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Editorial

Welcome to this edition of the Newsletter. As promised in Newsletter 104 we have a number of special contributions in this issue on the subject of Sir William Rowan Hamilton. This is to commemorate the bi-centenary of his birth. The bi-centenary celebrations took place last year—2005.

As you read through the Newsletter let your mind wander on a flight of fancy. Imagine, if you will, that you are walking with Hamilton along the banks of the Royal Canal. While you walk you are regaled with stories of his youth, his genius and his humanity by Dr. Fiacre Ó Cairbre. As you come to the spot where Hamilton experienced his 'electric shock' of a brainwave you will see a gathering crowd and hear them singing a rousing ditty and, of course, you may join in — music and words are supplied by Jack Gannon. Listen quietly, too, to a paean composed by June Robinson in Hamilton's honour.

It will be time, then, to get out the pen and scratch, not on stone, but head and paper as you follow a few simple rules derived from quaternions and perhaps succeed in impressing those many friends with majestic sums of two and four squares. Professor Anthony G. O'Farrell has indeed supplied a gem of mathematics which glistens with elegance.

Still there is more to come regarding Hamilton. For next we find that despite his great mathematics he is more famous yet for his contributions to physics. Indeed, he is seen as a man who changed the world and ushered in the scientific developments which typified the twentieth century and continue today. Professor Siddhartha Sen gives a glimpse of the regard with which Hamilton is viewed all over the world. Few are the people who have such magnitude of influence as Hamilton and it is a joy to celebrate the fact of his birth in this country.

Leaving the nineteenth and twentieth centuries behind we find ourselves back once more with some modern applications of mathematics. There is the fascinating insight into the work of an actuary given by a recent mathematics graduate, Philip Mullen. This is followed by a delightful description of mathematics within a card-trick supplied by a Young Science Exhibition winner, Eamonn Flannery. Now you can practice, practice, practice both your magic and your mathematics!

Also, as promised, there is a comprehensive look at the mathematics of voting in our country. A close eye to those additions and subtractions will obviously pay off if you are an astute vote-manager. But you will have to be careful! Too many votes for your candidate and they may lose their seat! Can it be possible? The mathematics prove it to be true. In a splendid monograph Dr. Helmer Aslaksen and Dr. Gary McGuire explore the detail which is within our voting system. In the near future, as parcel and sub-parcel of ballots tumble from the tally-person's hand you may be sure that the election scrutineers will keep an eagle eye on the sub-parcels which go on top of the existing piles.

To finish, there are the 'bread-and-butter' issues of Leaving Certificate Mathematics where Paul Holland provides an insight into 'Tables Page 9' and Applied Mathematics Solutions from John Maher which are much appreciated.

All in all, the contributions to this Newsletter should form a valuable source of reference as well as of fun and recreation. With sincere thanks to all contributors.

Indeed, contributions continue apace, I am glad to say, and in the next issue you may look forward to articles on prime numbers as well as how to better type your mathematics onto a computer word-processor. We should have results of Maths Competitions around the country as well as many other items.
All contributions are welcome.
Send by e-mail to : hallinann@eircom.net
or St. Mary's, Holy Faith, Glasnevin, Dublin 11.

Neil Hallinan

Branches : Contacts

Cork (Sec.): *Brendan O'Sullivan*, bos4@esatclear.ie
Donegal (Sec.): *Joe English,* mathsjc@eircom.net
Dublin (Sec.): *Barbara Grace,* barbara_grace_2004@yahoo.ie
Galway (Tr.): *Mary McMullin*, cmcm@iol.ie
Kerry (Tr.): *John O'Flaherty,* flahjohn@eircom.net
Limerick (Chair): *Gary Ryan,* theboyryan@hotmail.com
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Midlands (Sec.): *Dominic Guinan,* domguinan@eircom.net
Tipperary (Chair): *Donal Coughlan,* donal.coughlan@esatclear.ie
Waterford (Sec.): *Michael Brennan*, brennanwaterford@iol.ie
Wexford (Sec.): *Sean McCormack*, seanmaccormaic@yahoo.co.uk

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William Rowan Hamilton and The Quaternion Walk

Every year on October 16 the Department of Mathematics at NUI, Maynooth organise a public walk which retraces William Rowan Hamilton's steps from Dunsink Observatory to Broombridge where he conceived his famous Quaternions. It was on this day in 1843 that Hamilton created Quaternions as he walked along the banks of the Royal Canal.

In 1990 Professor Anthony G. O'Farrell (NUI, Maynooth) initiated the annual walk and it now attracts about 150 people across a wide spectrum of academics, second level teachers and pupils and members of the general public. In recent times the walk has typically been launched in Dunsink by a Nobel Prize winner or a Fields Medallist. In 2003 Andrew Wiles who proved Fermat's Last Theorem launched the walk to great media attention including the RTÉ 6 o'clock news. In 2005 Hamilton's great-great grandson, Mike O'Regan, participated in the walk. The local community around Broombridge in Cabra have really embraced the walk and last year (16th October, 2005) they had a large banner draped across the bridge and provided refreshments for the walkers there. All are welcome to participate in the walk and it is particularly suitable for a Transition year field trip. Contact the Department of Mathematics at NUI, Maynooth for details.

Anthony G. O'Farrell speaking at Broombridge during the walk on October 16, 2005

William Rowan Hamilton (1805-1865) is Ireland's greatest mathematician and one of the world's most outstanding mathematicians ever. The year 2005 marked the bicentenary of his birth and the Irish Government designated it `**Hamilton Year – Celebrating Irish Science'**. Many events took place across the country. There was a commemorative stamp and

coin. Léargas produced a television documentary on Hamilton on RTÉ and there was also a radio programme about him as part of the Icons of Science series. On August 4 the Cabra community celebrated his birthday with a huge party. The Royal Irish Academy put on a series of public Hamilton lectures in addition to their annual Hamilton lecture which occurs around October 16. A poem about Hamilton entitled *`The Benefactor'* was written by June Robinson. Back in 2003 a song called *`The Ballad of Rowan Hamilton'* was written by Jack Gannon.

In 1865, shortly before he died, Hamilton was elected as the first Foreign Associate in the newly established National Academy of Sciences of the US. This meant that the Academy considered him to be the greatest living scientist. This honour was richly deserved as Hamilton has emerged as a giant in the history of science.

Quaternions

The quaternions are Hamilton's most celebrated contribution to mathematics. Number couples *(a,b),* where *a* and *b* are real numbers, had been important in mathematics and science, and he was trying to extend his theory of number couples to a theory of `number triples' (or triplets) which are of the form *(a,b,c),* where *a, b* and *c* are real numbers. He hoped these triplets would provide a natural mathematical structure and a new way for describing our three-dimensional world in the same way that the number couples played a fundamental role in two-dimensional geometry. He was having a hard time in his search for a suitable theory of triplets (we now know it's impossible to find a suitable theory of triplets). Then, on Monday 16 October 1843, his mind gave birth to quaternions as he walked on the banks of the Royal Canal. He recalls the event, in a letter to his son Archibald, one month before his death in 1865:

> *Although your mother talked with me now and then, yet an undercurrent of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once an importance. An electric current seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work... Nor could I resist the impulse — unphilosophical as it may have been — to cut with a knife on a stone of Brougham Bridge as we passed it, the fundamental formula ...*

Hamilton realised that if he worked with `number quadruples' and an unusual multiplication operation, then he would obtain all that he desired. He called his new system of numbers `Quaternions' because each number quadruple had four components. He had created a completely new structure in mathematics. The mathematical community was stunned by his audacity in creating a system of numbers that did not satisfy the usual commutative *(xy = yx)* rule of multiplication.

A quaternion is an expression of the form $a+ bi+ cj+ dk$ where a, b, c, d are real numbers. Addition is defined componentwise and multiplication is defined using the fundamental formula

$$i^2 = j^2 = k^2 = ijk = -1.$$

The commutative rule for multiplication is not satisfied because one can show that $ij \neq ji$. In fact $ij = -ji = k$.

Quaternions played an important role in Maxwell's theory in 1864 which predicted the existence of electromagnetic waves. Maxwell was a Scottish mathematician whose work led him to predict the existence of this new phenomenon which nobody had ever imagined. Not many people believed him and it took twenty-two years before the German physicist, Hertz, verified his theory by detecting radio waves experimentally.

Radio waves are just one class of electromagnetic waves. Nine years later, in 1895, Marconi sent the first wireless telegraph, and then in 1901, he conducted the first transatlantic wireless communication by transmitting and receiving radio signals. Radio, television, radar and telecommunications all followed from this. Thus, the inventions of these and countless other significant products of our time, are directly related to quaternions. Also, the theory of `vector analysis', which is now indispensable in physics and its applications, is an offspring of quaternions. Hamilton's quaternions also play a significant role in the entertainment industry because they are important in computer graphics, designing computer games and special effects in movies. For example, they were used to create the famous character, Lara Croft, in the computer game Tomb Raider, and also some of the special effects in the movie, The Matrix Reloaded.

He has been called `The Liberator of Algebra' because his quaternions smashed the previously accepted notion that a useful algebraic number system should satisfy the rules of ordinary numbers in arithmetic (his quaternions did not satisfy the commutative rule of multiplication). His quaternions opened up a whole new mathematical world in which mathematicians were now free to conceive new algebraic number systems that were not shackled by the rules of ordinary numbers. Consequently, one may say that modern algebra was born on 16 October 1843, on the banks of the Royal Canal in Dublin.

Another renowned Irishman, Brendan Behan, would later also create a famous link between the Royal Canal and a mathematical concept!! The following lines were penned by Behan in his play The Quare Fellow:

The oul' triangle
Goes jingle jangle
Along the banks of the Royal Canal

The sound of the triangle indicated that it was unlocking time in Mountjoy Prison, where Behan was once an inmate.

All traces of Hamilton's carving are gone, but there is a plaque on the bridge commemorating the event. The plaque is on the west side of the bridge at Broombridge train station in Cabra and was unveiled by the Taoiseach, Eamon De Valera, in 1958. De Valera greatly admired Hamilton and had a degree in mathematics himself. He lectured in mathematics in Maynooth for two years (1912 -1914). De Valera paid homage to quaternions with a little graffiti of his own by scratching the fundamental formula on the wall of his prison cell in Kilmainham. Our current Taoiseach can see a statue of Hamilton near his office. The statue is on the steps of Government Buildings in Merrion Street in Dublin and dates back to the time when the buildings housed the College of Science.

Early days

William Rowan Hamilton was born on August 4 1805 at 29 Dominick St. in Dublin. His father, Archibald Hamilton, was a solicitor in Dominick St. and his mother, Sarah Hutton, came from a coachbuilder's family in Summerhill, Dublin. He had four sisters: Grace, Eliza, Sydney and Archianna.

In 1808 William was sent to Trim, Co. Meath, where he was educated by his uncle, James Hamilton, who was a curate and managed the Meath Diocesan school. William would spend the next fifteen years in Trim. Later on in his life he would recall these early days with great fondness. They lived in a big house that is now called St. Mary's Abbey. The house is beautifully situated beside the Yellow Steeple, on the banks of the Boyne, across from the spectacular ruins of Trim castle. A commemorative plaque was erected on the house in 2004. The house also served as the school. It provided an ideal environment for nurturing genius as it offered a vibrant and stimulating atmosphere for the young William. Uncle James was a well respected educator and an honours graduate in classics from Trinity College, Dublin. William wrote:

> *I would cherish the fondness for classical and for elegant literature which was early infused into me by my uncle to whom I owe my education.*

Aunt Sydney was very perceptive and enthusiastic. She wrote many letters to William's parents and these provide an invaluable account of his early years. William was a playful prodigy in the sense that he played around like any child of his age but he was a prodigy on account of the number of languages he mastered at a very early age.

William's mother died in 1817 and his father died two years later. William was now nominally responsible for his four orphaned sisters and this responsibility played a very

significant role in his career decisions later on.

Shortly after his father's death, he began to write poetry and this interest played a prominent part in the rest of his life. He would later win the Chancellor's poetry prize twice in Trinity and publish his literary work in journals and magazines. His sister, Eliza, also developed a keen interest in poetry. She became a poet and published a volume of her work in 1839. From his early days, he enjoyed rambling and playing in the countryside around Trim, and visiting the many ancient monuments in the area. He had an interest in the rich history and archaeology of the locality.

During his early years he displayed outstanding talent for languages and the classics. His mathematical education was good but he had not yet developed as keen an interest in mathematics as he had in languages and the classics. The turning point came in 1818 when a fourteen year old American child prodigy, Zerah Colburn, visited Dublin. Colburn was travelling around Europe demonstrating his exceptionally quick computational skills and William was deemed good enough to be invited to compete against him. William lost, but the loss seems to have triggered his interest in mathematics. He later wrote:

> *I was amused this morning, looking back on the eagerness with which I began the different branches of mathematics and how I always thought my present pursuit the most interesting. I believe it was seeing Zerah Colburn that first gave me an interest in these things.*

His first invention came when he was fifteen, with his own design of a semaphore system. Semaphores were of interest to the scientific community and his design was a big improvement on previous semaphores. With the aid of telescopes, people were able to use semaphores to communicate messages between two heights which were far apart.

He was reading Newton's Principia in 1820 and in 1821 he solved a problem that his tutor, Charles Boyton, a distinguished mathematician in Trinity, was unable to solve. In 1821 he corrected a mistake in Laplace's book, *Mécanique Céleste*, that had not been noticed before. As Laplace was an eminent French mathematician at the time, this was quite an achievement for a sixteen year old and news spread quickly. He now had a fervent ambition to excel as a mathematician. We see his aspiration to greatness in a letter to his aunt Mary in 1822:

> *Mighty minds in all ages have combined to rear upon a lofty eminence the vast and beautiful temple of Science and inscribed their names upon it in imperishable characters; but the edifice is not yet completed: it is not yet too late to add another pillar or another ornament. I have scarcely arrived at its foot, but I may*

aspire one day to reach the summit.

This youthful aspiration was to be richly realised.

Beauty in mathematics

Like many great mathematicians Hamilton sought and found beauty in mathematics. He was not necessarily interested in the practical uses of his mathematics and yet his work has had very powerful applications in the physical world. The beauty in mathematics lies in the beauty of ideas because mathematics is basically made up of ideas. For example, number, triangle, proof and derivative are all ideas in mathematics and are extremely powerful in the practical world.

