disciplines.

## 8. "Try again. Fail Again. Fail Better."

For many people mathematics is seen as a perfect discipline. Its results are beyond doubt. It

thrives on completion and success. The ability to produce correct results is often portrayed as the definitive standard of a successful mathematician, especially at earlier levels. Up until the last century mathematicians were employed to do accurate calculations and were known as calculators and computers. However, as we have seen, the outstanding characteristic of many mathematicians is to accommodate failure, even to produce failure. This is a concept which is missing from pedagogy and the mind-set of our Irish culture.

In a sense we have to redefine the meanings of 'success' and 'failure'. What is success? For many, it is the winning of a competition, the acquisition of a trophy. For others, it is the accumulation of wealth, the revelling in the status promulgated by peers. And failure is then defined as the obverse of success. Failure is often seen as not winning, not triumphing over a beaten opponent, not having the 'trappings of success'. But the best definition of success that I have elicited from young students is that success means 'taking part'. Without competitors there can be no winner; without participation there is only marginalisation.

Those who participate have already succeeded. Failure becomes part of a developmental strategy within success. The Beckettian mantra 'Try again. Fail again. Fail better.' becomes a badge of success. This approach is already being built into the thinking of venture capitalists in the technology industry, especially in California. Companies looking for financial support will receive it only if they produce evidence of having already failed in a number of enterprises.

## 9. Progress through Failure

In mathematics this approach through failures is rarely rewarded in school systems. The number of times a student attempts a problem is sometimes overlooked; 'rough work' is cast aside - perhaps by the student him- or herself. The 'correct' version is applauded or rewarded more vigorously. In examinations, one attempt is generally advised at a problem, with a sharp eye kept to the clock in order to 'attempt all questions'. A candidate who struggles with a concept in a variety of ways is thereby penalised. What a poor approach to real mathematics!

Real mathematics entails an approach through successive failures (pun intended). The latest advances towards a proof of the Twin Prime Hypothesis illustrate this clearly. Early in 2013 a paper showed that the bounds of gaps within primes could be limited to some millions. This bound was rapidly reduced so that now it stands at a tempting 246. The succession of failures has been quite a success.

## 10. The Next Best Failure

From times of antiquity 'till the present day, the annals of mathematics are littered with the progress of failures. Conceptual failures, notational failures, failures of artefact. Throughout it all the underlying drive has been to produce a contribution to the development of the Queen of Sciences. The contribution of the next best failure is more important than mere immediate success. This concept should somehow be reflected in our presentation of mathematics and recognised in its awards.

The Fail Better Exhibition was held in Trinity College, Dublin, in February 2014. Neil Hallinan

# Developing problem-solving approaches to teaching: Theory and Practice

## Four corner-stones in the cultivation of mathematical problem-solving

Aoibhinn Ní Shuilleabháin

### Polya's Plan

George Polya's book, *How to solve it* (1945), is likely to have been one of the first books to focus on building students' skills as problem solvers. Polya, a Hungarian professor of mathematics, realised that it was not sufficient that his students knew their mathematical facts — they also needed to have a relational understanding of the subject in order to use mathematics as a tool. While Polya's book has provided much food for thought for mathematics educators at all levels throughout the decades, the legacy of his writing is in defining a heuristic or framework for students to solve problems:

1. Understanding the problem:
   What is the unknown? What are the data? Can you write them down?
2. Devising a plan:
   Have you seen it before or do you know a related problem? Do you know a more general problem? Can you derive anything useful from the question?
3. Carrying out the plan:
   Check each step and ensure you can prove it is correct.
4. Looking back:
   Check the result. Can you derive the result differently?

This heuristic has been used in varying forms in classrooms all over the world for students encountering questions or activities that are unfamiliar to them. However, providing students with this guiding framework is not necessarily enough in teaching students to become 'problem solvers'. Educational research is pointing out more and more the need for a classroom environment that nurtures students' thinking.

