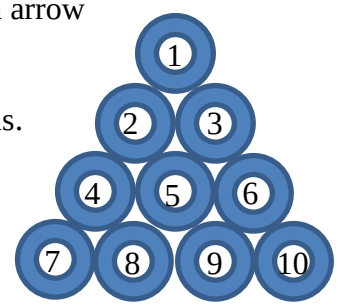
Peter's Problem 2019 Suggested Solution

Problem No. 1

Ten identical coins are arranged in a triangle as shown in the diagram as an arrow pointing upwards.



(i) Move 3 of the coins to new positions so that the arrow points downwards. Draw a diagram (with numbers) of the coins in their new positions. Highlight the numbers for the 3 coins which you have moved.

A further row of 5 coins is added to the given diagram to form a triangle of 15 coins in 5 rows.

(ii) What is the minimum number of coins that must now be moved into new positions in order to reverse the direction of the arrow?

Problem No. 1, Solution

(i) Move coins No. 1, 7 and 10 so that they take the positions indicated in the diagram. (Note: 1, 7 and 10 may be reordered in these positions.)

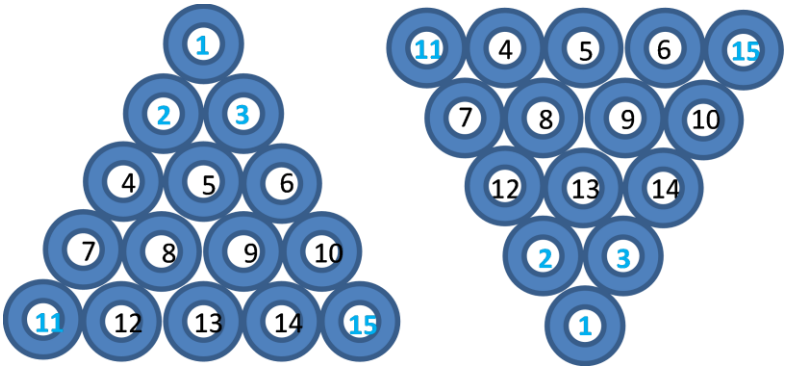
7 2 3 10

4 5 6

8 9

1

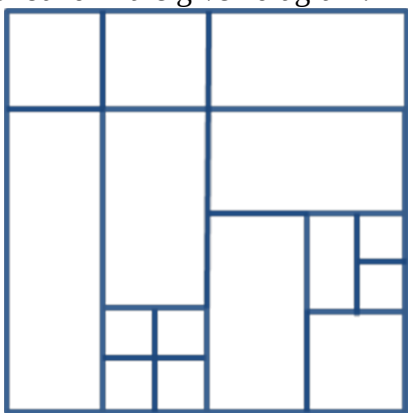
(ii) Move 5 coins – 1, 2, 3, 11 and 15. Moving a different set of 5 coins also reverses the arrow.



Additional information: In general, for N coins in such a triangular formation the number of coins required to move is given by dividing N by 3 and ignoring remainders.

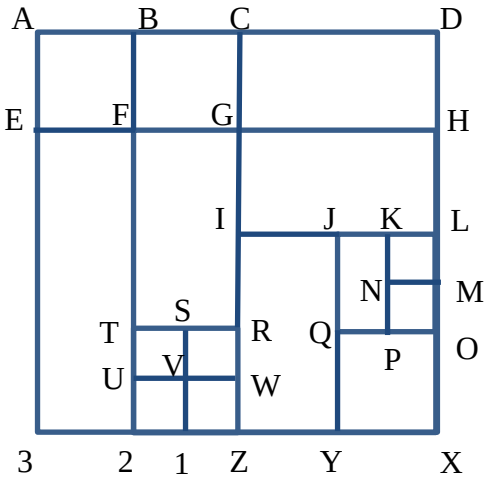
Problem No. 2

How many rectangles can be found in the given diagram?



Problem No. 2, Solution

A: ABFE, AB23, ACGE, ACZ3, ADHE, ADH3 (6)
B: BCGF, BCRT, BCWU, BCZ2, BDHF, BDX2 (6)
C: CDHG, CDLI, CDXZ (3)
E: EF23, EGZ3, EHX3 (3)
F: FGRT, FGWU, FGZ2, FHX2 (4)
G: GHLI, GHXZ (2)
I: IJYZ, ILX2 (2)
J: JKPQ, JLOQ, JLXY (3)
K: KLMN, KLOQ (2)
N: NMOP (1)
Q: QOXY (1)
S: SRWV, SRZ1 (2)
T: TXVU, TS12, TRWU, TRZ2 (4)
U: UV12, UWZ2 (2)
V: VWZ1 (1)



Total: 6 + 6 + 3 + 3 + 4 + 2 + 2 + 3 + 2 + 1 + 1 + 2 + 4 + 2 + 1 = 42 rectangles.

Problem No. 3

Beginning with 4, insert the mathematical signs +, –, × and ÷ along with brackets as appropriate between each of the digits so that this expression is correct:

4 6 5 3 8 7 9 2 1 = 58

Note: Brackets may be placed before the '4' or after the '1' if required.

Problem No. 3, Solution

Two examples from many possible:

$(4+6) \div 5+3+8 \times 7 - 9 \div (2 + 1) = 58$

$(((4 - 6 + 5) \times 3 \times 8 \times 7 \div 9) + 2) \times 1 = 58$

Problem No. 4
Replace the asterisks with the digits, 1, 2, 3, 4, 5, 6, 7, 8 and 9, one for each, so that the mathematical statement of subtractions is correct.

$$*** - *** - *** = 100$$

Problem No. 4, Solution
There are 168 possible answers. Answers may be found by trial and error or otherwise.

Examples: 536 – 147 – 289 = 100; 689 – 132 – 457 = 100; 968 – 123 – 745 = 100; etc.