I will now give an example of beauty in mathematics. Karl Friedrich Gauss (1777-1855) was a German mathematician and in his first arithmetic class the teacher had to leave for a while and gave the pupils a `busy work' problem to keep them occupied. The problem was to find the sum $S = 1 + 2 + 3 + 4 + \ldots + 100$. Gauss solved the problem almost immediately and put up his hand before the teacher left the room. Gauss noticed that $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, … $49 + 52 = 101$, $50 + 51 = 101$ and thus $S = 50 \times 101 = 5050$. This solution has a certain beauty and notice how it effortlessly falls into place once the symmetry within the problem is observed. The solution is much more elegant and powerful than adding up all the numbers directly and also the idea behind Gauss' solution can be used in more general situations.

Hamilton considered mathematics '*as an aesthetic creation, akin to poetry, with its own mysteries and moments of profound revelation'*.

Science to Hamilton meant mathematics and he once wrote:

For science, as well as poetry, has its own enthusiasm and holds its own communion, with the sublimity and beauty of the universe.

The renowned mathematician, G. H. Hardy, wrote:

The mathematician's patterns, like the painter's or poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics... It may be very hard to define mathematical beauty, but that is just as true of beauty of any kind — we may not quite know what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it.

In order to `see' mathematics in the physical world one must look with the `eyes of the mind' because that is how one `sees' ideas. This explains why many people don't see mathematics in the physical world. It's because they are not looking with the eyes of the mind. If one looks at the physical world with the eyes of the mind one will see mathematics in many places. For example, if you look at a car with your physical eyes you don't necessarily see the mathematics but if you look with the eyes of your mind you will see an abundance of mathematical ideas that are crucial for the design and operation of the car.

In relation to seeing mathematics, it's important to realise that, while mathematical notation plays an important role in mathematics, it is nowhere near as important or interesting as the ideas that are represented by the notation. Think of musical notation compared to the music itself. The musical notation is nowhere near as important or interesting as the music that is represented by the notation.

College Days

Hamilton published a paper called *`Theory of System of Rays'* in 1827. This was his first important contribution to science. It was a major breakthrough in optics. He applied algebra to describe the path of light. He incorporated all the properties of any optical system into just one equation. The solution of this equation gives what he called his `characteristic function'. This function provides a complete description of the optical system, because if a ray of light enters the optical system, the function describes how the ray leaves the system. Hamilton was mainly interested in obtaining mathematical harmony and unity in optics. However, his work did have significant applications in optics and in the design of optical instruments. The seeds of his theory actually came to him in 1822 when he was only seventeen.

Anthony G. O'Farrell, Steven Weinberg (Nobel Prize winner) and Fiacre Ó Cairbre at Dunsink for the walk on October 16, 2005

In 1823, he obtained first place in the entrance exam for Trinity College. His examination performances in Trinity were outstanding and he received many awards and prizes in science, classics and poetry. In 1826, he received an *optime* (which was a mark off the scale, meaning that the student had a total mastery of the subject) in both science and classics. No undergraduate had ever achieved this before. His reputation in Dublin was quickly established

and during his college days he often socialised with important figures.

On 17 August 1824 Uncle James introduced him to the Disney family in Summerhill, Co. Meath. This was when he first met Catherine Disney. He was captivated and instantly fell in love with her. They met several times afterwards and then in 1825 he made his feelings for Catherine explicit in a Valentine ode. Soon after sending this ode, he received word that she was to marry William Barlow, who was fifteen years older than her. It appears that this was an arranged marriage, because later letters show that she had always loved Hamilton. He never forgot her. In 1848 he wrote to his friend, the poet, Aubrey de Vere:

> *The same remembrance has run like a river through my life, hidden seemingly for intervals, but breaking forth again with an occasional power which terrifies me — a really frightful degree of force and vividness.*

Life at Dunsink Observatory

The position of Royal Astronomer for Ireland, based at Dunsink Observatory, became vacant in 1827 and Hamilton was strongly encouraged to apply for the post, even though he was still an undergraduate in Trinity. There were many renowned people interested in the post, but he was offered the position. He accepted and consequently became a Professor while still an undergraduate!

There were two important considerations for him in deciding to accept the post. Firstly, it offered an ideal residence for him and his four sisters, whom he had vowed to support when their parents died. It would take a long time before he could afford such a house if he had stayed at Trinity. Secondly, the post involved minimal teaching, with only one annual University course of 12 lectures on astronomy. Therefore, he could focus on his mathematical research.

In 1827, Alexander Nimmo, the engineer for the Western District of Ireland, took Hamilton on a two-month tour of engineering feats around Ireland and England. It was while staying in the Lake District that Hamilton first met Wordsworth. They found each other's company quite stimulating and became good friends. Hamilton wrote to Eliza:

> *He (Wordsworth) walked back with our party as far as their lodge; and then, on our bidding Mrs Harrison goodnight, I offered to walk back with him, while my party proceeded to the hotel. This offer he accepted and our conversation had become so interesting that when we arrived at his house, a distance of about a mile, he proposed to walk back with me on my way, a proposal which you may be sure that I did not reject; so far from it that when he came to turn once more*

towards his home, I also turned once more along with him. It was very late when I reached the hotel after all this walking.

Wordsworth wrote:

Seldom have I parted — never, I was going to say — with one whom, after so short an acquaintance, I lost sight of with more regret. I trust we shall meet again.

Hamilton's assertion that mathematics was an aesthetic creation with beauty and fascination, was *`to Wordsworth an entirely new revelation and had the effect of raising his conception, which had before been unduly depreciatory, of the dignity of science itself and its most eminent votaries'*. Hamilton was the godfather of Wordsworth's son William. He was also a good friend of the famous novelist, Maria Edgeworth.

The international scientific community was stunned by Hamilton's prediction of conical refraction in 1832. His purely mathematical theory predicted that a ray of light, entering certain biaxial crystals under certain conditions, would emerge as a hollow cone light beam or a hollow cylinder light beam. This phenomenon of conical refraction was totally unexpected and nobody had ever imagined it. His prediction was verified quantitatively and qualitatively by experiment.

Here was a purely mathematical prediction of a totally unexpected natural phenomenon being experimentally verified quantitatively and qualitatively. Mathematics had always been fundamental in understanding how nature works, but this apparently was the first time that a mathematical prediction of an unimagined physical phenomenon had been verified so precisely. He was awarded the Royal Medal of the Royal Society of London in 1835 for his work on conical refraction. There have been more famous mathematical predictions of unforeseen physical phenomena since Hamilton's time. Perhaps the most important one was the prediction of electromagnetic waves by Maxwell which was mentioned earlier.

In 1833 Hamilton married Helen Bayly from Nenagh, Co. Tipperary. They had three children: William, Archibald and Helen. In 1869 his daughter Helen married John O'Regan but tragically died a year later, after the birth of her son, John. The only descendants of Hamilton come from this son, John.

General Method in Dynamics

Following on from his ideas in his '*Theory of System of Rays'*, Hamilton was able to reveal that the laws of optics (i.e. light) and the laws for dynamics (i.e. the motion of objects) could be expressed in the same mathematical form. These were two totally different areas in

physics, optics and dynamics, and yet he constructed a beautiful mathematical theory that united the two at a highly abstract level. His new theory appeared in 1834 in two papers called `*General Method in Dynamics'*. This work was his most important contribution to science. It is necessary to have a theory of dynamics if we want to understand how objects (e.g. planets, satellites, trains, cars etc.) move and this understanding of motion is fundamental to our understanding and operation of the physical world around us.

Hamilton sought beauty in mathematics and was not necessarily concerned with its practical use. He makes this point in the introduction to his work on dynamics, saying that he finds it `intellectually pleasing' that he has reduced the study of forces and motion of an object from many equations down to just two equations. He wrote:

> *The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another; and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function, the unfolding of one central relation.*

The opinion among the majority of scientists was that he had developed a very elegant mathematical theory, but they had no real practical use for it. Hamilton's new method of dynamics was way ahead of its time as it would take over eighty years before his deep mathematics in his dynamics would play a vital part in a major breakthrough for our understanding of the physical world. Early in the twentieth century physicists were beginning to realise that Newton's laws of motion did not work well at the atomic level where very small particles like electrons and atoms are moving around. Hence, they were attempting to develop a new theory for the atomic level. This new theory was called Quantum Mechanics (or Wave Mechanics) and Hamilton's method of dynamics was indispensable for it. His aesthetically pleasing and highly abstract mathematics from 1834 became the cornerstone of the modern understanding of the physical world in the twentieth century.

Schrödinger, who won the Nobel prize in 1933, was one of the leading scientists working on Quantum Mechanics in the 1920s and he wrote:

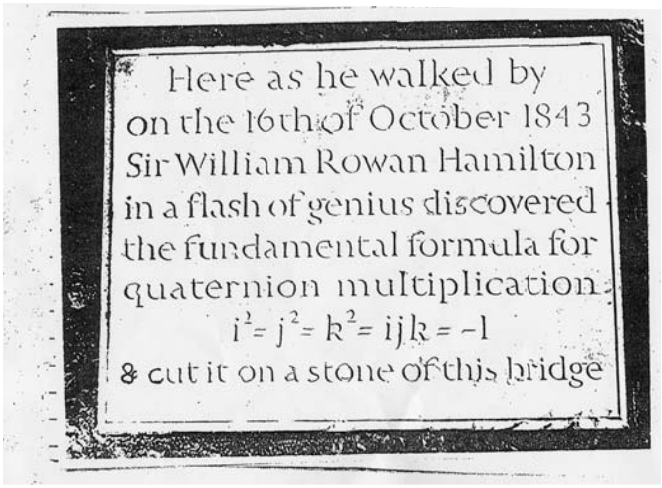
> *I daresay not a day passes — and seldom an hour — without somebody, somewhere on the globe, pronouncing or reading or writing Hamilton's name. That is due to his fundamental discoveries in general dynamics. The Hamiltonian Principle has become the cornerstone of modern physics, the thing with which a physicist expects every physical phenomenon to be in conformity... The modern development of physics is continually enhancing Hamilton's name. His famous analogy*

between dynamics and optics virtually anticipated wave mechanics, which did not have to add much to his ideas... The central conception of all modern theory in physics is `the Hamiltonian'. If you wish to apply modern theory to any physical problem, you must start with putting the problem `in Hamiltonian form'. Thus Hamilton is one of the greatest men of science the world has produced.

Hamilton was elected President of the Royal Irish Academy in 1837. He proved a committed and competent president until his term in office ended in 1846.

Hamilton died on 2 September 1865. The causes of death were gout and bronchitis. He is buried in Mount Jerome Cemetery in Dublin. He once wrote:

I have very long admired Ptolemy's description of his great astronomical master, Hipparchus, as a labour-loving and truth-loving man. Be such my epitaph.



The plaque at Broombridge

Acknowledgements

I wish to thank Ríocht na Midhe (Meath Archaeological and Historical Society) for permission to reproduce some parts of my journal article on Hamilton in Vol XI, 2000.

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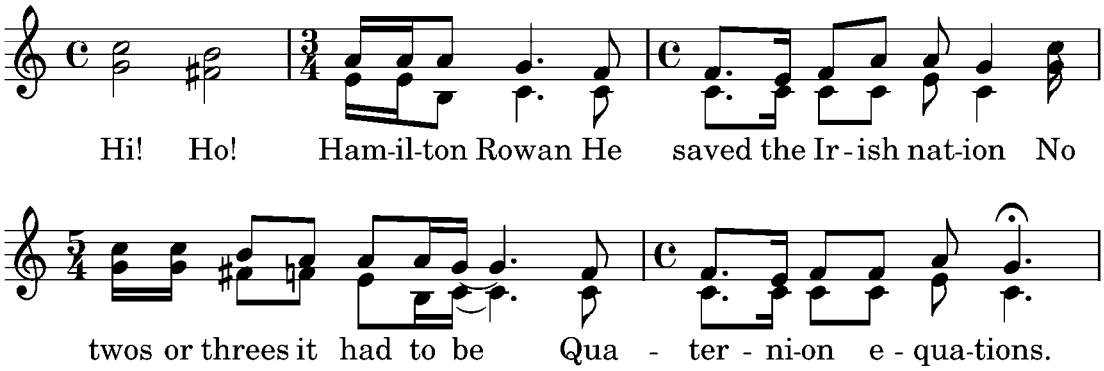
Fiacre Ó Cairbre
NUI, Maynooth

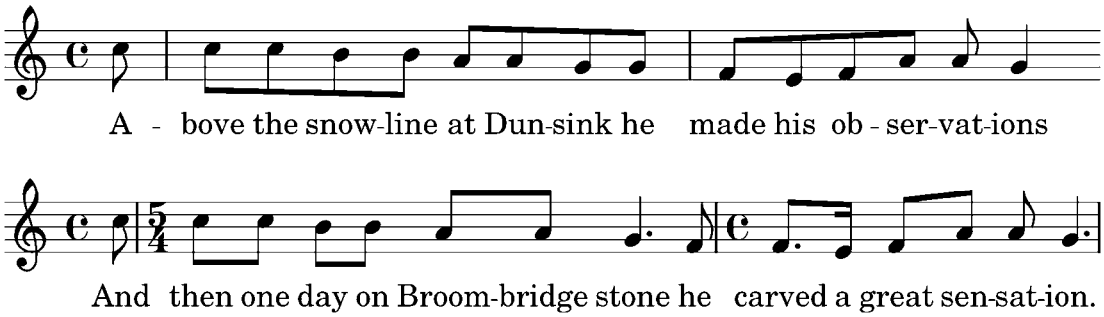
The Ballad of Rowan Hamilton

Jack Gannon

Refrain



Verses



Many were the grateful ones, They thought it was a pity

That oonsigh, eejits and the rest, were foremost in the city.

So praise the gallant scientists who favour rhyme and reason

At Broombridge you can see it there, it always is in season.

Music engraving by LilyPond 2.6.5—www.lilypond.org

The Benefactor

Two hundred years ago this year
A man was born who had no fear
Of numbers, science or the like
In fact, they brought him great delight!

His destiny was to become
The famous William Hamilton
To change our lives beyond repair
With discoveries extraordinaire

When energies he elsewhere channelled,
Fruits of labour soon were pummelled
By Wordsworth—friend—or was he foe?
Who said Will's poetry should go!

Back in the heyday of his youth
William's lust for life was fuelled
Primarily by love for one
Whom sadly he could never own

In leisure times he had his vice
The demon drink he found so nice
Distracted him in times of stress
When numerics proved a mess

His theories touch so many parts
Of science and even here, the Arts
He's left a lasting legacy
In daily life for all to see

Two hundred light years on today
His brilliant work still on display
Hamilton has left us all
In wonderment and great enthrall

So when you visit UGC
And gaze upon that silver screen
Give a thought to Hamilton
Whose 4-D vectors still play on

Hold your popcorn out to share
E'en though it seems to be thin air
He may have popped in to admire
His work in modern day attire!