### The Results of Research: The Four Corner-stones

Amalgamating this literature, it seems there are four corner-stones to cultivating mathematical problem solvers. From Goos' (2004) study of developing a classroom community, Carpenter et al.'s (1989) research on teachers focusing on students' mathematical thinking and Schoenfeld's extensive writings on the teaching of mathematics these are summarised as the following:

- Building a classroom community which incorporates communication of students' mathematical thinking (Boaler, 1998; Schoenfeld, 1992)
- Focusing on the processes of students' thinking (van Es & Sherin, 2008)(Carpenter, Fennema, Peterson, Chiang, & Loef, 1989)
- Encouraging students to reflect on their work (Mason, 2002; Schoenfeld, 1994)
- Incorporating problems that are relevant to students' learning (Stanic & Kilpatrick, 1989)

Encouraging more communication of mathematical thinking is an important element of developing classroom community. Teachers elicit whole class discussions and provide opportunity for students to work together on problems but also give students time to read, think about and attempt problems on their own (Goos, 2004). Encouraging students to collaborate on attempting problems also develops their mathematical understanding through necessitating students to articulate and validate their thinking (Boaler, 1999).

By incorporating more of a focus on how students are thinking within the mathematics classroom, teachers build on their own pedagogical content knowledge and can better equip themselves in facilitating classroom discussions and in guiding student learning. Noticing and interpreting the strategies and language that students use in attempting to solve a problem helps teachers in further building students' mathematical thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; van Es & Sherin, 2008).

Reflecting on conjectures and answers is, according to Schoenfeld and Mason, one of the most important features of becoming a problem solver and builds a student's ability to perceive structures, see connections, capture patterns symbolically, conjecture and prove, abstract and generalise.

Finally, providing students with questions that are culturally relevant and provide meaning for their mathematical learning is important in order to engage students in solving unfamiliar problems (Galbraith, 2013; Stanic & Kilpatrick, 1989).

These points form the basis of my teaching to pre-service teachers who are studying Mathematics Pedagogy as part of their PME with the School of Mathematical Sciences in UCD.

Three such activities are suggested here which can be introduced to students as unfamiliar questions, but where students have enough prior knowledge to attempt the activity.

The first example may be presented as an initial introduction or further enquiry into quadratic patterns (modified from Driscoll, Mathematics Teacher magazine, 1999). The second example incorporates arithmetic series and the geometry of the circle and can be extended within calculus (modified from Paul Stephenson's activity in the Journal of the Association of Teachers of Mathematics, 2014 (242)). A third example is suggested by Schoenfeld (1994) (originally from the work of Seymour Papert) as a basis for building classroom discussions.

## The Problems

These activities are not proposed as a 'finished product' but as a starting point from which teachers can modify or extend for their own purposes and for their own students.

**Window Business Task**

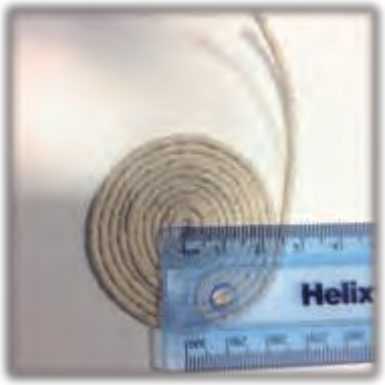
Liberty Hall building in Dublin is 16 stories high and is covered entirely with windows on all four sides. Each floor has 38 windows and once a year all the windows are washed. The cost for washing the windows is €2.00 for each first floor window, €2.50 for each second floor window and this pricing pattern continues for each floor.

How much will it cost to wash all of the windows of this building?

What if the building were 30 stories high?

What if it were *n* stories high?

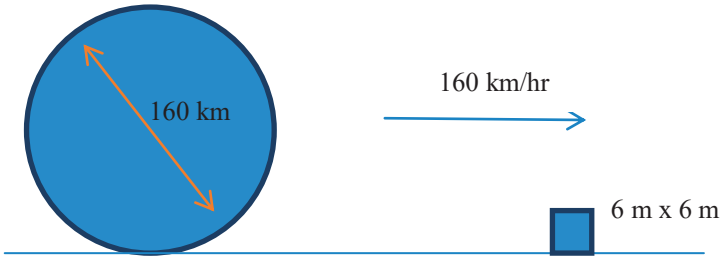
**How long is the string?**




**The Concrete Wheel Problem**

You are sitting in a room at ground level facing a floor-to-ceiling window which is 6 metres square. A huge solid concrete wheel, 160 kilometres in diameter, is rolling down the street and is about to pass right in front of the window from left to right. The centre of the wheel is moving to the right at 160 km per hour.

What does the view look like from inside the room as the wheel passes by?