An alternative method:

Use letters a, b, c, d, e, f, g, h, i for the asterisks. Then abc – def – ghi = 100.
Rewrite as: abc = def + ghi + 100
Rewrite vertically as:

$$\begin{array}{r} def \\ +\ ghi \\ \underline{+100} \\ abc \end{array}$$

We can form the equations:
A) i + f = c or B) i + f = 10 + c
C) e + h = b or D) e + h = 10 + b or E) e + h + 1 = b or F) e + h +1 = 10 +b
G) d + g + 1 = a or H) d + g + 2 = a since there cannot be a 'carry' of more than 1.

Thus, we have Case 1: Solving simultaneously A), C), G)

$$\begin{array}{l} i + f = c \\ e + h = b \\ \underline{d + g + 1 = a} \\ d + e + f + g + h + i + 1 = a + b + c \ (I) \end{array}$$

Introduce a further equation which can link the digits together
Since a, b, … , h, i are digits the sum of the digits from 1 to 9 is 45.
Therefore a + b + c + d + e + f + g + h + i + 1 = 46 (J)
So, d + e + f + g + h + i + 1 = 46 – (a + b + c)
Substituting (J) in (I) we get
46 – (a + b + c) = a + b + c
Therefore, 46 = 2(a + b + c)
Which implies a + b + c = 23.

Use the information gained from this equation to find three digits which add to 23.
The three digits are 9 + 8 + 6 which total 23. This is the only possibility.
The remaining digits are 1, 2, 3, 4, 5, 7.

Problem No. 4, Solution, continued

The number abc is composed of the digits 9, 8, 6. These may be in any order. Thus the possible values of abc are

986, 968, 869, 896, 698, 689.

The sum of def + ghi must then be 100 less than this. ie,

886, 868, 769, 796, 598, 589.

The sums of pairs of numbers in def + ghi must give the values 9, 8, 7, 6 or 5.

Using the set of 6 digits 1, 2, 3, 4, 5, 7

we see

9: 7 + 2; 5 + 4

8: 7 + 1; 5 + 3

7: 5 + 2; 4 + 3

6: 5 + 1; 4 + 2

5: 4 + 1; 3 + 2

There are 10 sums of pair values which are shown as follows:

132	312	123	321	321	312	123	241	132	214
+754	+574	+745	+547	+475	+457	+475	+357	+457	+375
886	886	868	868	796	769	598	598	589	589
+100	+100	+100	+100	+100	+100	+100	+100	+100	+100
986	986	968	968	896	869	698	698	689	689

In each of these sums the pairs may be inverted and still provide the same total although with different number values. This may happen in 2x2 = 4 ways since inverting all 3 pairs inverts the whole sum and provides no new number values.

For instance,

132
+754
886

may be given as

732 752
+154 or +134
886 886 etc.

Thus, there will be 4x10 or 40 different number solutions to the problem in this case.

Furthermore, the order of the numbers may be swopped in the subtraction using the format of the question. For instance 986 – 132 – 754 = 100 or 986 – 754 – 132 = 100.

Thus we have a total of 80 solutions here.

Further solutions:

The other combinations of equations (A) to (H) which could be combined are ADH, BEG and BFH since the 'carries' do not combine in the other combinations of equations.

Both ADH and BEG lead to the equation 18½ = a + b + c which is impossible.

Problem No. 4, Solution, continued

Equations BFH lead to the solution a + b + c = 14.
There are 8 combinations of 3 digits which produce this total.
1+4+9 =14; Remaining digits: 2, 3, 5, 6, 7, 8.
1+5+8=14; Remaining digits: 2, 3, 4, 6, 7, 9.
1+6+7=14; Remaining digits: 2, 3, 4, 5, 8,9.
2+3+9=14; Remaining digits: 1, 4, 5, 6, 7, 8.
2+4+8=14; Remaining digits: 1, 3, 5, 6, 7, 9.
2+5+7=14; Remaining digits: 1, 3, 4, 6, 8, 9.
3+4+7=14; Remaining digits: 1, 2, 5, 6, 8, 9.
3+5+6=14; Remaining digits: 1, 2, 4, 7, 8, 9.
There are 44 solutions produced in this group and with reversals of subtractions a total of 88.

Thus there are 80 + 88 = 168 solutions in total.

Problem No. 5

Each of the symbols represents a number. The addition totals for each row and column are given, except one.
Find the number represented by the * symbol.

20	27	20	17	
Θ	Θ	Θ	¥	29
¥	η	¥	%	*
%	η	%	%	13
¥	¥	¥	Θ	23

Problem No. 5, Solution

3 Θ + ¥ = 29 and Θ + 3¥ = 23. Thus 8 ¥ = 40 so ¥ = 5 and therefore Θ = 8.
Then Θ + ¥ + 2% = 17 , so 2% = 4, therefore % = 2.
Θ + ¥ + 2 η = 27 , so 2 η = 14, therefore η = 7.
Finally, 2 ¥ + η + % = *, so 10 + 7 + 2 = *, therefore 19 = *.

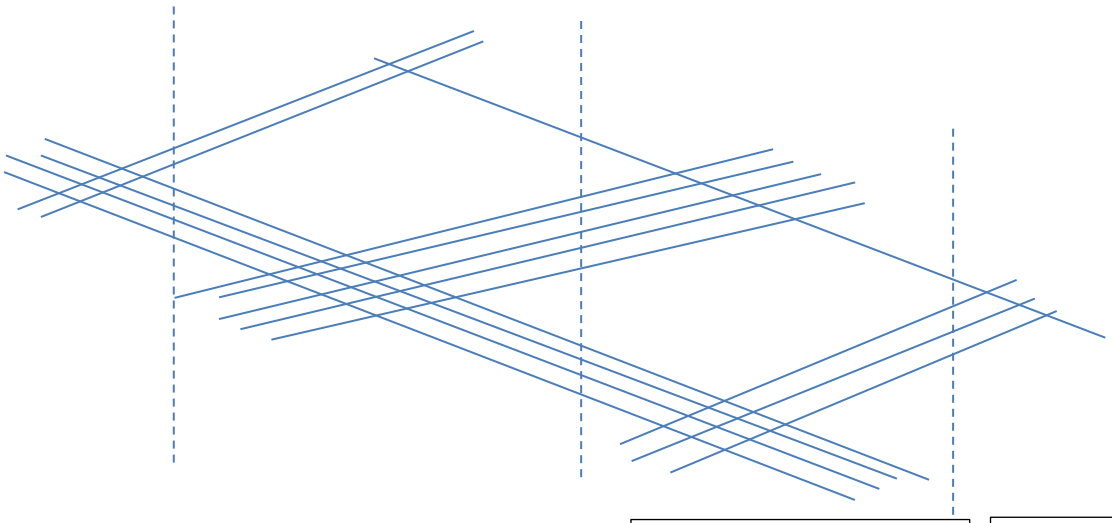
20	27	20	17	
8	8	8	5	29
5	7	5	2	*19
2	7	2	2	13
5	5	5	8	23