June Robinson
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A Quaternionic Gem

Anthony G. O'Farrell[1]
Mathematics Department
NUI, Maynooth
Co. Kildare
Ireland
e-mail: anthonyg.ofarrell@gmail.com

1 Introduction

It is generally accepted nowadays that Hamilton's greatest achievement is his general theory of dynamics. By comparison, quaternions have had less impact. At the same time, people continue to use and develop the theory and techniques of quaternionic algebra and anlalysis, and they continue to find new applications, so one cannot say what the verdict may be on the relative importance of the two inventions, in the long run.

It is in the nature of mathematics that its abstract concepts find use far from their origins. Quaternions were invented because Hamilton wanted an algebra that would facilitate geometric work in three dimensions. Number theory (the theory of whole numbers) is quite a different area of mathematics, so I particularly like the fact that quaternions may be used to make an important step in the proof of a theorem in number theory. Here is the theorem:

Theorem 1.1 (Lagrange) *Each positive integer is the sum of at most four square positive integers.*
(Fermat claimed to have a proof of this theorem, but we usually give the credit to the first person who publishes a proof, and, as in other cases, Fermat did not reveal his proof.)

It is slightly more convenient to discuss this in the equivalent form: *Each nonnegative integer is the sum of four square integers.* The point is that, by allowing zero into consideration, we can always use exactly four numbers to represent a given number. For instance, 6 is not the sum of four positive squares, but it is $2^2 + 1^2 + 1^2 + 0^2$.

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2 Sums of Squares

The squares $1, 4, 9, \ldots$ are relatively rare among positive integers. The sums of at most two squares are

$0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$, $4 = 0^2 + 2^2$, $5 = 1^2 + 2^2$, $8 = 2^2 + 2^2$, and so on. It is a useful exercise for a programming class to write code to generate the numbers of this form that are less than or equal to 100 (or $1000, 10000, \ldots$). For instance, using Maple:

```
> S2S:= proc(N)
                 # returns a list of the sums of 2 squares <= N

                local A,B,C,n,m;

                A:= seq( seq( m^2+n^2,
                              m=0..floor(sqrt(N-n^2))
                            ),
                          n=0..sqrt(N)
                        );  #  sequence of all the sums <= N

                B:= {A}; #  cast to set;  removes duplicates

                C:= sort(  [B[]]  );  #  cast to list and sort.

             end proc;

> S2S(100):  #  output suppressed.
```

The sorting step is probably unnecessary, in practice, since Maple implementations will probably store the elements of a set of numbers in the natural order.

Inspection of this list, and longer ones, reveals that there are substantially more sums of two squares than of one square, but there are gaps: the numbers $3, 6, 7, 11, 12, 14, 15$, etc. are missing.

What is the pattern?

Mathematicians always look for structure, and the key structure here is the *semigroup*. By definition, a set S of nonnegative numbers is a *multiplicative semigroup* if and only if it has the product of each pair of elements as another element, i.e.

$$\left.\begin{array}{rcl} x & \in & S \\ y & \in & S \end{array}\right\} \Rightarrow xy \in S$$

For example:

1. The set $\mathbb{N}$ of all positive integers is a multiplicative semigroup.
2. The set $\mathbb{Z}^+$ of all nonnegative integers is a multiplicative semigroup.
3. The set $\mathbf{10^\bullet}$ of all nonnegative powers of 10 is a multiplicative semigroup.
4. The set E of all even positive numbers is a multiplicative semigroup.
5. The set O of all odd positive numbers is a multiplicative semigroup.
6. The set $\langle 6 \rangle$ of all positive multiples of 6 is a multiplicative semigroup.
7. The only finite mulitplicative semigroups are the empty set $\emptyset$, and $\{0\}, \{1\}$, and $\{0, 1\}$.

If S is a multiplicative semigroup, then so are

$$S \cup \{0\}, \qquad S \cup \{1\}, \qquad S \sim \{0\}, \qquad S \sim \{1\},$$

i.e. a semigroup remains a semigroup if 0 or 1 (or both) are added or removed.

The following fact is basic:

Lemma 2.1 *If S is a multiplicative semigroup of nonnegative integers, and all the primes belong to S, then all integers greater than 1 belong to S.*

Proof. This is just a restatement of part of the Fundamental Theorem of Arithmetic: every integer greater than 1 is a product of prime numbers. ∎

The connection between semigroups and sums of two squares is:

Theorem 2.2 *The set*

$$S_2 = \{m^2 + n^2 : n \in \mathbb{Z}^+, m \in \mathbb{Z}^+\}$$

of sums of two nonnegative squares is a multiplicative semigroup.

Proof. We use complex numbers $c = a+ib$, and recall that the modulus, or absolute value is given by

$$|c| = |a + ib| = \sqrt{a^2 + b^2},$$

where $\sqrt{\ }$ denotes the nonnegative square root. Also, the complex conjugate is

$$\overline{c} = \overline{a + ib} = a - ib,$$

and is related to the absolute value by

$$|a + ib|^2 = (a + ib)(a - ib),$$

i.e. $|c|^2 = c\overline{c}$.

Recall that a is the real part of c, denoted $\Re c$, and b the imaginary part, $\Im c$.

Finally, note that

$$\overline{c_1 c_2} = \overline{c_1} \cdot \overline{c_2},$$

i.e. the complex conjugate of a product is the product of the complex conjugates.

Now suppose m_1, n_1, m_2, n_2 are nonnegative integers. We want to show that

$$(m_1^2 + n_1^2)(m_2^2 + n_2^2)$$

is the sum of two squared integers, i.e. has the form

$$m_3^2 + n_3^2$$

for some integers m_3, n_3.

Take

$$\begin{aligned} c_1 &= m_1 + in_1, \\ c_2 &= m_2 + in_2, \\ m_3 &= \Re(c_1 c_2), \\ n_3 &= \Im(c_1 c_2). \end{aligned}$$

Then

$$m_3 = m_1 m_2 - n_1 n_2, \text{ and } n_3 = m_1 n_2 + m_2 n_1 \tag{1}$$

are obviously integers, and

$$\begin{aligned}(m_1^2+n_1^2)\cdot(m_2^2+n_2^2) &= |c_1^2|\cdot|c_2^2| \\ &= c_1\cdot\overline{c_1}\cdot c_2\cdot\overline{c_2} \\ &= (c_1\cdot c_2)\cdot(\overline{c_1}\cdot\overline{c_2}) \\ &= (c_1\cdot c_2)\cdot\overline{(c_1\cdot c_2)} \\ &= |c_1\cdot c_2|^2 \\ &= m_3^2+n_3^2.\end{aligned}$$

Thus we are done. ■

For instance, $5 = 1^2 + 2^2, 13 = 2^2 + 3^2$, so 65 is the sum of two squares, and the formula (1) gives us two numbers that do the trick:

$$m_3 = 1\times 2 - 2\times 3 = -4, \quad n_3 = 1\times 3 + 2\times 2 = 7,$$
$$65 = 4^2 + 7^2.$$

A person could perhaps impress their friends by pushing the boat a bit further out:

$$85 = 9^2 + 2^2, \quad 97 = 9^2 + 4^2,$$

so

$$8245 = (9\times 9 - 2\times 4)^2 + (9\times 4 + 2\times 9)^2 = 73^2 + 54^2.$$

Starting from this point, standard texts on number theory go on to characterise the numbers that are sums of two squares. The end result is:

Theorem 2.3 (Euler) *A number $n \in \mathbb{Z}^+$ belongs to S_2 if and only if*

$$n = r^2 p_1 \cdots p_k,$$

where r is an integer and the p_i are primes congruent to 1 or 2 modulo 4. ■

Note that 2 is the only prime congruent to 2 modulo 4. The primes excluded are those congruent to 3 modulo 4, such as 3, 7, 11, 19, etc.

We omit the proof of this result. If you wish to pursue it, you could look at one of the references at the end of the paper.

3 Quaternions

We would like to use a similar approach for

$$S_4 = \{m_1^2 + m_2^2 + m_3^2 + m_4^2 : m_i \in \mathbb{Z}\},$$

the set of all numbers expressible as the sum of four nonnegative square integers, and in fact we can do this by replacing the complex numbers by the quaternions.

A quaternion q takes the form

$$q = a + bi + cj + dk,$$

where a, b, c and d are real numbers, and the (distinct!) quaternion units i, j, k satisfy

$$\begin{gathered} i^2 = j^2 = k^2 = -1, \\ ij = k, \quad jk = i, \quad ki = j, \\ ji = -k, \quad kj = -i, \quad ik = -j. \end{gathered}$$

Apart from the famous oddity that the multiplication of i, j, and k is non-commutative, the usual rules of arithmetic apply:

$$\begin{aligned} q(q_1 + q_2) &= qq_1 + qq_2 \\ (q_1 + q_2)q &= q_1 q + q_2 q, \end{aligned}$$

and, provided a is real,

$$(aq)q_1 = a(qq_1) = q(aq_1) = (qq_1)a.$$

Thus if $q = a + bi + cj + dk$ and $q_1 = a_1 + b_1 i + c_1 j + d_1 k$, then we calculate

$$\begin{aligned} qq_1 &= (a + bi + cj + dk)(a_1 + b_1 i + c_1 j + d_1 k) \\ &= aa_1 + ab_1 i + ac_1 j + ad_1 k \\ &\quad + ba_1 i + bb_1 i^2 + bc_1 ij + bd_1 ik \\ &\quad + ca_1 j + cb_1 ji + cc_1 j^2 + cd_1 jk \\ &\quad + da_1 k + db_1 ki + dc_1 kj + dd_1 k^2 \\ \\ &= (aa_1 - bb_1 - cc_1 - dd_1) \\ &\quad + (ab_1 + ba_1 + cd_1 - dc_1)i \\ &\quad + (ac_1 - bd_1 + ca_1 + db_1)j \\ &\quad + (ad_1 + bc_1 - cb_1 + da_1)k. \end{aligned}$$

The conjugate q is defined as

$$\overline{q} = a - bi - cj - dk,$$

and a calculation gives

$$q\overline{q} = a^2 + b^2 + c^2 + d^2.$$

The *norm* of q is defined as

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2},$$

so $|q|^2 = q\overline{q}$. Note also that $|\overline{q}| = |q|$.

The conjugate of a sum is the sum of the conjugates, and $\overline{aq} = a\overline{q}$ whenever a is real, but it is not usually true that the conjugate of a product is the product of the conjugates; the correct formula is

$$\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1}.$$

This can be verified by another horrendous direct calculation, but the elegant way to see it is to observe that both sides are linear (over the reals) in both factors q_1, q_2, so it suffices to check the equation for the 16 cases in which q_1, q_2 are drawn from the basis $\{1, i, j, k\}$. The cases involving equal factors or the factor 1 are trivial, so given the symmetries of the system, this boils down to one equation

$$\overline{i \cdot j} = \overline{k} = -k = j \cdot i = \overline{j} \cdot \overline{i}$$

This allows us to imitate the calculations for complex numbers (taking care about the order of factors!), and get

$$\begin{array}{rcrcl} |q_1 \cdot q_2|^2 & = & (q_1 \cdot q_2) \cdot \overline{(q_1 \cdot q_2)} & = & q_1 \cdot q_2 \cdot \overline{q_2} \cdot \overline{q_1} \\ & & & = & q_1 \cdot |q_2|^2 \cdot \overline{q_1} \\ & & & = & q_1 \cdot \overline{q_1} \cdot |q_2|^2 \\ & & & = & |q_1|^2 \cdot |q_2|^2 \\ & & & = & (|q_1| \cdot |q_2|)^2, \end{array}$$

so taking square roots,

$$|q_1 \cdot q_2| = |q_1| \cdot |q_2|,$$

4 Sums of Four Squares

We are not going to prove Lagrange's Theorem, but we can now prove one of the main ingredients in the proof:

Lemma 4.1 (Euler) *The set S_4 of sums of four square integers is a multplicative semigroup.*

Proof. Suppose we are given integers $m_1, m_2, m_3, m_4, n_1, n_2, n_3$, and n_4. We have to show that

$$(m_1^2 + m_2^2 + m_3^2 + m_4^2) \cdot (n_1^2 + n_2^2 + n_3^2 + n_4^2)$$

is the sum of four squares.

Take $q_1 = m_1 + m_2 i + m_3 j + m_4 k$ and $q_2 = n_1 + n_2 i + n_3 j + n_4 k$, and let

$$q_1 \cdot q_2 = p_1 + p_2 i + p_3 j + p_4 k.$$

Then from the formula (2) it is clear that the p_i are all integers, and

$$(m_1^2 + m_2^2 + m_3^2 + m_4^2) \cdot (n_1^2 + n_2^2 + n_3^2 + n_4^2) = |q_1|^2 \cdot |q_2|^2 = |q_1 \cdot q_2|^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2,$$

so we are done. ∎

This lemma reduces the problem of proving Lagrange's Theorem to the problem of proving that each prime is the sum of four squares. This is not a trivial remainder, and in fact it defeated Euler.

If you wish to get to the end of the story, you can read an account of the rest in Burton [B] or Herstein [H], for instance.

It is remarkable that Euler found his lemma long before the invention of quaternions. The above proof makes it really transparent, but he actually came up with what is effectively the same formula for the components p_i by sheer ingenuity. Armed with a knowledge of quaternions, you can easily reconstruct his formulas by substituting q_1 and q_2 into the formulas (2), getting:

$$\begin{aligned} & (m_1^2+m_2^2+m_3^2+m_4^2)\cdot(n_1^2+n_2^2+n_3^2+n_4^2) \\ = \quad & (m_1n_1-m_2n_2-m_3n_3-m_4n_4)^2 \\ & +(m_1n_2+m_2n_1+m_3n_4-m_4n_3)^2 \\ & +(m_1n_3+m_3n_1-m_2n_4+m_4n_2)^2 \\ & +(m_1n_4+m_4n_1+m_2n_3-m_3n_2)^2. \end{aligned}$$

Now you can really amaze your friends by writing huge numbers as the sum of four squares. For instance, take 6469693230, which is the product of the first ten primes. We have

$$\begin{aligned} 2 &= 0^2+0^2+1^2+1^2 \\ 3 &= 0^2+1^2+1^2+1^2 \\ 5 &= 0^2+0^2+1^2+2^2 \\ 7 &= 1^2+1^2+1^2+2^2 \\ 11 &= 0^2+1^2+1^2+3^2 \\ 13 &= 0^2+0^2+2^2+3^2 \\ 17 &= 0^2+0^2+1^2+4^2 \\ 19 &= 1^2+1^2+1^2+4^2 \\ 23 &= 1^2+2^2+3^2+3^2 \\ 29 &= 0^2+0^2+2^2+5^2 \end{aligned}$$

Following the quaternion method we used to prove Euler's Lemma, we form the quaternions

$$\begin{aligned} a_2 &= 0+0i+1j+1k \\ a_3 &= 0+1i+1j+1k \\ a_5 &= 0+0i+1j+2k \\ a_7 &= 1+1i+1j+2k \\ a_{11} &= 0+1i+1j+3k \\ a_{13} &= 0+0i+2j+3k \\ a_{17} &= 0+0i+1j+4k \\ a_{19} &= 1+1i+1j+4k \\ a_{23} &= 1+2i+3j+3k \\ a_{29} &= 0+0i+2j+5k. \end{aligned}$$

Then we multiply them all together, getting

$$-70176 - 37594i + 6997j + 9097k.$$

A quick calculation verifies that

$$70176^2 + 37594^2 + 6997^2 + 9097^2 = 6469693230,$$

as expected.