## Conclusion

For scientists, mathematicians, programmers, and people in all manner of other jobs, solving problems is an integral part of their work. Over the past two decades, mathematics curricula all over the world have reformed to incorporate more of a focus on problem-solving in order to provide students with opportunities to build their knowledge of content through solving problems but also to improve their skills in attempting unfamiliar questions. Building a classroom community where ideas are shared, conjectures made and suggestions trialled is an important part of preparing our students for an environment which rewards problem solvers who learn from failing as much as from succeeding in their mathematical endeavours.

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Aoibhinn Ní Shuilleabháin Dublin Branch

# The IMTA Archives

The IMTA have succeeded in establishing an archive of documents in the NUI Maynooth Library. Donated material becomes the property of the University. The material will be documented by the Library. The IMTA have full access and may borrow material. Other users and researchers may view the material by appointment.

A sub-committee of the Council viewed the current set of donated material before approving its lodgement in the archive. This chiefly consists of Minute-books of Council meetings and the Newsletter Editorial Committee from 1964 to 1970s as well as correspondence with Branches regarding the organisation of courses during the same time-period. A full set of Newsletters from 1964 to 2013 has also been donated to the archive and will be lodged in the near future.

Donations of material for the archive are, at present, coordinated by Neil Hallinan.
Email: hallinann@gmail.com

# The Poisoned Wine Problem - Solution

See page 64 for the problem. Various solutions are presented here.

**1)** Use 1000 slaves. Do one test on Day 1. Each slave is assigned a barrel which they taste. Wait 30 days. One of the slaves dies. Result: One slave dies and the barrel is identified.

**2)** Variation of Solution No. 1: Use 999 slaves. If all survive then the one barrel not tested is the poisoned one. Result: Either zero or 1 slave dies and the barrel is identified in 30 days.

**3)** (Provided by a 5th year student). Use a time-slot of 1 minute for tests. Use 1 slave. Since 5 days are available for tests this amounts to 5x24x60 = 7200 minutes. Only 999 minutes are required (not 1000!). Tests may be carried out for 200 minutes per day (and 199 minutes on the final day). Wait the required 30 days, timed to the minute. When the slave dies the barrel may be identified. Result: One slave dies and the barrel is identified between 30 and 35 days.

**4)** Variation of Solution No. 3: There are 60x24 = 1440 minutes in one day. The slave may test the 1000 barrels in one day. Wait for 30 days, timed to the minute. The minute the slave dies will indicate the poisoned barrel. Result: A slave dies and the barrel is identified in 30 days.

**5)** Use 500 slaves. Since 500x2 = 1000, put 2 barrels in front of each slave. Each slave tests a barrel on Day 1 and a barrel on Day 2. Wait 30 days or 31 days. Result: One slave dies and the poisoned barrel is identified.

**6)** Variation of Solution No. 5: Use 200 slaves. Since 200x5 = 1000, put 5 barrels in front of each slave. Each slave tests a barrel each day for 5 days. Wait between 30 and 35 days. Result: A slave dies and the barrel is identified.

| Day 1 | Barrel No. 200 | **Barrel No. 1** | **Barrel No. 2** | Barrel No. 3 | Etc. |
|---|---|---|---|---|---|
| Tested by | Slave No. 100 | **Slave No.1** Slave No. 100 | **Slave No. 1** | **Slave No. 1** Slave No. 2 | Etc. |

Table 1

**7)** (See Table 1) Use 100 slaves. Since 100x10 = 1000, put 10 barrels in front of each slave, which is 2 per day in front of each slave for 5 days. The slaves stand in a circle. Each slave tests 3 barrels each day - their own two plus one belonging to the slave on their right.

If Barrel No. 1 is poisoned then Slave No. 100 and Slave No. 1 die 30 days later. If Barrel No. 2 is poisoned then only Slave No. 1 dies. Similarly, for each barrel, either one or two slaves die after the required 30 days. Result: One or two slaves die and thus the poisoned barrel may be identified between 30 and 35 days.

**8)** Variation of Solution No. 7: Use 50 slaves with 20 barrels in front of each slave (50x20 = 1000), which is 4 barrels per day for 5 days. Each slave tests 10 barrels each day – their own 4 plus 3, 2 and 1 from the next three slaves on their right. Up to 4 slaves may die in order to identify the poisoned barrel. Result: One, two, three or four slaves die and the poisoned barrel is identified between 30 and 35 days.