Problem No. 6
Log onto https://nrich.maths.org/5612 (Method in Multiplying Madness?)

(i) Multiply 253 by 41 using the "Multiplying with Lines" method, showing your work.
(ii) Write down two advantages and two disadvantages of using this method of multiplication.

Problem No. 6, Solution

253 x 41 = 10373



Crossing point: 2x4 = 8 +2 carried in Total: **10**	Crossing points: 2x1 = 2 4x5 = 20 +1 carried in:Total: 23. Unit: **3**; Carry 2	Crossing points: 5x1 = 5 4x3= 12 Total: 17 Unit: **7**, Carry 1	Crossing points 3x1 = 3 Unit: **3**;Carry: 0

Total: answer = 10373

Advantages: Accessible; Depends on addition or simple multiplication; Novel; Fun, etc.
Disadvantages: Cumbersome; Alignment of angle of lines to allow for addition of intersection points; Error prone if drawing not good, etc.

Problem No. 7
Write down any 3 digits and write down the same 3 digits again in the same order to make a 6-digit number. e.g. 123123 or 389389 or 306306, etc.

Explain why 7, 11 and 13 all divide evenly into all numbers of this form.

Problem No. 7, Solution
$7 \times 11 \times 13 = 1001$

For any 3-digit number, abc, we have (abc)(1001) = abcabc.

=>abcabc is divisible 1001 and hence by 7, 11 and 13.

Problem No. 8
In a triangle, one angle is twice the size of another and one angle is three times the size of another.

List the measure in degrees of all of the smallest angles which are possible.

Problem No. 8, Solution
Let the angles be labelled A, B and C where A+B+C = 180^o.
Then the possibilities for the sizes are:

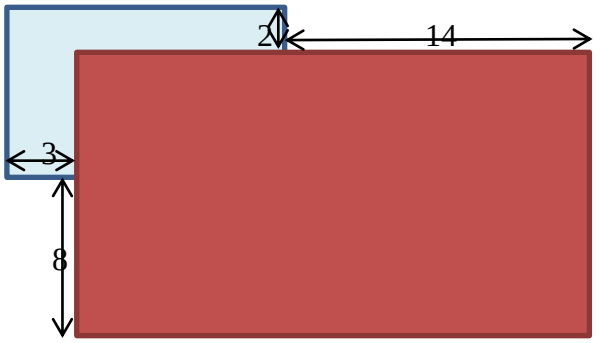
(1) A, B=2A, C=3A; => 6A = 180^o.
(2) A, B=2A, C= 3(B) = 3(2A) =6A; => 9A = 180^o.
(3) A, B=3A, C=2(B) = 2(3A) =6A; => 10A = 180^o.
(4) A, B=(3/2)A, C=2(B) = 2(3/2)A=3A; => (11/2)A = 180^o.

Solving each of these gives the results A = 30^o; A = 20^o; A= 18^o; A= $\frac{360^o}{11} = 32\frac{8}{11}^o$.

Alternative solution:
Let two of the angles be x and 2x.
Then the third angle is either 3x, 6x, $\frac{x}{3}$ or $\frac{2x}{3}$
If x,2x and 3x, x+2x+3x=180 => 6x=180 =>x=30^0.
If x, 2x and 6x, x+2x+6x=180 =>9x=180 => x=20^0
If x, 2x and $\frac{x}{3}$, x+2x+$\frac{x}{3}$ =180 => $\frac{10}{3}$x = 180 => x=54 and $\frac{x}{3}$ = 18^0
If x, 2x and $\frac{2x}{3}$, x+2x+$\frac{2x}{3}$ = 180=> $\frac{11}{3}$x= 180 => x=$\frac{540}{11}$ and $\frac{2x}{3}$ = $\frac{360}{11}$ = $32\frac{8}{11}^0$
Smallest angles are 18^0, 20^0, 30^0 and $32\frac{8}{11}^0$.

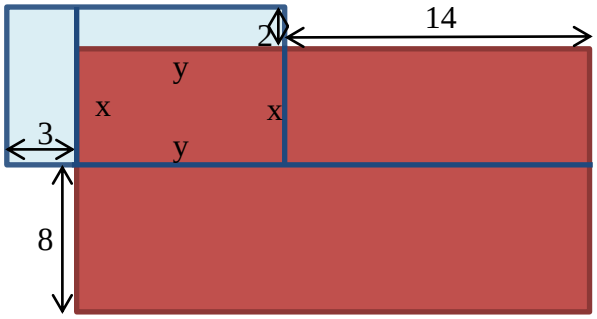
Problem No. 9

One rectangular sheet of paper is placed on top of another as shown.
Given that the dimensions of the larger sheet are double those of the smaller sheet, find the total area, in cm^2, covered by the sheets in the position shown. [Measurements are given in cm].