Here are a few exercises that can be tackled using the methods developed above:

1. Express the following numbers as the sum of four squares:
   (a) 2640330.
   (b) 200560490130.
   (c) 277945762500.
2. **(a)** 1462500.
   (b) 2371330.
   (c) 44240625.

(Hint: Start by factorising the number, in each case.)

We close by remarking that the sums of *three* squares cannot be tackled in this kind of way, and the theorem that characterises these (due to Gauss) is quite deep. You can find it in Serre [S].

Acknowledgement

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References

[B] D.M. Burton. Elementary Number Theory. Revised printing. Allyn and Bacon. 1980.
[H] I.N. Herstein. Topics in Algebra. Blaisdell. 1964.
[S] J.-P. Serre. A Course in Arithmetic. Springer. 1973.

William Rowan Hamilton
Contributions to Physics

William Rowan Hamilton is one of the small number of scientists whose work has profoundly changed the way we look at nature and the way we try to find the underlying laws that govern the Universe. At the age of seventeen Hamilton had a great insight which made him write:

> Mighty minds in all ages have combined to rear upon a lofty eminence the vast and beautiful temple of Science, and inscribed their names upon it in imperishable characters; but the edifice is not completed: it is not yet too late to add another pillar or another ornament. I have yet scarcely arrived at its foot, but may aspire one day to reach its summit.

He had a vision of the way the properties of light rays could be described in a truly original way. Instead of looking at rays individually Hamilton looked at the entire system through which light was travelling and formulated the properties of the system through one function: The Characteristic Function. This function contained in it all the optical information which described the way light would behave when sent through the system. Furthermore, the Characteristic Function satisfied an extremal property—i.e. for the actual path followed, it was either a maximum or a minimum.

The path followed by light was thus very special. This maximum or minimum property lead to an equation for the Characteristic Function. Solving this equation is then a way of determining the way light behaves when it goes through the system. Hamilton realised soon, by the time he was 21, that the same idea works for particles - i.e., there was a way in which the equations describing particle motion could be encoded in one Characteristic Function.

In 1834 he wrote out the details of this idea. One equation—the Hamilton Jacobi Equation—described the way a complicated system of many particles behaved. From this one equation one could determine the motion of each individual particle. Furthermore, as in the case of light, there was an underlying maximum/minimum property. This remarkable insight of Hamilton has changed the way laws of nature are formulated. The maximum/minimum property under the name Hamilton's Principle is a cornerstone of modern physics. It is the

approach used to formulate fundamental theories of nature.

The insight of Hamilton regarding the mathematical similarity between the way light rays travel through optical systems and the way particles move through their mutual interaction led Erwin Schrödinger in 1926 to discover Wave Mechanics – the fundamental theory of non-relativistic atomic scale systems, a theory which is used to understand the way electronic systems work and is currently used to make new materials at the atomic scale: the subject of nanoscience.

Applying his idea of the Characteristic function to the wave theory of light Hamilton made a remarkable prediction in October 1832. He predicted, on the basis of calculations, that a single ray of light when sent through a certain kind of crystal could emerge as a circular cone! This was something no one had ever seen! This was the first time that a prediction of a natural phenomenon was made based on a theoretical calculation. Hamilton's Trinity colleague, Humphrey Lloyd, was able to carry out an experiment and observed what had been predicted by Hamilton on the 14th of December 1832. It was a special moment in the history of science.

William Rowan Hamilton, born in Dublin, buried in Mount Jerome, is a figure respected in the world of science. His contributions continue to be explored in modern times. A conference organised in Dublin in October 2005 brought together leading mathematicians of the world to talk about Symplectic Geometry, Contact Geometry and Hamiltonian Systems. All of these were notions contained in Hamilton's great work on Optics and Dynamics. It was clear from the meeting that much remains to be done.

We hope the story of Hamilton will spur young people with an interest in science to try to scale the lofty heights of the edifice of science, like Hamilton, and make contributions that will allow us to glimpse the wonders of nature.

Siddhartha Sen

Hamilton Mathematics Institute

TCD, Dublin, Ireland

Mathematics in Practice

Philip Mullen works as a trainee actuary with an Investment Company in Dublin. Recently I asked him to describe his work and the mathematics that is involved in it.

Neil: Philip, what courses must you take in order to become an actuary?
Philip: Many of my colleagues enter the profession by studying an Actuarial Mathematics and Finance degree in either UCD or DCU. These are the only two institutions in Ireland which offer this course and the exams which are covered in the undergraduate course offer exemptions from 8 of the 14 exams which must be passed in order to qualify as an actuary. The structure of the exams for qualification is 8 mathematical based exams, 1 Communications exam and 5 exams on Legislation and Practice. The Actuarial degree courses offer exemptions from the 8 maths based exams.

Neil: Where can you do actuarial exams? How long does it take to become an actuary?
Philip: There is no Irish Actuarial exam system and so Irish members of the profession currently sit the exams set by the Institute & Faculty of Actuaries which is the UK body. It is an amalgamation of the Institute (English) and Faculty (Scottish) of Actuaries. There are two exam sittings each year – April and September. Students typically sit two exams at each sitting. The percentage of people passing at each sitting is usually between 30% and 50%. So, for someone without the 8 exemptions from the actuarial degree it can take anywhere from 4 years to 15 years to get through the exams. Most people aim to complete the exams in 6 or 7 years. However, if you have the 8 exemptions you would expect to qualify considerably faster.

Neil: Where did you do your degree, Philip? Did it prepare you for actuarial studies?
Philip: Some people, such as myself, study a Mathematics degree. I got my Mathematics and Statistics degree in Maynooth where I studied topics such as Calculus, Applied and Theoretical Statistics, Number Theory and other interesting topics (with less applied uses) such as the History of Mathematics and Euclidean & non-Euclidean Geometry. I also studied a Statistics Masters degree in UCD which was useful in giving me grounding in statistics which is useful in actuarial mathematics. I then submitted the course content of my degrees together with my results and obtained exemption from 1 of the 8 maths based exams.

Neil: What Secondary school maths does an actuary use? If any?
Philip: There are many topics in actuarial mathematics which are new to a college or school graduate. However, like all areas of mathematics you need to know the building blocks in order to understand more difficult topics. The main areas of secondary school maths which I still use would be calculus and probability. Calculus is a big part of many of the actuarial exams as is probability. The areas of actuarial work dictate what kind of mathematics you will do on a day-to-day basis.

Neil: What is the usual maths that you use every day? Are actuarial studies always linked with business and finance?

Philip: There are four main areas of actuarial work –

- Pensions, Life Assurance, General Insurance and Investment. I work in Pensions and so use indices, logs, probabilities and solving equations quite a lot, together with other basic tools such as averaging and interpolation which Excel is great for helping out with! We use a lot of mortality tables which tell us when people are expected to die and so calculate how much money needs to be set aside now in order to pay a person a pension until they die.

Life Assurance actuaries do similar work but also need to calculate how much of a reserve the company needs to hold back in order to maintain it's solvency.

General Assurance actuaries carry out similar work again but look at the probability of events occurring such as car crashes or fires as opposed to people dying. They also work in the whole area of reinsurance where the risk of insuring against a certain event is spread between several companies and so the impact of the event occurring is shared.

Investment actuaries work in the area of building portfolios of investments which offer the highest return with least amount of risk. These actuaries work a lot with derivatives.

Neil: Tell me Philip, is there a requirement of secrecy in your work?

Philip: All actuaries must comply with a code of ethics laid down by the Institute and Faculty of Actuaries and can be disbarred for breach of these. Secrecy and discretion is a big part of our job as we deal with sensitive data such as people's salaries, dates of birth, and so on.

Neil: Well, Philip, thank you for taking the time to tell me about your occupation of actuary. I have found it really informative . It's been a pleasure talking to you.

Neil Hallinan, Dublin

Speaking of Pi

With these simple guidelines you can sing-along 116 digits of π with Kate Bush (Pi, *Aerial).*
Here are the first 150 digits of π (Kate Bush does not sing the sections bracketed out):

3 ·1415926535 8979323846 2643383279 5028841971 6939937510 5820974944
5923078164 06286208 [99 8628034824 3421170679] 8214808651 3282306647
0938446095 5058223 [172 5359408128 …]

The Best Card Trick Ever

Introduction

In January of this year I entered an individual project at the BT Young Scientist Competition which, I'm happy to say, was awarded first prize in the Junior section of the Physics, Chemistry and Mathematics category. The project was entitled *Games and the Mathematics they Teach.* Its aim was to show how the study of the maths behind certain games/puzzles/tricks can make learning mathematics more interesting for young pupils.

The project looked at four major 'games' and a number of minor ones. The four major games I examined were

1. The Tower of Hanoi puzzle
2. Magic Squares
3. Chessboard puzzles
4. The Best Card Trick Ever

The minor games/puzzles/tricks that I examined were

1. The sum to 100 problem
2. The game of Mastermind
3. The Age Trick
4. Tangrams
5. The 21-card trick

The one that visitors to my stand liked the most was "The Best Card Trick Ever". Many of these visitors were teachers who asked me lots of tough questions. Over the next few pages I will explain how this trick works and point out how it teaches something about:

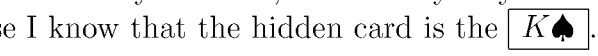
- the Multiplication and Pigeonhole Principles
- permutations
- a different idea of distance
- how to use a code to transmit information secretly

The Best Card Trick Ever

It was my Dad who showed me this trick and helped me to understand why it works. He read about it in a mathematics journal.[1]

Three people take part in this trick: you (the reader) who deals five cards to yourself from a deck of 52 cards; my assistant who takes these five cards from you after you have looked at them but without letting me see what they are. My assistant then shows four of the five cards face upwards on a table for us all to see. Then I amaze you by announcing what the hidden fifth card is.

Example of Performing the Trick: Deal a hand of five cards to yourself from a deck of 52 cards. Do not show them to me; instead show them to my assistant, who will now show me four of them, say

There is one card left in your hand, known only to you and my assistant. Not so! Because I know that the hidden card is the  .

Surely, this is impossible? My assistant showed me four cards, which means there are 48 cards left that could be the hidden one. I received the four cards in some specific order, and by varying that order my assistant could pass me one of

$$4 \times 3 \times 2 \times 1 = 4! = 24$$

messages. (This calculation uses the Multiplication principle.)

However, this is only half of the total number of possibilities. So how did I know?

Further Examples: If my assistant hands me the following four cards in the order shown from left to right:

then I say that the hidden fifth card is the . So again, how do I know?

[1] Michael Kleber, "The Best Card Trick Ever", *The Mathematical Intelligencer*, 24, no. 1, (Winter 2002): 9-11.

Similarly, when my assistant sees the five cards

she hides the

and displays the four cards

in that order. Why?

Explanation: Since there are five cards but only four suits it will always be the case that *at least* two of the cards are of the same suit. (Why? Because of the Pigeonhole Principle.) Thus my assistant is always able to choose a pair of cards of the same suit.

> It is one of this pair of cards which my assistant keeps hidden.

Step I: The *first* card my assistant shows me is of the same suit as the card that stays hidden.

For example, if the five cards you hand to my assistant are

then she will hide exactly one of

and place the other as the *first* card in the display of four cards.

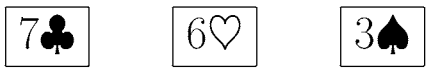
Once I see this first card, there are *only* 12 choices for the hidden card. So this first cards gives a lot of information. Dad told me how this informaton is measured exactly but I didn't go into this when I was defending my project because it uses logs which we haven't covered at school yet.

But how can the three remaining cards be used to single out one possibility from a possible twelve, when these cards can only be arranged in

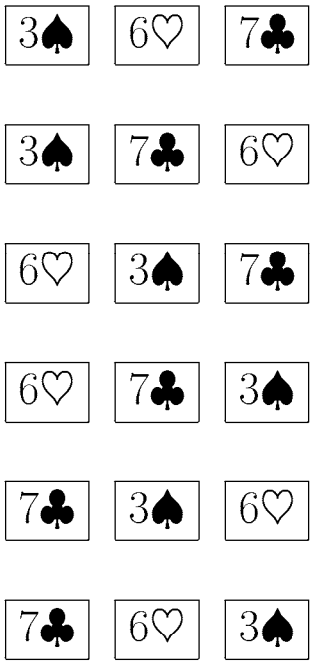
$$3 \times 2 \times 1 = 3! = 6$$

ways? (Multiplication Principle again.)

For example, the six different ways of arranging the three cards:



are



Thise is where the teacher can tell students about *permutations* (and also about *combinations*).

So the question is, how are the remaining three cards used to tell which of the twelve possible cards is hidden?

> Step II: This is achieved by my assistant carefully choosing which of the two cards from the same suit pair should be displayed.

Using the following ranking

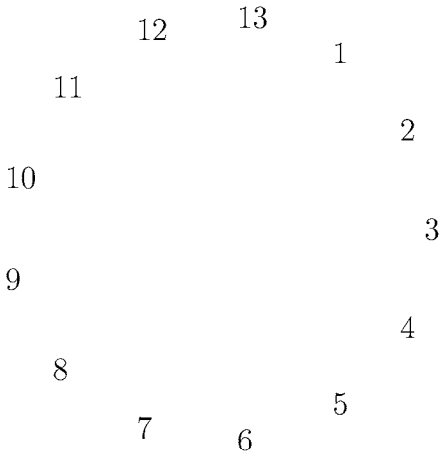
$$\boxed{A} \longleftrightarrow 1$$
$$\boxed{2} \longleftrightarrow 2$$
$$\boxed{3} \longleftrightarrow 3$$
$$\boxed{4} \longleftrightarrow 4$$
$$\boxed{5} \longleftrightarrow 5$$
$$\boxed{6} \longleftrightarrow 6$$
$$\boxed{7} \longleftrightarrow 7$$
$$\boxed{8} \longleftrightarrow 8$$
$$\boxed{9} \longleftrightarrow 9$$
$$\boxed{10} \longleftrightarrow 10$$
$$\boxed{J} \longleftrightarrow 11$$
$$\boxed{Q} \longleftrightarrow 12$$
$$\boxed{K} \longleftrightarrow 13$$

the two cards of the same suit correspond to two numbers between 1 and 13.