**9)** Further variation of Solution No. 7: Use 25 slaves. Since 25x40 = 1000, put 40 barrels in front of each slave, which is 8 barrels per day for 5 days. The slaves stand in a circle. Each slave tests 36 barrels each day – their own 8 plus 7, 6, 5, 4, 3, 2 and 1 from the next seven slaves on their right, respectively. From 1 to 8 slaves may die in order to identify the poisoned barrel.

**10)** Further variation of Solution No. 7: The limit of this process is when 15 slaves each has 14 barrels (one less than the number of slaves) before them each day. Since 15x14x5 = 1050, we can do this for 4 days. Each slave tests a total 14 + 13 + 12 + … + 3 + 2 + 1 barrels which is 105 barrels each day. Then 15x14x4 = 840 barrels are tested in 4 days. For the one remaining day only 160 barrels need to be tested. For this day 5 empty barrels are added to the stock to make

165 barrels. Since 15x11= 165, on that day each slave gets 11 barrels and each tests 66 barrels.

Result: From 1 to 14 slaves may die and the barrel is identified between 30 and 35 days.

| Day 1 | | | | | | | |
|---|---|---|---|---|---|---|---|
| Barrel No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Slave No. | 1 | 2 | 1, 2 | 3 | 1, 3 | 2, 3 | 1, 2, 3 |
| Binary total of slaves numbers | 1 | 10 | 11 | 100 | 101 | 110 | 111 |

Table 2

**11)** (See Table 2): Use 8 slaves. 200 barrels are tested each day for 5 days. Each barrel is assigned to a number of slaves. Barrel No.1 is given to Slave No.1. Barrel No.2 is given to Slave No. 2. Barrel No. 3 is given to Slaves No. 1 and 2. This is equivalent to counting the barrels in binary using the slaves as the counting pieces. Thus every barrel has a unique binary number. E.g. Barrel No. 15 is given as Barrel No. $1111_2$ which means it is tested by Slaves No. 1, 2, 3 and 4. Since $11111111_2 = 255$ we can see that 8 slaves are sufficient to test up to 200 barrels.

Result: From 1 to 8 slaves may die and the poisoned barrel is identified between 30 and 35 days.

**12)** Variation of Solution No. 11: Use 8 slaves. 255 barrels are tested each day for 3 days using the binary system of counting the barrels. This means that 255x3 = 765 barrels are tested in three days leaving 235 barrels to be tested on Day No. 4 which may be done in the same manner.

Result: From 1 to 8 slaves die and the poisoned barrel is identified between 30 and 34 days.

**13)** Further variation of Solution No. 11 (from *Algorithmic Puzzles*): Use 10 slaves. Test all 1000 barrels in one day. Each barrel is assigned a binary number from 0 to 999. This is from $0000000000_2$ to $1111100111_2$. Each slave is assigned a binary digit position and tests all barrels whose number contains that bit. The slaves who die will uniquely determine the barrel number. Notice that $0111111111_2$ has the highest number of 1-bits and therefore the highest number of slaves who may die is 9. One barrel is not tested. Result: From zero to 9 slaves die and the poisoned barrel is identified in 30 days.

**Summary of Results**

| Soln. | No. slaves used | Number of slaves who may or must die | | | | | | | | | | | | | | | No. Days |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1000 | | 1 | | | | | | | | | | | | | | 30 |
| 2 | 999 | 0 | 1 | | | | | | | | | | | | | | 30 |
| 3 | 1 | | 1 | | | | | | | | | | | | | | 30-35 |
| 4 | 1 | | 1 | | | | | | | | | | | | | | 30 |
| 5 | 500 | | 1 | | | | | | | | | | | | | | 30-31 |
| 6 | 200 | | 1 | | | | | | | | | | | | | | 30-35 |
| 7 | 100 | | 1 | 2 | | | | | | | | | | | | | 30-35 |
| 8 | 50 | | 1 | 2 | 3 | 4 | | | | | | | | | | | 30-35 |
| 9 | 25 | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | | | | | 30-35 |
| 10 | 15 | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 30-35 |
| 11 | 8 | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | | | | | 30-35 |
| 12 | 8 | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | | | | | 30-34 |
| 13 | 10 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | | | | | | 30 |

Table 3: A Summary of the Results

Note: No slaves were harmed in any way in the making of this article.

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