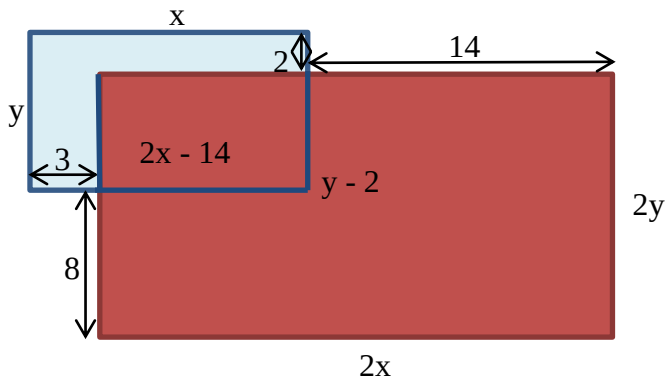


Problem No. 9, Solution

The shape may be divided into a number of sections as shown. 2(2+x) = 8 + x => 4 + 2x = 8 + x => x = 4
2(3+y) = 14 + y => 6 + 2y = 14 + y => y = 8
The smaller rectangle:(2+4) x (3+8) = 6x11.
The larger rectangle:(8 + 4) x (14 + 8) = 12x22.
For the five sections in the diagram the areas are:
3x6 = 18; 2x8 = 16; 4x8 = 32; 14x4 = 56; 8x22 = 176.
Total area = 18+16+32+56+176 = 298cm^2.

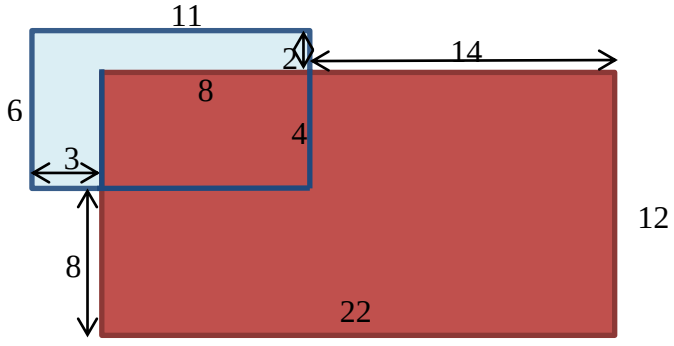


Alternative Solution:



2x – 14 + 3 = x
=> 2x – x = 14 – 3
=> x = 11.

2y = 8 + y – 2
=> 2y – y = 6
=> y = 6.

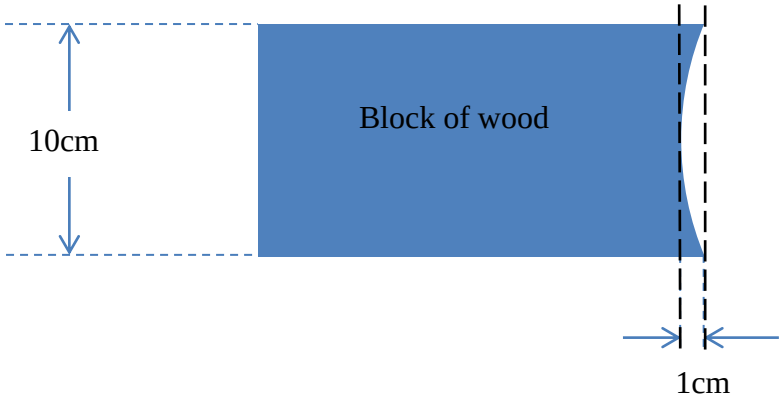


Area covered
= Area of 2 rectangles minus Intersection of the rectangles.
= 11x6 + 22x12 – 4x8
= 66 + 264 – 32
= 298cm^2.

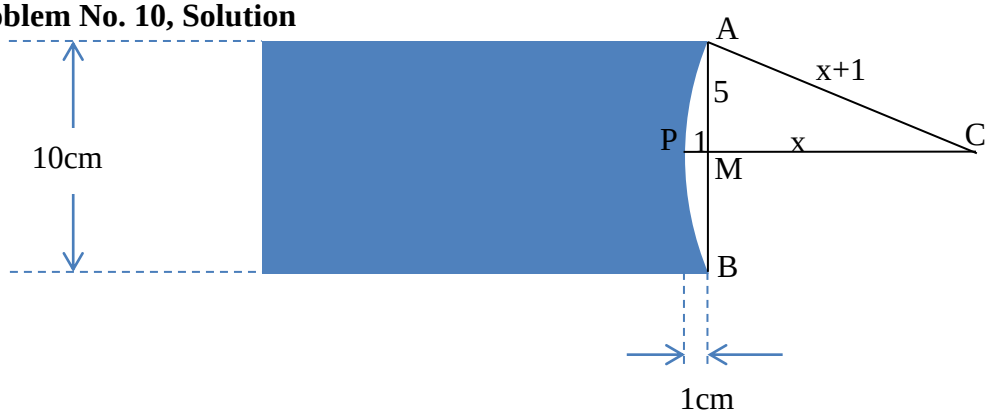
Problem No. 10

A circular cutting blade has been used to form the curve on a rectangular piece of wood as shown on the plan view given.

Using the dimensions given, find the radius of the cutting blade (in cm).



Problem No. 10, Solution



Draw the chord AB. Draw the bisector, D, of chord AB to meet the chord at M and the circle at P. Then |MP| = 1; |MA| = 5.

C is the centre of the circle

Join AC. Triangle CAM has a right angle at M.