Now we need to figure out a way to use one of these numbers to point what the other one is. This is where the idea of a *code* comes in.

I really enjoy the clever way this is done.

A New Type of Distance: Imagine the thirteen numbers $1-13$ arranged in a circle as they might appear on a 13-hour clock face:



How far is the number 2 from the number 10 on this clockface?

To go clockwise from the number 2 to the number 10 requires 8 steps but to go from 2 to 10 in the anti-clockwise direction requires only 5 steps. Thus, we may say that when distance is measured on this clock, the number 2 is 5 units (and not 8) from the number 10.

> Exercise: Explain why 6 units is the furthest any number is from any other number!

But 6 is exactly the number of permutations we can have with three cards.

Thus we can set up a correspondence between the six permutations and the numbers 1 to 6.

One way of doing this is as follows:

$$SML \longleftrightarrow 1$$

$$SLM \longleftrightarrow 2$$

$$MSL \longleftrightarrow 3$$

$$MLS \longleftrightarrow 4$$

$$LSM \longleftrightarrow 5$$

$$LMS \longleftrightarrow 6$$

Here S stands for the card among the three cards which has the smallest numerical value, M for the card with the middle numerical value and L for the card with the highest numerical value.

For example, the permutation

corresponds to MLS and so, according to the above correspondences, stands for the number 4.

What if two or more of the three cards have the same numerical value as in

?

In this case the suits are ordered lexiographically as follows:

♣ ♢ ♡ ♠

so that the above permutation corresponds to SLM since the 6♡ ranks higher as a 6 than the 6♣. Hence the permutation represents the number 2.

Summary: So now let us perform the trick from beginning to end.

Step I: Firstly we know that my assistant can always choose from the five cards dealt, a pair of the same suit.
If my assistant sees the five cards

she identifies the pair of the same suit as

Which of these does she hide and which does she display?

Step II: Because the distance between 3 and 9 is 6 she hides the

and displays the four cards

in that order.

Now the first 3♢ tells me that the hidden card is also a diamond. The permutation of the three remaining cards

corresponds to LMS and so has numerical value 6. I now add 6 to the 3 in the 3♢ and announce that the hidden card is the 9♢ !

Another Example: If on the other hand, my assistant sees the five cards

she identifies the pair of the same suit as

Which of these does she hide and which does she display?

Since the numerical equivalent of a Q is 12 and because $12 - 4 = 8$, there is no point hiding the Q♢ and displaying the 4♢. Why not?

Because to go from the 4 to the 12 requires 8 to be added to the 4, but the most we can add on is a 6. However, since the clock-distance between 4 and 12 is $13 - 8 = 5$, she hides the

in this case and displays the four cards

in that order.

The first card tells me that the hidden card is a diamond.
The permutation corresponds to LSM which has numerical equivalent 5. Therefore, I add 5 to 12 to get 4 on the 13-hour clockface and announce that the hidden card is the

Original Example: When you hand my assistant the following five cards:

$7\spadesuit$ $Q\heartsuit$ $K\spadesuit$ $8\clubsuit$ $3\diamondsuit$

she notes immediately that the suited-pair is

$7\spadesuit$ $K\spadesuit$

Since these cards have the numbers 7 and 13 she notes that the smaller is within six units of the larger one. Thus she decides to hide the

$K\spadesuit$

and displays

$7\spadesuit$ $Q\heartsuit$ $8\clubsuit$ $3\diamondsuit$

in this order.
The first card displayed, namely the $7\spadesuit$, tells me that the hidden card is also a spade. The permutation

$Q\heartsuit$ $8\clubsuit$ $3\diamondsuit$

is of the form LMS and so I add 6 to the 7 to get 13.
Since this is the numerical equivalent of a King, I announce that the hidden card is the $K\spadesuit$.

`Exercise:` What would you do as my assistant if you were to see the following combination of five cards?

$8\spadesuit$ $5\diamondsuit$ $5\heartsuit$ $Q\diamondsuit$ $9\spadesuit$?

Solution: In this case we have two suited-pairs,

and

It is a little easier to work with the first pair than to use the second pair because its smaller card is within six units of the larger one. Hence you would hide the

and display

in this order.

The first card displayed, the 8♠, tells me that the hidden card is also a spade. The permutation

is of the form SML since the 5♢ is ranked lower than the 5♡. Since SML corresponds to a 1, I add 1 to the 8 to get 9 and announce that the hidden card is the

.

Concluding Remark: The Best Card Trick was also my favourite of all the 'games' I studied. I found it unreal that with just four cards it is possible to point out 1 of 48 possibilities. It takes time to get good at performing it and your assistant is just as important as you are because she must understand the secret code used as well as you do so that the trick works properly. However, because you get to be a magician it is well worth the effort to learn the mathematical ideas behind this very impressive trick.

Eamonn Flannery
First Year
Scoil Mhuire gan Smál
Blarney, County Cork

Mathematical Aspects of Irish Elections

Helmer Aslaksen
Department of Mathematics
National University of Singapore
Singapore 117543
Singapore
aslaksen@math.nus.edu.sg
www.math.nus.edu.sg/aslaksen/

Gary McGuire
Department of Mathematics
National University of Ireland
Maynooth
Co. Kildare
Ireland
Gary.McGuire@nuim.ie
www.maths.nuim.ie/staff/gmg/gmg.html

1 Introduction

In this paper we will describe the Irish voting system by means of examples from the 2002 general election. We will also discuss the random element of the current counting procedure, the policy of only considering the last received sub-parcel when transferring, and the fact that STV is non-monotonic.

Ireland's electoral system is proportional representation by means of a single transferable vote (STV) in multi-seat constituencies. Voters are asked to rank the candidates in order of preference on the ballot paper. Proportional representation, in some shape or form, is the preferred electoral system throughout most of the democratic world. STV is also used in Australia, New Zealand, Malta and Northern Ireland. However, the preferred choice in the UK is the single member plurality or "first-past-the-post" system where only one winning candidate represents the constituency. This system can dramatically distort election results. For example, in the 1997 UK general election, the Labour party won 63% of the seats with a 43% share of the vote. In the 2002 general election in Ireland, Fianna Fáil won 48% of the seats with 41.5% of the first preference votes, and the outgoing Fianna Fáil/Progressive Democrats government together won 53% of the seats with 45.4% of the first preference votes. These two figures are often close in an STV election. Because of this, some argue that STV is more representative of the wishes of the people, and this is probably the main reason for using STV.

In 1959 and 1968 the Fianna Fáil government held a referendum trying to switch to the "straight vote" system as in the UK, but they were defeated. During a 1959 Dáil debate before the referendum Sean Lemass said: "There are not half the deputies in this house, much less half the electorate, who can give an intelligent explanation of what happens to a number three preference on a ballot paper." We hope this article may help readers in this regard.

All constituencies have three, four or five seats. Small three-seater constituencies favour the larger parties, and it is generally accepted that the optimal size for STV constituencies is at least five ([18]). In the 1977 election there were no less than 26 three-seaters, 10 four-seaters and only 6 five-seaters. Moreover, all the Dublin constituencies were three-seaters. There are reasonable arguments for three-seaters in rural areas, but not in a congested city. However, an independent boundary revision before the 1981 election removed these anomalies ([4]).

In Ireland, the candidates are listed alphabetically on the ballot paper, but they could also be listed by party or randomly. Before 1965 party affiliations were not given on the ballot paper. This change increased party loyalty among the voters. For a detailed discussion of transfer patterns in Irish elections see the book by Sinnott ([17]).

Irish political parties have long experience in vote management techniques like spreading the first preference vote among the party candidates. While this favours the big parties, it can also lead to intra-party factionalism, and Irish political culture is personified by a high degree of localism and a large number of independent candidates ([3,4,17]).

In order to simplify the counting process, only the last received sub-parcel is considered when transferring surpluses. We will give examples of how this policy can lead to paradoxical results in Section 8, and discuss whether this affected the result in Dublin Central in 2002. With the introduction of electronic voting, it would be possible to no longer just consider the last received sub-parcel. There are arguments for and against the introduction of electronic voting. These issues re widely discussed elsewhere, and we will not go into them further.

There is a vast literature on the mathematics of voting systems. One important objection against STV is that it is “non-monotonic”. This means that increasing your first preference votes can actually prevent you from being elected! We will give a simple example of this in Section 9.

We will not go into the finer details of cases of ties, when lots need to be drawn. For further details we refer the interested reader to the Electoral Act, 1992, which is available online ([19]). While writing this paper, we consulted a number of web sites explaining the rules for the counting. We discovered that several of them did not state the rules properly. The reader may enjoy looking at some of these sites and comparing them with this paper and the Electoral Act of 1992. A good source for additional information is the web site Irish Election Database ([16]), and of course, the excellent “How Ireland voted” series of books ([8, 9, 10, 12, 13]).

An interesting classroom exercise would be to present the counting procedure as an algorithm. This is relevant because of computerized voting. For details, see the information paper on electronic voting ([20]). We attach a summary flowchart as an appendix. More detailed flowcharts can be found in ([2]). An asterisk in the tables denotes a sitting TD.

2 Outline of the Counting Procedure

We will start with an outline of the procedure, and then discuss the finer points later on.

At the beginning of the count for a constituency, all valid ballot papers are thoroughly mixed together and then sorted according to first preferences. If we denote the total valid poll by P and let N be the number of seats, then the quota Q is calculated by the formula

$$Q = 1 + \frac{P}{N+1},$$

where we round down if this number is not an integer. This is the smallest number of votes that guarantees election.

If the number of votes for any candidates is equal to or greater than the quota they are deemed elected. For example, in Dublin Central (Table 1), N = 4 and P = 34034, so Q = 6807·8 by the above formula. Thus the quota is 6807 and Bertie Ahern with 10,882 votes is deemed elected on the first count.

The surplus of an elected candidate is the number of votes by which that candidate's total number of votes, original and transferred, exceeds the quota. When a candidate is elected, the surplus ballots may be transferred to the remaining candidates according to the next preference on the ballots. The ballots are physically divided into *sub-parcels* (literally, different piles of ballots). Some ballots may not have a next preference for a continuing candidate, and these ballots are called *non-transferable*. Ballots that do have a lower preference for a continuing candidate are called *transferable*.

The general idea is that a surplus is transferred if and only if at least one of the following conditions are met.

1. If transferring the surplus could elect the highest remaining candidate.

2. If transferring the surplus could bring the lowest remaining candidate equal to or above the second lowest remaining candidate.

3. If transferring the surplus could bring the lowest remaining candidate up to one quarter of the quota (candidates whose total reaches that figure qualify for reimbursement of some election expenses).

Candidate	Party	%	C1	C2	C3	C4	C5	C6	C7
Ahern, Bertie*	FF	32.0	10882	10882	10882	10882	10882	10882	10882
Costello, Joe	Lab	12.2	4136	4499	4539	5046	7870	7870	7870
Fitzpatrick, Dermot	FF	7.6	2590	4855	4968	5043	5560	6264	6418
Gregory, Tony*	Ind	16.7	5664	6477	6625	7242	7242	7242	7242
Kehoe, Nicky	SF	14.6	4972	5300	5343	5487	5818	6177	6339
Mitchell, Jim*	FG	11.1	3769	3976	4063	4268	-	-	-
O'Donnell, Patrick Noel	Ind	0.3	89	95	-	-	-	-	-
O'Loughlin, Paul Thomas	CSP	1.1	366	377	-	-	-	-	-
Prenderville, Tom	Ind	0.3	97	103	-	-	-	-	-
Simpson, Tom	GP	4.3	1469	1545	1652	-	-	-	-

Table 1: Dublin Central, 4 seats, quota 6807.

However, there are exceptions. As we will discuss in Section 4, there may be more than one surplus present. In that case we must see if the *sum* of the undistributed surpluses satisfies these conditions, so satisfying at least one of these conditions is not necessary. In addition, the surplus of another candidate may be larger, in which case that surplus is transferred first, so satisfying at least one of these conditions is not sufficient.

3 When is a Surplus Transferred?

A common misunderstanding is that the surplus is transferred if it could change the order of any of the lower-placed candidates or it could result in any candidate reaching one quarter of the quota. This is not true! We are only required to consider whether it could change the position of the *lowest*-placed candidate or result in the *lowest*-placed candidate reaching one quarter of the quota. For example, in Dublin Central (Table 1) after count 4, Gregory's surplus of 435 is not enough to move Mitchell up, so Mitchell is eliminated and the surplus is not distributed. However, Costello is only 3 ahead of Fitzpatrick, so distributing the surplus could have changed the order between them. Similarly, in Galway West (Table 5) after count 8, Fahey's surplus could have changed the order between Lyons and McDonagh, but it is not transferred at that point. It is transferred after count 10 when Callanan has been eliminated.

We now give examples of where these conditions are or are not met.

As an example where none of the conditions are met, we consider Carlow-Kilkenny, count 1 (Table 2). Aylward is elected but his surplus of 320 is not transferred because it does not satisfy any of these conditions. Let us check: Firstly, the highest continuing candidate is more than 320 from the quota. Secondly, the lowest candidate, Billy Nolan, is more than 320 below the

Candidate	Party	%	C1	C2	C3	C4	C5	C6	C7	C8
Aylward, Liam*	FF	20.5	12489	12489	12489	12489	12489	12489	12489	12489
Browne, Fergal	FG	9.0	5468	5530	5670	5762	5894	-	-	-
Collins-Hughes, Eddie	Ind	2.7	1614	1644	-	-	-	-	-	-
Hogan, Phil*	FG	12.9	7841	7852	8074	8239	8903	12238	12238	12238
Kiernan, Tom	SF	3.4	2078	2098	2216	-	-	-	-	-
McGuinness, John*	FF	15.4	9343	9371	9657	10085	10527	10654	10878	12111
Nolan, Billy	Ind	0.6	335	-	-	-	-	-	-	-
Nolan, M.J.	FF	14.3	8711	8772	8887	9140	9204	9832	9895	11009
O'Brien, Michael	Lab	6.1	3732	3739	3946	4220	-	-	-	-
Pattison, Seamus*	Lab	0.0	-	-	-	-	-	-	-	-
Townsend, Jim	Lab	7.0	4272	4330	4415	4568	6625	7646	7653	10615
White, Mary	GP	8.2	4961	5003	5401	6055	6696	7329	7355	-

Table 2: Carlow-Kilkenny, 5 seats, quota 12169. Note that Seamus Pattison is the Ceann Comhairle and is returned unopposed.

second-lowest candidate, Collins-Hughes. And finally, adding 320 to the lowest candidate's total would not bring Billy Nolan up to one quarter of the quota. Note that in fact Aylward's surplus is not transferred until count 7!