Let |CM|=x cm => Radius is x+1 = |CA|

$(x+1)^2 = x^2 + 5^2$

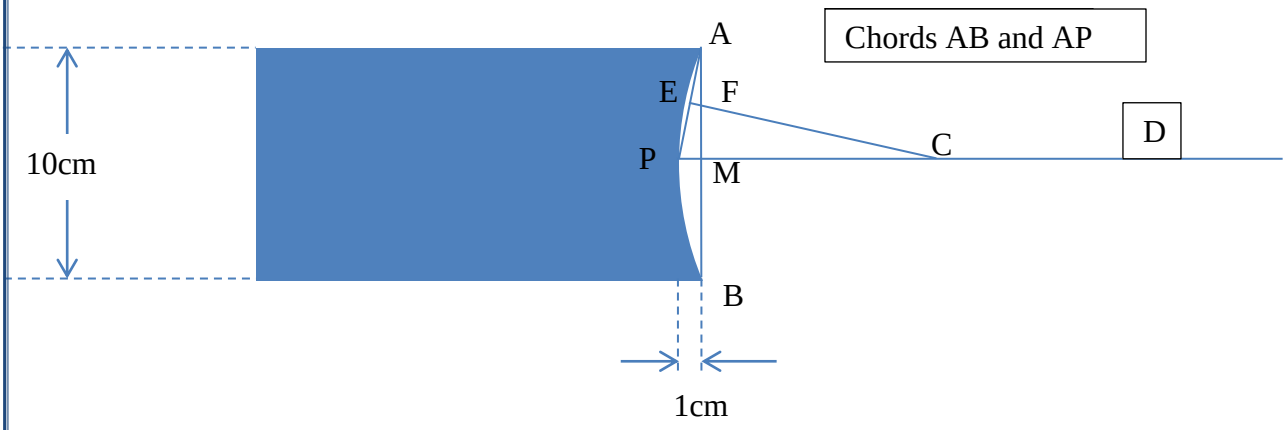
$x^2 + 2x + 1 = x^2 + 25$

$2x + 1 = 25$

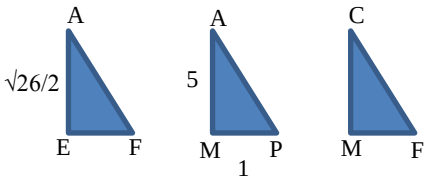
$2x = 24$

x =12 and the Radius is 13cm.

Problem No. 10, Solution, continued. Alternative solution:



Draw the chord AB. Draw the bisector, D, of the chord AB to meet the chord at M and the circle at P.
Then |MP| = 1; |AM| = 5.
Draw EC the bisector of chord AP to intersect line D at centre C. By Pythagoras |AP| = √26 and |AE| = √26/2.
AEF, AMP and CMF are similar triangles with right angles at E, M and M respectively and equal smallest angles at A, A and C respectively.



From triangles AEF and AMP, |MP| : |EF| = |AM|:|AE|
1 : |EF| = 5: √26/2; Therefore, |EF| = √26/10.
By Pythagoras |AF| = 26/10; |FM| = 5 – 26/10 = 24/10 = 12/5.
From triangles AMP and CMF, |MP|: |AM| = |FM|: |CM|
1:5 = 12/5 : |CM|
Therefore, |CM| = 60/5 = 12.
So, the radius of the circle is 12+1 = 13cm.

Problem No. 10, Further alternative solution:
Using PQ as diameter = 2R and similar triangles AMP and QAP (right angles at M and A, respectively) we have |MP|: |AP| = |AP|:|PQ|
1:√26 = √26: 2R
Hence 2R = 26
So, R = 13.
That is, the radius of the circle is 13cm.

Problem No. 11
Log onto www.steps.ie

(i) From the 12-15 section, list three types of engineers engaged in the production of chocolate and three functions of engineers in the presentation of the finished product.

(ii) From the 16-18 section list the 6 problems that engineers of the future will have to solve.

Problem No. 11, Solution

(i) Chemical engineer; biochemical engineer; environmental engineer.
Functions: Design, upgrade, maintain high speed machines.

(ii) Environmental pollution; Global warming; Healthcare improvements; Renewable energy; Water treatment; Flood prevention.

ProblemNo. 12
A chef made 4 mixes of orange squash and water in the following ratios
Orange squash : Water = 1:2, 1:3, 1:4 and 1:5.
The 4 mixes were placed in jars marked A, B, C and D respectively.

The chef then removed *k* ml from each of the jars marked A and D and poured each into a jar labelled X. She also removed *k* ml from each of the jars marked B and C and poured each into a jar labelled Y.

(i) Which of the jars, X or Y, contains the higher concentration of orange squash?
(ii) Why?

Problem No. 12, Solution
(i) Jar X has the higher concentration of orange squash.
(ii) Let the amount of each mix of squash removed be k = 600ml.
For jar X: A mix in the ratio 1:2 => Orange : Water = 200ml and 400ml
A mix in the ratio 1:5 => Orange : Water =100ml and 500ml
Therefore when these two are mixed there is 300 ml of orange and 900 ml of water

⇨ In this mix there is 25 % ($\frac{300}{1200}(100)$) orange

For Jar Y: A mix in the ratio 1:3 => Orange : Water = 150ml and 450ml
A mix in the ratio 1:4 => Orange : Water =120ml and 480ml
Therefore when these two are mixed there is 270 ml of orange and 930 ml of water

⇨ In this mix there is 22.5 % ($\frac{270}{1200} X\ 100$)orange

⇨ Jar X has higher concentration.

Problem No. 12, Alternative Solution to (ii)

(ii)

Consider two mixes of items Q and W, where the first is mixed in the ratio 1:A and the second is mixed in the ratio 1:B.

A combination of equal amounts of both mixes will have a ratio Q:W of $\left(\frac{1}{A+1}+\frac{1}{B+1}\right):\left(\frac{A}{A+1}+\frac{B}{B+1}\right)$ $= (A+1+B+1):(A(B+1)+B(A+1))$

This is Q:W = $(A+B+2):(2AB+A+B)$

For jar X, A= 2 and B = 5 so the ratio Q:W will be (2+5+2):(20+2+5) = 9:27.

For jar Y, A= 3 and B= 4 so the ratio Q:W will be (3+4+2):(24+3+4) = 9:31.

The ratio 9:27 shows that the squash is less diluted and therefore has a greater concentration of orange squash.

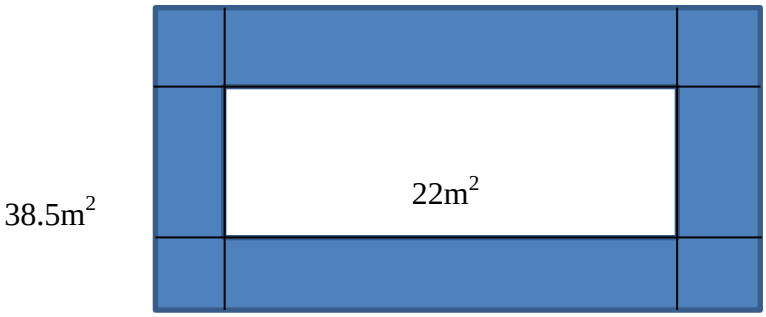
Alternatively, the ratio of squash (Q) in jars X and Y will be in the ratio 9(31):9(27). That is, in the ratio 31:27. Therefore there is more orange squash in jar X.

Problem No. 13
A prisoner was helping a contractor to repair the flat roof on his cell when he noticed that edges of the roof were flush with the outer surface of the walls of the cell.

(i) If the cell measured 4m by 5.5 metres internally and the roof had an area of $38.5m^2$, calculate the width of the wall, in cm, if all of the walls were the same thickness.

(ii) If the prisoner wished to escape by tunnelling through the wall at in average rate of 1.3cm per day, on what date should he begin the tunnel if he is to escape on March 2th, in time for the beginning of Engineers Week, 2019?

Problem No. 13, Solution



(i) Let x = Width of a wall.
Length of the roof = 5·5 + 2x; Width of the roof = 4 + 2x.
Area of the roof = $(2x + 5{\cdot}5)(2x + 4) = 38{\cdot}5$
$4x^2 + 8x + 11x + 22 – 38{\cdot}5 = 0$
$4x^2 + 19x – 16.5 = 0$
$8x^2 + 38x – 33 = 0$
$(2x + 11)(4x – 3) = 0$
$2x = – 11$ or $4x = 3$. The negative answer is rejected.
x = 0·75m = 75cm which is the width of the wall.

(ii) 75÷1·3 = 57·69.
It will take the prisoner 58 days to tunnel through the wall. The prisoner has to start tunnelling on the 4th of January, 2019 (not a leap year).
Note on calculation of dates: Jan 4th counts as Day 1. Therefore digging 28 days in Jan + 28 days in Feb + 2 days in March to exit on 2nd March.

Alternative solution to (i):
(i) Let x be the combined width of the walls on two sides. Then the dimensions of the roof are (x+4) by (x+5·5).
$(x+4)(x+5{\cdot}5) = 38{\cdot}5 \Rightarrow x^2 + 9{\cdot}5x + 22 = 38{\cdot}5 \Rightarrow x^2 + 9{\cdot}5x - 16{\cdot}5 = 0$
Then $2x^2 + 19x – 33 = 0$ so $(2x-3)(x+11) = 0$.
$2x = 3$ or $x = – 11$. The negative answer is rejected.
Therefore, x = 1·5 which is the width of two walls. The width of one wall is 0·75cm.

Problem No. 14
Choose any two natural numbers each less than 10.
Select one of the numbers and add 1, multiply by 5, add 1 again, double your answer and subtract 1. Add the second number to your answer, add 2, double again, subtract 8 and halve this number and write down your answer.

Write down the rule that identifies from the answer the two numbers selected at the start.

Problem No. 14, Solution
Choose 3 and 7. Select 3.
Add 1: 4; Multiply by 5: 20; Add 1: 21; Double: 42; Subtract 1: 41; Add 2nd(7):48; Add 2: 50; Double: 100; Subtract 8: 92; Half: 46.
Use 46 to identify original numbers: The 1st number chosen is 1 less than the digit in the 10's column; The 2nd number chosen is 1 more than the digit in the units column.
Justification (not asked): The algorithm entails that if x is the 1st number and y is the 2nd number then the result is 10x + y + 9. Multiplying a single digit number (x) by 10 (10x) will place that digit in the 10's column. Adding 9 to a single digit number (y+9) will give one less than that digit plus 10 (y-1+10). The '1' of this 10 is added to the 10's column thus giving a digit one more than the 1st digit chosen (x+1 in the 10's column). The digit in the units column will be one less than the second digit chosen (y-1 in the units column).

Problem No. 15
How many Natural numbers less than 1000 have the same remainder when divided by 3, 4 and 5?

Problem No. 15, Solution

Remainders for division by 3, 4, 5 coincide for multiples of 60, multiples of 60 plus 1; multiples of 60 plus 2. There are 17 multiples of 60 (including multiple by 0) less than 1000. Therefore there are 3x17 = 51 such numbers including 0. For Natural numbers this gives a total of 50 numbers.

A list of these numbers: 1, 2; 60, 61, 62; 120, 121, 122; … … 960, 961, 962.

Problem No. 16

A student began to count on the fingers of his left hand. He started by calling the thumb 1, the first finger 2, middle finger 3, ring finger 4,little finger 5, then reversed direction calling the ring finger 6, middle finger 7, first finger 8, thumb 9, first finger 10 and so on.

(i) On which finger did he reach 50?

(ii) On which finger would he stop on a count to 2019?

Problem No. 16, Solution

(i) A direct count shows that the count of 50 ends on the 1st finger.
Alternatively, using the formula produced below, 50÷8 = 6R2. Remainder 2 indicates that the count will end on the 1st finger.

Note: Given any number N the formula is 'Divide the given number by 8 and use the Remainder to indicate the required finger'. This is given in the 3rd column in the chart.

	Observed pattern n = 0, 1, 2, …	**Formula for any given number N: N ÷8 + Remainder, R**
Thumb	8n + 1	**R = 1: Thumb**
1st Finger	8n (n>0) and 8n + 2	**R = 0 or 2: 1st finger**
Middle Finger	8n + 3 and 8n + 7	**R = 3 or 7: Middle finger**
Ring Finger	8n + 4 and 8n + 6	**R = 4 or 6: Ring finger**
Little Finger	8n + 5	**R = 5: Little finger**

(ii) Using the formula 2019÷8 = 252 with Remainder 3 indicates that the count will end on the Middle finger.