Next an example of where only condition 1 holds: Cork East, count 1 (Table 3). Ned O'Keeffe is elected on count 1 with a surplus of 10574 – 9158 = 1416. The highest remaining candidate has 8340, which is 9158 – 8340 = 818 away from the quota. The surplus is therefore transferred on the next count because it satisfies condition 1. However, it does not satisfy conditions 2 or 3.

Next an example of where only condition 2 holds: Carlow-Kilkenny, count 7 (Table 2). Aylward's surplus of 320 from count 1 is finally transferred, but just barely! Mary White from the Green Party has 7329 votes while Jim Townsend from Labour has 7646. The difference is 317. If White had received 4 votes less, she would have been eliminated, and Carlow-Kilkenny would have finished the count without a single surplus transfer!

Finally an example of where only condition 3 holds: Kerry South, count 4 (Table 6). O'Donoghue is elected on count 1 with a surplus of 9445 – 9162 = 383. One quarter of the quota is 2290.5 and Casey has 2216 after count 3. The surplus is not transferred on counts 2 or 3, but it meets condition 3 (and does not meet conditions 1 or 2) after count 3 so it is transferred on count 4.

Candidate	Party	%	C1	C2	C3	C4
Ahern, Michael*	FF	18.2	8340	9204	9204	9204
Bradford, Paul*	FG	15.4	7053	7224	7718	8231
Manning, Patrick	CSP	0.4	187	189	-	-
Mulvihill, John	Lab	10.5	4813	4838	5559	-
Murphy, June	SF	5.7	2624	2685	-	-
O'Keeffe, Martin	GP	2.5	1136	1151	-	-
O'Keeffe, Ned*	FF	23.1	10574	10574	10574	10574
Sherlock, Joe	Lab	10.5	4792	5015	6192	8660
Stanton, David*	FG	13.7	6269	6324	6978	8438

Table 3: Cork East, 4 seats, quota 9158.

Candidate	Party	%	C1	C2	C3	C4	C5	C6	C7
Brolchain, Niall	GP	4.4	2193	2216	2225	2241	2308	2416	2450
Callanan, Danny	SF	3.0	1468	1485	1492	1498	1509	2166	2200
Coistealbha, Sean AC	SF	2.7	1311	1373	1379	1383	1397	-	-
Cox, Margaret	FF	6.6	3269	3697	3700	3704	3752	3801	3870
Cuiv, Eamon*	FF	20.1	9947	9947	9947	9947	9947	9947	9947
Fahey, Frank*	FF	14.6	7226	7902	7903	7904	7993	8120	8318
Grealish, Noel	PD	5.5	2735	2787	2789	2791	2858	2876	3121
Healy-Eames, Fidelma	FG	2.7	1320	1340	1343	1347	-	-	-
Higgins, Michael D.*	Lab	10.5	5213	5315	5330	5343	5510	5646	5772
Lyons, Donal	PD	4.0	1995	2017	2017	2026	2058	2068	2545
Manning, Eileen	Ind	0.2	96	97	100	-	-	-	-
McCormack, Padraic*	FG	9.6	4760	4850	4853	4859	5260	5318	5502
McDonagh, Michael	FG	4.6	2279	2300	2301	2306	2619	2632	2670
McDonnell, Declan	PD	3.0	1462	1479	1480	1484	1499	1506	-
Nulty, Joseph The Ice Man	Ind	0.2	76	77	-	-	-	-	-
Scallon, Dana Rosemary	Ind	3.4	1677	1727	1731	1740	1812	1872	1920
Walsh, Seamus	Ind	4.9	2439	2559	2573	2588	2626	2742	2774

Table 4: Galway West, 5 seats, counts 1 to 7, quota 8245.

	C8	C9	C10	C11	C12	C13	C14	C15	C16
Brolchain, Niall	2651	3108	3111	3267	3391	3749	3775	-	-
Callanan, Danny	2306	-	-	-	-	-	-	-	-
Cox, Margaret	4082	4297	4314	4623	4782	5244	5273	5585	5906
Cuiv, Eamon*	9947	9947	9947	9947	9947	9947	9947	9947	9947
Fahey, Frank*	8318	8318	8318	8318	8318	8318	8318	8318	8318
Grealish, Noel	3233	3315	3323	4544	5098	5325	5325	5684	6215
Higgins, Michael D.*	6095	6664	6682	7171	7498	8151	8244	10580	10580
Lyons, Donal	2713	2787	2798	-	-	-	-	-	-
McCormack, Padraic*	5865	6001	6012	6285	7623	8420	8420	8420	8420
McDonagh, Michael	2748	2808	2812	2888	-	-	-	-	-
Walsh, Seamus	2944	3154	3155	3234	3328	-	-	-	-

Table 5: Galway West, 5 seats, counts 8 to 16, quota 8245.

Candidate	Party	%	C1	C2	C3	C4	C5	C6	C7
Barry, Donal	Ind	2.5	934	-	-	-	-	-	-
Casey, Sheila	FG	5.3	1934	2033	2216	2220	-	-	-
Fitzgerald, Seamus	FG	12.4	4539	4732	4791	4811	6083	-	-
Fleming, Tom	FF	18.9	6912	6995	7255	7407	7593	8276	8381
Grady, Donal	Ind	3.7	1346	1449	-	-	-	-	-
Healy Rae, Jackie*	Ind	17.0	6229	6455	6819	6893	7147	8409	8584
Moynihan-Cronin, Breeda*	Lab	14.5	5307	5480	5946	5979	6432	9442	9442
O'Donoghue, John*	FF	25.8	9445	9445	9445	9445	9445	9445	9445

Table 6: Kerry South, 3 seats, quota 9162.

4 Multiple Surpluses

There could be more than one surplus. For example, this would happen if an elected candidate's surplus was not transferred, but then another candidate was elected. An example of this is Dublin Central (Table 1). On count 4 Gregory was elected, but his surplus was not transferred

then. Instead, Mitchell was eliminated. This resulted in Costello being elected on count 5. There were then two surpluses present.

There could also be more than one candidate reaching the quota in the first count as in Cavan-Monaghan, where two people were elected on the fist count. Again there were two surpluses present.

When more than one surplus is present, the largest is checked against the three conditions. If the largest surplus on its own does not meet any of the three conditions, the largest surplus will be transferred on its own if the *sum* of the surpluses meets one or more of the conditions. Otherwise we proceed with the elimination of candidates. We consider the sum of all untransferred surpluses to make sure that we have exhausted all possibilities before a candidate is eliminated.

Note that it does not matter who reached the quota first. The only time an earlier surplus gets priority is if the two are equal.

The question of whether a surplus should be transferred is considered again after each count. For example, in Carlow-Kilkenny (Table 2), Aylward was elected on the first count but the surplus was not transferred until the 7^{th} count.

5 Elimination

If none of the three conditions for transferring a surplus are met for any of the surpluses or the sum of the untransferred surpluses, we eliminate the candidate with the lowest number of votes. All the transferable votes of the eliminated candidate are transferred to the remaining candidates according to the next preference. The non-transferable votes are classified as “non-transferable not effective”, since they did not have any effect on the election, and are set aside.

It is important to consider any untransferred surplus. Suppose the three lowest candidates are A with 2004 votes, B with 2002 votes and C with 1 vote. If there is as surplus of 2000 present, then it will not be transferred because it cannot move C ahead of B. But when eliminating C, it may seem like we could eliminate B and C together, since just eliminating C alone could not move B ahead of A. However, we must not forget the untransferred surplus. We must first eliminate C and then transfer the surplus. The transferred surplus could result in B moving ahead of A.

It is more efficient to exclude several candidates in one count. For example, in Dublin Central (Table 1) three people were eliminated at count 3 and in Westmeath four were

eliminated in one count. The law says that this may be done when we have checked two conditions. Firstly, the combined vote of the candidates to be eliminated together with any untransferred surplus must be less than the vote of the candidate above them. In that case, separate elimination of some of them and transfer of any surpluses would not be enough to bring the highest of them above any of the candidates above them. Secondly, we must make sure that if any of the candidates to be eliminated are below one quarter of the quota, separate elimination of some of them and transfer of any surpluses would not bring any more of them over one quarter of the quota. In other words, nobody should be eliminated before it is clear whether they can claim reimbursement of some expenses.

In Galway West (Table 4), Nulty and Manning are eliminated separately. Nulty is eliminated after count 2 and Manning is eliminated after count 3. According to the law, they could have been eliminated together. One quarter of the quota is 2061 and it is true that Nulty's votes could have brought Lyons over that, but that is not the question. Lyons is not about to be eliminated at this stage. Notice that we could not eliminate Healy-Eames at this stage, because the votes from Nulty and Manning could have brought her ahead of McDonnell. Galway West was the longest count in 2002 with 16 counts, but that was partially unnecessary.

6 Filling of Last Seats

As the count proceeds, the non-transferable ballots are being set aside. In effect, the total poll is decreasing and the quota is being reduced. It can thus happen at the end that candidates are elected without reaching the official quota. After the final count, the candidate or candidates who have the highest votes among those remaining will be deemed elected, even if they have failed to secure a quota. Carlow-Kilkenny (Table 2) is a 5-seater, but since Pattison was the Ceann Comhairle and automatically elected, only four seats were up for election. Only two of the remaining five candidates had reached the quota in the end, so two were elected without making the quota.

After all the seats have been filled, if a candidate could possibly reach one quarter of the quota then surpluses should still be transferred. In Donegal South-West in 1982, three candidates were elected on the first count. However, the counting continued until the fourth count, with the surplus of all three elected candidates being transferred, in order to determine whether the Workers Party candidate could claim reimbursement of some expenses, which he did ([16]).

7 How is a Surplus Transferred?

The surplus transfer is where the counting rules get more complicated. When a candidate is elected on the first count, their transferable papers are divided into sub-parcels (literally, piles of ballots) according to the second preference. These sub-parcels are distributed as we explain below. On later counts, when the votes credited to an elected candidate consist of original and transferred votes, the *last received sub-parcel* consists of those ballots received in the count that resulted in the candidate reaching the quota. In this case, the returning officer examines the papers contained in the last received sub-parcel of the candidate who has just been deemed to be elected, and arranges the transferable papers in this sub-parcel in further sub-parcels according to the next available preferences for continuing candidates recorded on those papers. Precisely how these subparcels are distributed is the topic of this section. A new sub-parcel is also made for the non-transferable papers.

If candidate A is elected on the first count, then the last sub-parcel consists of all A's first preferences. However, if A is elected on a later count, the last sub-parcel will *not* involve the ballots that have A as a first preference, so the transfers do not reflect the preferences of the voters who put A as their first preference. See Section 8 for further discussion on the last sub-parcel.

Note that if the number of transferable papers is less than the surplus, some of the non-transferable votes of that elected candidate are used to make up the remainder of the surplus. These ballots are called *"non-transferable not effective"* and are set aside. The remainder of the non-transferable votes are retained as part of the candidate's quota and placed with that candidate's papers. These are called *"non-transferable effective"* votes. (For example, if A is 50 votes below the quota and receives a surplus distribution of 100 votes, then A's surplus is 50. Suppose that the 100 votes received have 40 transferable and 60 non-transferable. The surplus of 50 consists of 40 transferable and 10 non-transferable not effective ballots.) This terminology is not entirely self-explanatory! You may think of it as follows. If a non-transferable vote is included in the quota, it was effective and is called "non-transferable effective". But if it has no further preferences for continuing candidates and was not used in bringing the last candidate over the quota, then it was not effective in any way and is called "non-transferable not effective".

Unfortunately, it gets even more confusing. Some detailed voting tables give the number of "non-transferable votes". However, that is not really the number of non-transferable votes! It is the number of non-transferable not effective votes. To understand this, it is necessary to understand why the number of "non-transferable votes" is listed. If you add the number of votes after the last count is usually quite a lot smaller than the total number of votes cast. The

To illustrate, let us look at Dublin Central, count 6 (Table 1). Costello is elected on count 5 with a surplus of 7870 – 6807 = 1063. His last received sub-parcel of votes, which took him over the quota, contains 2824 votes. On the next count, his surplus is transferred. 704 go to Fitzpatrick and 359 to Kehoe. So we see that the whole surplus is transferred and there are no non-transferable non-effective papers. Were there any non-transferable effective votes in the 2824? We do not know. If there were, they were put in a separate sub-parcel and the remaining votes were arranged according to the next preferences for the remaining candidates. After the proportions are calculated, the required numbers of papers were taken form the top of these piles and transferred to Fitzpatrick and Kehoe.

Let us sum up the three examples we have considered. For Ahern and Costello the number of transferable papers is greater than or equal to the surplus. Therefore we need to guess the number of non-transferable papers and compute ratios. For Gregory, the number of transferable papers is less than the surplus, so we know that all the ones that were used to make up the quota were also non-transferable, and we do not need to worry about ratios.

What is the difference between Ahern, Costello and Gregory in this respect? First of all, it depends on which count we are at. All of Ahern's papers are first preference votes, and although some people may only vote for him, it would be extremely unlikely if we were in Case 1 on the first count. However, once the papers have been transferred around a few times, we are more likely to run out of further preferences. There are at least two reasons for this. Firstly, we are further down on the list of preferences – for Ahern we are looking at second preferences, while for Costello an Gregory it is at least third preference. Secondly, the candidates listed as preferences may already have been eliminated.

Another factor is the candidate's party. Gregory runs as an Independent. Independent voters do not necessarily follow vote management, and may have fewer preferences, or they may have preferences for other Independent or small-party candidates who have been eliminated in earlier counts. Costello's (Labour) surplus is the result of a Fine Gael to Labour transfer, so they are likely to be "well-managed" votes with lots of preferences. Gregory's surplus comes from the Green Party candidate Simpson, and only about half of them have a vote for Fitzpatrick (FF) and Kehoe (SF). However, voter transfers can be surprising. 34% of the 1063 papers in the Mitchell to Costello parcel that are transferred have a Fine Gael, Labour, Sin Féin transfer.

Traditionally, the transfer system has worked against Fianna Fáil ([10]), as supporters of virtually all other parties used their preference votes against them and Fianna Fáil candidates who had done well on the first count were overhauled by candidates from other parties.