Problem No. 17

A railway depot is fitted with a track for trainee engine drivers. It contains two freight carriers, A and B, and an engine, E. The track is circular in shape with two sidings as shown in the diagram. It has a tunnel, T, which is large enough for the engine to pass through but not the freight carriages. At the start of the training exercise the engine is placed at the entrance to a siding (X) with the carriages A and B next to Tunnel North in the order shown in the first diagram. The trainee's task is to use the engine to bring the carriages and the engine to new positions as shown in the second diagram, with the carriages next to Tunnel South in the order shown and the engine at the entrance to Siding South, Y.

In order to push or pull a carrier the driver has to hitch the engine onto it. When finished moving that carrier the driver has to unhitch the engine from it.

(i) How many '**hitch-unhitch the engine'** operations are required, at minimum. ie, how many times the engine has to be hitched to a carriage, some movement or movements carried out and then the engine unhitched from that carriage.

(ii) Describe, with the help of diagrams similar to the ones given, the moves required for the driver to move from the position shown in the first diagram to that in the second diagram giving an account of the 'hitch-unhitch the engine' operations.

[Notes:
1. From inside the cab the driver may operate the track siding-switch to allow a change to the rail from the circular track to the straight track.

2. There is no left-hand turn into the top siding travelling clockwise nor is there a right-hand turn into the lower siding travelling anti-clockwise.

3. For descriptive purposes the labels North Siding (SN), South Siding (SS), Track switch point entrance to Siding North (X) and Track switch point entrance to Siding South (Y), North side of Tunnel (TN) and South side of Tunnel (TS) may be used as in the third diagram.]

Diagram No. 1

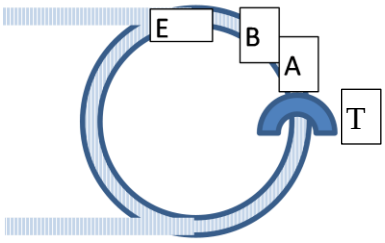


Diagram No. 2

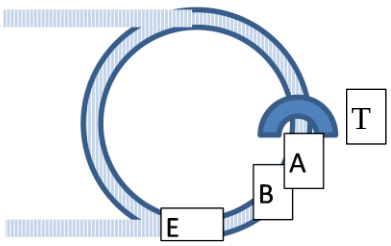
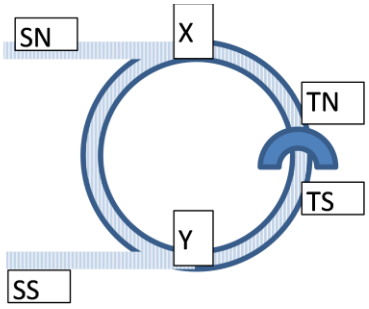


Diagram No. 3



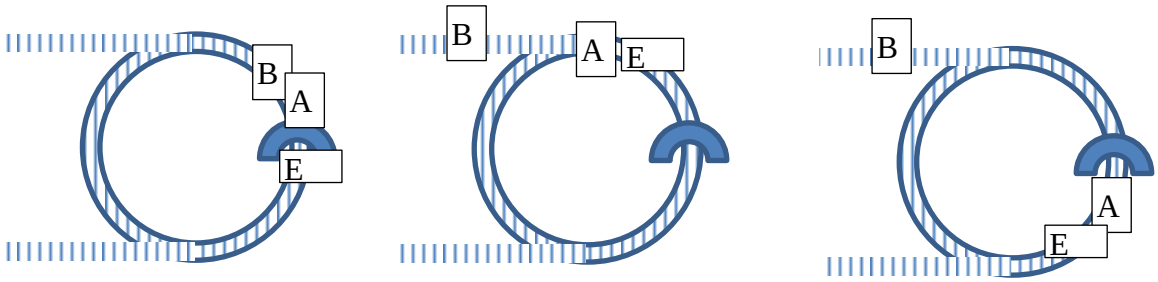
Problem No. 17, Solution

(i) There are 2 hitch-unhitch the engine operations required, at minimum.

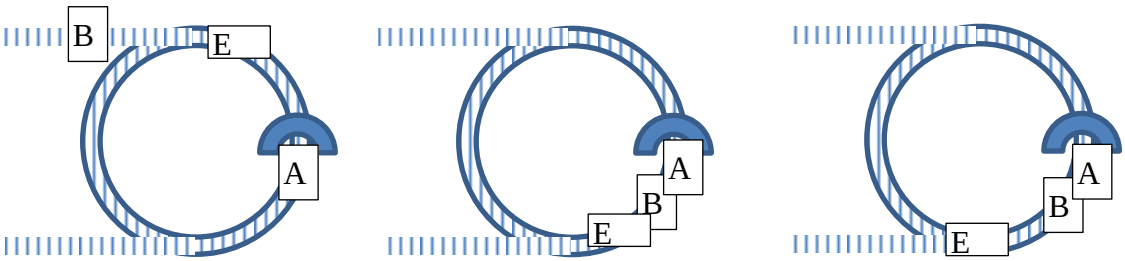
(ii)

1. Move E anticlockwise and go through the tunnel and **hitch the engine to A**. A and B should be safely connected. Push A and B into siding SN. Disconnect A from B. Pull A to X. Push A to Tunnel South. **Unhitch the engine from A**. [1]
2. Move E clockwise to X. Move to Siding North. **Hitch the engine to B**. Pull B to X. Push B anticlockwise to meet A at Tunnel South. **Unhitch the engine from B** [2]
3. Move E to position Y.

There are 2 'hitch-unhitch the engine' operations involved.



Stages of Step 1



Stages of Step 2

Stages of Step 3

Problem No. 18
Table-tennis balls come in boxes of 8 and in boxes of15.
Viz., an order for 38 balls may be filled without breaking open any box by sending 2 boxes of 15 and 1 box of 8 while an order for 37 balls cannot be filled without breaking open a box.