8 The Last Received Sub-parcel Problem

There are alternative methods of distributing the surplus to the method used in the Republic of Ireland. Each of these methods has its own advantages and disadvantages. We discuss this issue in this section.

In order to save time, only the last received sub-parcel is considered when transferring surpluses in an Irish general election, as we discussed in the previous section. Unfortunately, this could cause fundamental distortion. There is a discussion of this in the book by Sinnott ([17]), see also ([6]).

In order to understand how this policy can distort the results, we have constructed some hypothetical examples. Suppose County X is a three-seater with 39996 votes and hence a quota of 10000, and that the first preference vote is given as in Table 7. Assume that voters for candidate A all give their second preference to candidate B and their third preference to candidate C. The surplus from A will then first push B over the quota, and is then transferred to C who also is elected.

There are two things to notice. First of all, the same sub-parcel of votes is involved in electing all three candidates. In a sense, the votes cast for A become "transferable super-effective"! Secondly, this happens regardless of the preferences of B. Even if all first preference votes for B have D as their second preference, there will be zero transfers to D, since we only consider the sub-parcel transferred from A. However, if we look at all of B's votes, as much as 82% of them could have a next preference for D, so D could have received 819 votes instead of 0 if all of B's votes were taken into account! Obviously one can argue that this outcome distorts the will of the public.

We will now consider a variation of this. Suppose that there is a fifth candidate E, and that the first count figures are as in Table 8. Suppose that all of the (first preference) votes for E have A as their second preference, B as their third preference, and C as their fourth preference. When E is eliminated, the transfers will then first elect A, then B and finally C. Again, this happens regardless of the preferences of A, B and C. Suppose that D has wide cross-party appeal, and that all the voters for A, B and C have D as their second preference. The only problem is that the 5% who vote for E do not like D. So D is the first choice of (just under) 25% and the second choice of 70% and still fails to get elected! Notice how the current system totally ignores the fact that 70% have D as their second preference, while the ballots with E as first preference are "transferable super-effective"! In fact, in this example the only ballots where further preferences are considered are the votes for E. This election result would not be very representative of the public will, and some argue against STV because things like this can

difference is the number of non-transferable not effective votes. Listing the number of "non-transferable votes" shows that no votes have disappeared.

So in particular, if the table says that there were zero "non-transferable votes", that does *not* necessarily mean that there were no non-transferable papers! It just means that there were no non-transferable not effective papers. In other words, the surplus was smaller than the number of transferable papers. There could still be non-transferable effective papers that were used to make up the quota. So there is no need to account for any reduction in the total number of votes. However, when studying transfer patterns ([17]), the actual number of non-transferable (effective or not effective) papers is of interest.

This sounds complicated, but let us consider some simple examples. There are two cases.

Case 1: Number of transferable papers ≤ surplus. In this case all transferable papers are transferred. Consider Dublin Central, count 7 (Table 1). Gregory is elected on count 4 with a surplus of 7242 – 6807 = 435 votes. His surplus is distributed on count 7, and 154 votes are transferred to Fitzpatrick and 162 to Kehoe. His last received sub-parcel, which took him over the quota, has 617 votes. The number of non-transferable effective votes can be calculated by subtracting the surplus from the number of ballots in the last received sub-parcel. In this case we have 617 – 435 = 182 non-transferable effective votes. The number of non-transferable not effective votes can be calculated by subtracting the number of papers transferred from the surplus. In this case we have 435 – (154 + 162) = 119 non-transferable not effective votes. Thus, in the last received sub-parcel of 617 votes, there are in fact a total of 301 non-transferable votes. Of these, 182 are effective (retained as part of the quota) and 119 are not effective (used as part of the surplus). This leaves 617 – 301 = 316 votes to be transferred.

Case 2: Number of transferable papers > surplus. In this case transferable papers are transferred in proportion, from the top. The number of papers to be transferred from each sub-parcel (which is X, say) bears the same proportion to the number of papers in the sub-parcel Y as the surplus S bears to the total number of transferable papers T. In other words,

$$\frac{X}{Y} = \frac{S}{T}$$

and we use this formula to calculate X. This is what generally happens if someone is elected on the first count, but theoretically a candidate whose votes do not give any other preferences but who makes the quota on the first count could be a counter example. In that case the number of transferable votes would be 0 and hence less than the surplus. (We remark that a surplus can be 0, as happened with O'Flynn in Cork North-Central in 2002.)

Let us look at Dublin Central, count 1 (Table 1). Bertie Ahern gets 10882 first preference

votes and is elected on the first count. His surplus is 10882 – 6807 = 4075. All Ahern's papers are put into sub-parcels according to the second preferences, and the non-transferable papers are also put into a sub-parcel. Since these are first preference votes and no candidates have been eliminated, there probably are not any non-transferable papers, so we will assume that all his papers are transferable. How many are there in each sub-parcel? We do not know. All we know is the ratio in which the surplus is transferred. For the sake of illustration, let us guess that Fitzpatrick's sub-parcel has, say, 6050 papers and Kehoe's sub-parcel has, say, 878 papers. How did we guess these figures? We know the numbers of transferred ballots. The ratio *S/T*, where *T* is the total transferable vote for Ahern and *S* is his surplus is

$$\frac{S}{T} = \frac{4075}{10882} = 0 \cdot 374471 \quad ...$$

If Fitzpatrick's sub-parcel contains 6050 papers, then we should therefore transfer

$$(6050) \times (0 \cdot 374471) = 2265 \cdot 55 ...$$

papers to him, and transfer

$$(878) \times (0 \cdot 374471) = 328 \cdot 786 ...$$

papers for Kehoe.

[Note: How do we deal with the fractions? We compute the corresponding numbers for all the other candidates and round up the answers that have the largest fractional part, and we round up as many of these as are needed in order to make the total number of transferred ballots equal to the surplus (for further details see ([19]). It is precisely at this point that a mistake was discovered in the program used for counting the votes electronically in 2002, see [2]. The program rounded down all these numbers, whereas some should be rounded up.]

After calculating the proportions for each candidate, we transfer 2265 papers from Fitzpatrick's sub-parcel to the top of his pile of first preference votes. A key point here is that we transfer the *top* 2265 papers as a *sub-parcel*, from the Fitzpatrick pile of Ahern's first preferences to the *top* of the Fitzpatrick pile of first preferences. It turns out here that we also round up the figure for Kehoe, so we transfer a sub-parcel of the top 329 papers to his pile.

If we are in Case 2 on a count later than the first, the surplus is transferred in proportion as in the above example, but the transferred papers are only taken from the top of the last received sub-parcel of votes that brought the elected candidate over the quota. The papers for inclusion in the surplus are taken from the top of each candidate's sub-parcel of next preferences contained in the sub-parcel of papers last received by the elected candidate. See Section 8 for further discussion of this.

	C1	C2	C3
A	12000	12000	12000
B	9001	11001	11001
C	9000	9000	10001
D	9995	9995	9995

Table 7: County X, 3 seats, quota 10000.

	C1	C2	C3	C4
A	9994	12000	12000	12000
B	9001	9001	11001	11001
C	9000	9000	9000	10001
D	9995	9995	9995	9995
E	2006	-	-	

Table 8: County Y, 3 seats, quota 10000.

happen. However there is a solution, as we will discuss below.

Let us consider first a real life example where the transfer patterns are probably complex, although we cannot know for sure. After count 5 in Dublin Central (Table 1) Kehoe (Sinn Féin) was ahead of Fitzpatrick (Fianna Fáil) before the transfer of the surplus from Costello (Labour). However, the papers that brought Costello over the quota on count 5 were all from the elimination of Mitchell (Fine Gael). Now consider the transfer of Costello's surplus. The ballots in this surplus would mostly have Mitchell (FG) as the number 1 preference. The result was that 704 ballots were transferred to Fitzpatrick (FF) and 359 to Kehoe (SF). These Fine Gael voters preferred FF to SF in the ratio 2 to 1. What would have happened if all of Costello's votes had been considered? Perhaps a majority of these Labour voters would have ranked Kehoe (SF) above Fitzpatrick (FF), and the final outcome would have been different.

We remark that this was the only example in the 2002 election when transfers from Fine Gael or Labour helped elect a Fianna Fáil candidate.

With electronic voting there is no longer any reason for considering the last received subparcel only. All the votes with all their further preferences could be taken into consideration by transferring all the papers, but at a fraction of their value. This is called the Gregory

method, named after J.B. Gregory who proposed it in 1880. It is used in Northern Ireland for local and European elections, but not for Westminster elections ([4]). In Ireland it is only used for the Seanad panel elections, where the number of votes is about 900. It is being proposed for use in local elections in Scotland, and appears to have been passed by parliament there. To use this method here (in a Republic of Ireland general election) would require a change in legislation.

The exact details of how to implement the Gregory method turn out to be quite subtle, and there are many different approaches, including the Meek method ([6, 14, 15]). Some methods involve changing the rules on transfers of votes to allow transfers to candidates who have already been elected. Of course, additional surpluses are then created and these, in turn, may be transferred. Some methods also recalculate the quota in the course of the count.

Due to limitations of space we will just give some pointers and comments to relevant papers. The article by Farrell and McAllister ([5]) discusses the changes in the law in Australia in 1983, and introduces the terminology *inclusive Gregory method* and *weighted inclusive Gregory method*. We feel that using the term inclusive Gregory method to describe the method introduced in the 1983 law is somewhat unfortunate. The legislators probably intended to implement the *weighted* inclusive Gregory method, but did not phrase the law correctly. It is easy to see that from a mathematical point of view, the inclusive Gregory method is just a mistake, and does not deserve serious attention. However, it is curious to see that it was actually implemented, and the paper by Farrell and McAllister is important reading.

The briefing note by James Gilmour ([6]) for the Scottish local government and transport committee compares different methods of counting. His distinction between the philosophies of the "inclusive" and "exclusive" approaches in valuable and seems not to be mentioned elsewhere. It would be interesting to know if such considerations were previously brought up by Gregory or others. In the past, it seems likely that the practical limitations of the manual vote count has dictated the choice of methods, rather than intentional and fundamental difference of philosophy. However, with electronic voting it is possible to implement a wider range of methods, and it makes sense to consider the distinction.

Since the ballots are thoroughly mixed at the start of the counting process, the transfers should represent a random sample of the total number of papers which have the same preferences up to that candidate. Even with the preliminary mixing, there is still this random element in the counting procedure. This is discussed by Gallagher and Unwin ([7]). They constructed a probability model and studied past election results. The only relevant cases are when the final difference in votes between the last elected candidate and the runner-up is less than or equal to twice the number of randomly selected surplus votes that have reached these two candidates via other candidates. Their conclusion is that about one in five of all results could theoretically have

been affected. However, the probability of a result being affected is usually very small. They only found 5 cases where the probability was greater than 10%, but see also ([12, p176]).

As we have said, with the introduction of electronic voting the need to take a random sample could be removed because a computer could count all the preferences on all the ballots instead of taking a random selection. In the 2002 election, some constituencies did have electronic voting. However, in those constituencies the "random sample" legislation had to be obeyed. This involved the computer ordering all the votes in order to choose some at random. According to the government information paper ([20]), the computer was programmed to randomly order the papers by means of a random number generator called the Lehmer algorithm ([11]). This is a simple procedure involving modular arithmetic.

We remark that the last two paragraphs have been concerned with two important areas of mathematics, probability theory and number theory!

9 STV is Non-monotonic

One odd aspect of STV is that it is non-monotonic. This means that getting more votes may prevent you from being elected. This is discussed extensively in the literature on voting ([1, 4]). In this section we will give a simple hypothetical example that shows that STV is non-monotonic and briefly mention possible real-life examples.

X1	X2	X3
7997	4002	4001
9998	6002	

X1	X2	X3
7999	4000	4001
7999		8001

Table 9: County Z, quota 8000, original count on the left, recount on the right.

Three candidates from party X are fighting for the last seat in county Z. When X3 is eliminated, the transfers are evenly split among X1 and X2 and X1 is elected. But X2 does not like X1, and demands a recount. The recount shows that two X1 ballots had been placed among the X2 ballots, so at the second to last count X1 actually has 7999 and X2 4000. But when X2 is eliminated, it turns out that all the votes have X3 as their next preference, and X3 is elected!

Proponents of STV say that these occurrences are rare. It is hard to tell when this actually happen in Ireland, but Irish elections are often close and mathematically surprising things do happen frequently. According to "How Ireland Voted" ([9, 12, 13[) there were possible instances in 1987, 1992 and 1997. In 1989 in Dublin North-East, after the ninth count, Sean Haughey (Fianna Fáil) had 5,356 votes, Pat McCartan (Workers Party) had 4,995 and Neil Holman (Progressive Democrats) 4, 971. Holman was eliminated, and his transfers took McCartan ahead of Haughey. If 25 of Haughey's votes had been cast instead for Holman, McCartan would have been eliminated instead of Holman and McCartan's transfers might have elected Haughey ahead of Holman ([12, p.87]). In 1992 in Dublin North-East, Cosgrave might have been elected if 25 of his votes had been cast instead for McCartan ([9, p.78]). In 1997 in Limerick East, Jackman might have been elected instead of Kemmy if 150 of his votes had been cast instead for Ryan leading to the elimination of Kemmy ([13, p.145]).