What is the maximum number of balls that cannot be bought without breaking open a box?

Problem No. 18, Solution
Answer: 97 is the maximum.
Explanation: 8x: **multiples of 8** ($x \geq 0$)
0, 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, …
15y: multiples of 15 ($y \geq 0$)
0, 15, 30, 45, 60, 75, 90, 105, …

8x + 15y: combinations of multiples of 8 with multiples of 15. Missing numbers cannot be expressed as a combination of multiples of 8 and 15. These can be ordered as follows:
0, __8, **Two Groups of 1; 7 numbers missing between both and 6 missing after 2nd group**
15, 16,___23, 24, **Two groups of two numbers; 6 numbers missing between and 5 after**
30, 31, 32,___38, 39, 40, **Groups of 3 numbers; 5 numbers missing between and 4 after**
45, 46, 47, 48, ___53, 54, 55, 56, **Groups of 4; 4 missing between and 3 after**
60, 61, 62, 63, 64,___68, 69, 70, 71, 72, **Groups of 5; missing 3 between and 2 after**
75, 76, 77, 78, 79, 80, ___83, 84, 85, 86, 87, 88, **Groups of 6; missing 2 between and 1 after**
90, 91, 92, 93, 94, 95, 96,___98, 99, 100, 101, 102, 103, 104, **missing 1 [97] and none after.**
105, 106, 107, 108, 109, 110, 111, 112,__113, 114, 115, 116, 117, 118, 119, 120, **none missing between groups or after the groups**
120, 121, 122, 123, 124, 125, 126, 127, 128, 128, 129, 130, 131, 132, 133, 134, 135, 136, 135, ….
Thus, 97 (between 96 and 98) is the maximum number of balls which cannot be bought without breaking open a box.

Alternative solution
Any number which leaves Remainder 1 on division by 8 cannot be expressed as a combination of multiples of 15 with multiples of 8 until the number itself is a multiple of 15. This number is 105.
The previous such number leaving Remainder 1on division by 8 is 105 – 8 = 97.
The complete set of numbers which are not expressible as a combination of multiples of 8 and multiples of 15 can be found by examining remainders in this manner and are given in the following list.

Remainders on division by 8 until a multiple of 15 is reached														
0														
1	9	17	25	33	41	49	57	65	73	81	89	97	**105=7x15**	
2	10	18	26	34	42	50	58	66	74	82	**90=6x15**			
3	11	19	27	35	43	51	59	67	**75=5x15**					
4	12	20	28	36	44	52	**60=4x15**							
5	13	21	29	37	**45=3x15**									
6	14	22	**30=2x15**											
7	**15=1x15**													

Problem No. 19

Three people – yourself and two friends – are faced with the problem of crossing a rope bridge which is only strong enough to hold any two at a time. Furthermore, a single special token used to open and close the gates at either end as people pass through must be brought by whichever person or group crosses the bridge. It must then be brought back before the next person or group can cross. Individually, the crossing times are 1 minute, 2 minutes and 10 minutes for yourself and your two friends, respectively.

Before you can cross the bridge a third friend arrives who also wishes to cross the bridge with you. You ask this third friend how long it takes them to cross the bridge if they were to cross on their own. When you are told you are happy that when the four of you cross as a group it will only add a further 4 minutes to the minimum crossing time that you had calculated for yourself and your two friends at the start.

What are the smallest and the greatest lengths of time (in whole numbers of minutes) that your third friend could have told you that he or she would take to individually cross the bridge?

Problem No. 19, Solution

We can refer to the people as A who takes 1 minute, Bwho takes 2mins., C who takes 10 mins. and D whose time of crossing we have to find.

A(1) and B(2) cross the bridge. Time taken = 2 mins.
A(1) crosses back. Time taken = 1 min.
A(1) and C(10) cross the bridge. Time taken = 10 mins.
Total time taken = 13 minutes.

The newcomer (D) adds 4 minutes to their time.
They could cross in 17 minutes.
The smallest length of time D could take is 2 minutes.
The greatest length of time D could take is 10 minutes.

Reasons:

A(1) and B(2) cross the bridge. Time taken = 2 mins.
A(1) crosses back. Time taken = 1 min.
C(10) and **D(2, 3, 4, 5, 6, 7, 8, 9, or 10**) cross the bridge. Time taken = 10 mins.

Notice that it does not matter what time D takes once it is between 2 and 10 minutes since he or she is 'carried' by C who takes 10 minutes anyway.

B(2) crosses back. Time taken = 2 mins.
A(1) and B(2) cross the bridge. Time taken = 2 mins.
Total time taken = 2 + 1 + 10 + 2 + 2 = 17 minutes.

Problem No. 20
Prove that every odd integer lies exactly halfway between a multiple of 3 and a multiple of 4.

Problem No. 20, Solution
Any odd number, n, can be written as $n = 2k + 1$, $k \varepsilon Z$.
We note that n lies half way between 6 and $2n - 6$ since $(6 + 2n - 6)/2 = n$.
6 is a multiple of 3.
Consider $2n - 6$
$2n - 6 = 2(2k + 1) - 6 = 4k + 2 - 6 = 4k - 4 = 4(k - 1)$ which is a multiple of 4.
Therefore the odd number n lies halfway between a multiple of both 3 and 4.

Problem No. 21
Log on to www.math.harvard.edu/~knill/mathmovies/swf/maandpakettleaddition.html and view the Ma and Pa Kettle film (Length 2:15).

(i) What answer does Pa get when he divides 25 by 5?
(ii) What mathematical error does Pa make in his calculation?

Problem No. 21, Solution
(i) 14.
(ii) Pa transforms the Unit digit 1 (which he correctly got from division of 5 by 5) into a Tens digit 1 by placing the 4 next to it which is then combined with the Unit digit 4 to give 14 instead of placing the 4 under the 1 and adding 1 and 4 to get the correct answer 5.