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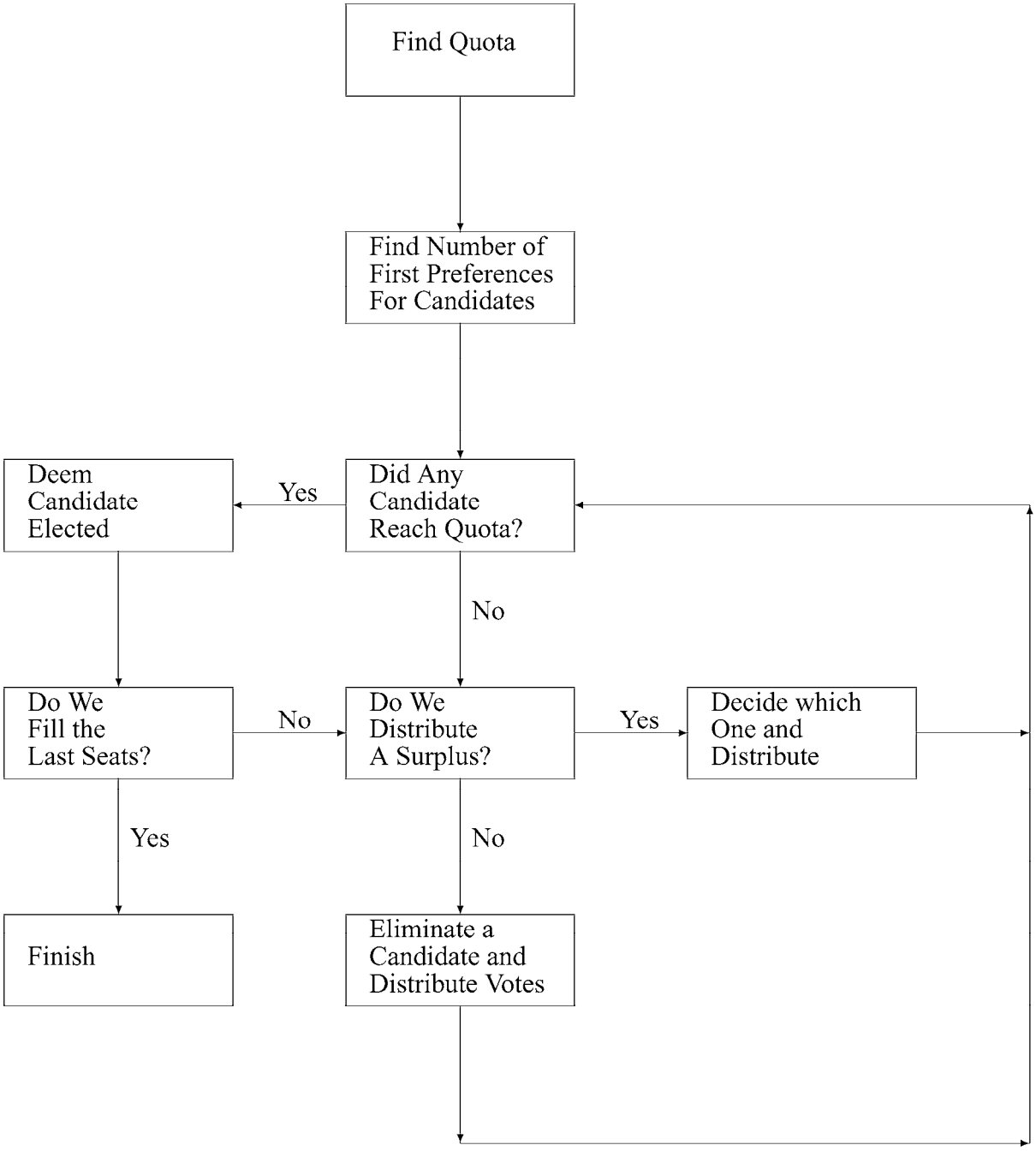
Appendix : Flowchart (More detailed flowcharts in [2])



Further Calculus and Series

You may have looked at the end of page 9, Mathematics Tables, and silently uttered something unprintable, while simultaneously thanking providence that you didn't have to teach it:

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Well, you have! Or, more precisely, you've taught everything about it that matters,that is, if you've done the further calculus option in the Leaving Cert Higher Maths.
The application of DeMoivre's Theorem is self-explanatory, or should be. But what about the $e^{in\theta}$?

Consider the following Maclaurin series or, better still, get your pupils to consider them:

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \qquad \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

Replacing x with $i\theta$ we get: $$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots$$

Replace x with θ to get: $$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

Replace x with θ, and multiply by i: $$i\sin\theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \ldots$$

Note from the three series just given that: $e^{i\theta} = \cos\theta + i\sin\theta$

Now $\underline{e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$. That is the original formula shown.

Nor does the fun end there! We can use this formula to evaluate i^i.

$$i = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} \text{ (polar form)}$$

$$i^i = (\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2})^i = (e^{i\frac{\pi}{2}})^i = e^{-\frac{\pi}{2}}$$

So, in this case, an imaginary number to the power of an imaginary number gives us a real number. Two wrongs give us a right or, more seriously, it indicates that the word "imaginary" is a misnomer.

Paul Holland
Presentation College
Galway

Applied Mathematics – Leaving Certificate Higher 2005 – Solutions

1. **(a)** Car A and car B travel in the same direction along a horizontal straight road.
Each car is travelling at a uniform speed of 20 m/s.
Car A is at a distance of d metres in front of car B.
At a certain instant car A starts to brake with a constant retardation of 6 m/s^2.
0.5 s later car B starts to brake with a constant retardation of 3 m/s^2.

Find

(i) the distance travelled by car A before it comes to rest

(ii) the minimum value of d for car B not to collide with car A.

(i) $$v^2 = u^2 + 2as$$

$$0 = 20^2 + 2(-6)s$$

$$s = \frac{100}{3} \quad \text{or} \quad 33.3$$

(ii) Car B $$s = ut + \tfrac{1}{2}at^2$$

$$s = 20(0.5) + 0$$

$$s = 10$$

When brakes are applied :

$$v^2 = u^2 + 2as$$

$$0 = 20^2 + 2(-3)s$$

$$s = \frac{400}{6} \quad \text{or} \quad 66.6$$

$$\text{minimum value of } d = 66.6 + 10 - 33.3$$

$$= 43.3$$

1 **(*b*)** A mass of 8 kg falls freely from rest. After 5 s the mass penetrates sand. The sand offers a constant resistance and brings the mass to rest in 0.01 s.

Find

(i) the constant resistance of the sand

(ii) the distance the mass penetrates into the sand.

(i) Speed after 5 s

$$v = u + at$$
$$v = 0 + 9.8(5)$$
$$v = 49$$

Sand

$$v = u + at$$
$$0 = 49 + a(0.01)$$
$$a = -4900$$

$$mg - R = ma$$
$$8(9.8) - R = 8(-4900)$$

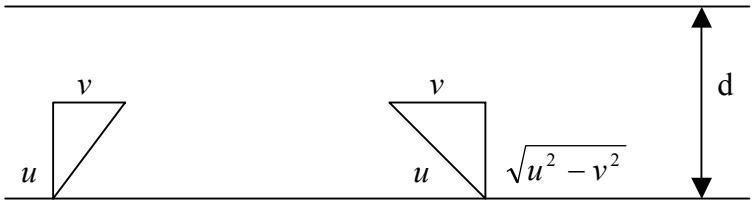
$$R = 39278.4 \text{ N}$$

(ii)

$$v^2 = u^2 + 2as$$
$$0 = (49)^2 + 2(-4900)s$$
$$s = 0.245$$

2. **(a)** A woman can swim at u m/s in still water.
She swims across a river of width d metres. The river flows with a constant speed of v m/s parallel to the straight banks, where $v < u$.
Crossing the river in the shortest time takes the woman 10 seconds.

Find, in terms of u and v, the time it takes the woman to cross the river by the shortest path.



shortest time :

$$10 = \frac{d}{u} \quad \text{or} \quad d = 10u$$

shortest path :

$$\text{speed} = \sqrt{u^2 - v^2}$$

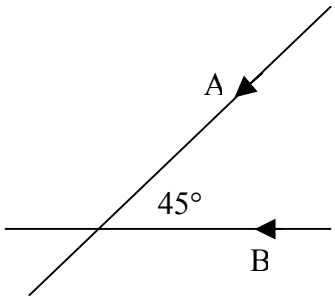
$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{d}{\sqrt{u^2 - v^2}}$$

$$= \frac{10u}{\sqrt{u^2 - v^2}}$$

.

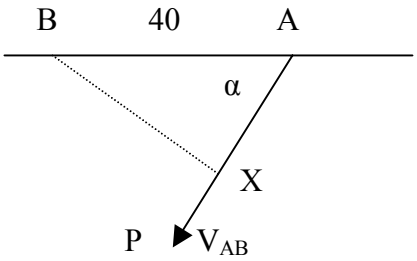
2 **(b)** Two straight roads intersect at an angle of 45°. Car A is moving towards the intersection with a uniform speed of p m/s. Car B is moving towards the intersection with a uniform speed of 8 m/s. The velocity of car A relative to car B is $-2\vec{i} - 10\vec{j}$.



At a certain instant car A is $220\sqrt{2}$ m from the intersection and car B is 136 m from the intersection.

(i) Find the value of p.

(ii) How far is car A from the intersection at the instant when the cars are nearest to each other?
Give your answer correct to the nearest metre.



(i) $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$

$$-2\ \vec{i}\ -\ 10\ \vec{j} = \left(-\frac{p}{\sqrt{2}}\ \vec{i}\ -\ \frac{p}{\sqrt{2}}\ \vec{j}\right) - \left(-8\ \vec{i}\ \right)$$

$$-2\ \vec{i}\ -\ 10\ \vec{j} = \left(8 - \frac{p}{\sqrt{2}}\right)\ \vec{i}\ -\ \frac{p}{\sqrt{2}}\ \vec{j}$$

$$\Rightarrow \quad p = 10\sqrt{2}$$

(ii) A reaches the intersection in $\dfrac{220\sqrt{2}}{10\sqrt{2}}$ or 22 s

In this time B travels $8(22)$ or 176 m
and is now 40 m from the intersection

$$\alpha = 90 - \tan^{-1}\left(\frac{2}{10}\right) = 78.69°$$

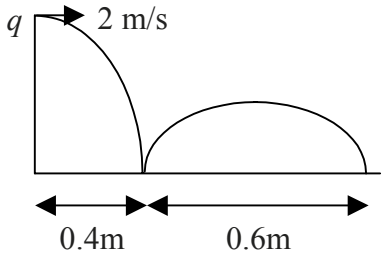
$$|\mathrm{AX}| = 40\cos\alpha$$
$$= 40(0.1961) \text{ or } 7.844$$

$$\text{time} = \frac{|\mathrm{AX}|}{V_{AB}} = \frac{7.844}{\sqrt{104}} \text{ or } 0.769$$

In this time A travels $10\sqrt{2}(0.769)$ or 10.8 m

Ans 11 m.

3. **(a)** A ball is projected horizontally from a point q above a smooth horizontal plane with speed 2 m/s.
The ball first hits the plane at a point whose horizontal displacement from q is 0.4 m.
The ball next strikes the plane at a horizontal displacement of 1 m from q.
The coefficient of restitution between the ball and the plane is e.



Find the value of e.

First stage $\quad \text{r}\,\vec{\text{i}} = 0.4 \quad \Rightarrow 2(t) = 0.4$

$\Rightarrow \quad t = 0.2$

$$v\vec{j} = u + at$$
$$= 0 - 9.8(0.2)$$
$$= -1.96$$

Second stage $\quad \text{r}\,\vec{\text{i}} = 0.6 \quad \Rightarrow 2(t) = 0.6$

$\Rightarrow \quad t = 0.3$

Rebound velocity $= 2\,\vec{i} \; + \; 1.96\,e\vec{j}$

$$r\vec{j} = 0$$
$$1.96e\,t - \tfrac{1}{2}gt^2 = 0$$
$$1.96e(0.3) - \tfrac{1}{2}g(0.3)^2 = 0$$
$$e = \tfrac{3}{4}$$

3 **(*b*)** A plane is inclined at an angle β to the horizontal. A particle is projected up the plane with initial velocity *u* at an angle α to the inclined plane.
The plane of projection is vertical and contains the line of greatest slope.

(*i*) Find the range of the particle on the inclined plane in terms of u, α and β.

(*ii*) Show that for a constant value of u the range is a maximum when

$$\alpha = 45° - \frac{\beta}{2}.$$

$$r_j = 0$$

$$u\sin\alpha.t \; - \; \tfrac{1}{2}g\cos\beta.t^2 \; = \; 0$$

$$\Rightarrow \quad t \; = \; \frac{2u\sin\alpha}{g\cos\beta}$$

$$\text{Range} \; = \; u\cos\alpha.t \; - \; \tfrac{1}{2}g\sin\beta.t^2$$

$$= \; u\cos\alpha.\left(\frac{2u\sin\alpha}{g\cos\beta}\right) \; - \; \tfrac{1}{2}g\sin\beta..\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$= \frac{u^2}{g\cos\beta}\sin 2\alpha. \; - \; \frac{2u^2\sin\beta}{g\cos^2\beta}.\sin^2\alpha$$

$$\frac{dR}{d\alpha} \; = \; \frac{u^2}{g\cos\beta}2\cos 2\alpha. \; - \; \frac{2u^2\sin\beta}{g\cos^2\beta}.2\sin\alpha\cos\alpha$$

$$= \; \frac{2u^2}{g\cos^2\beta}\{\cos 2\alpha.\cos\beta \; - \; \sin 2\alpha\sin\beta\}$$

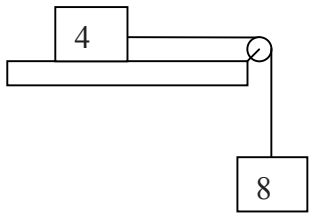
$$= \; \frac{2u^2}{g\cos^2\beta}\{\cos(2\alpha+\beta)\}$$

$$\frac{dR}{d\alpha} \; = \; 0 \quad \Rightarrow \quad \cos(2\alpha+\beta) = 0$$

$$\Rightarrow \quad 2\alpha+\beta = 90°$$

$$\Rightarrow \quad \alpha = 45° - \frac{\beta}{2}$$

4. **(a)** A particle of mass 4 kg rests on a rough horizontal table. It is connected by a light inextensible string which passes over a smooth, light, fixed pulley at the edge of the table to a particle of mass 8 kg which hangs freely under gravity.



The coefficient of friction between the 4 kg mass and the table is $\frac{1}{4}$.

The system starts from rest and the 8 kg mass moves vertically downwards.

Find

(i) the tension in the string

(ii) the force exerted by the string on the pulley.

(i) $8g - T = 8f$

$T - g = 4f$

$8g - T = 2T - 2g$

$10g = 3T$

$T = \frac{10g}{3}$

(ii) $\text{Force} = \sqrt{T^2 + T^2}$

$= T\sqrt{2}$

$= \frac{10g\sqrt{2}}{3}$ or 46.2 N

4 **(b)** Two particles of masses 3 kg and 5 kg are connected by a light inextensible string, of length 4 m, passing over a light smooth peg of **negligible radius**. The 5 kg mass rests on a smooth horizontal table. The peg is 2.5 m directly above the 5 kg mass.
The 3 kg mass is held next to the peg and is allowed to fall vertically a distance 1.5 m before the string becomes taut.

(i) Show that when the string becomes taut the speed of each particle is $\frac{3\sqrt{3g}}{8}$ m/s.

(ii) Show that the 3 kg mass will not reach the table.

(i)
$$\begin{aligned} v_1^{\,2} &= u^2 + 2as \\ &= 0 + 2(g)(1.5) \\ v_1 &= \sqrt{3g} \end{aligned}$$

$$3\left(\sqrt{3g}\right) = 3v + 5v$$

$$v = \frac{3\sqrt{3g}}{8}$$

(ii)
$$3g - T = 3a$$
$$T - 5g = 5a$$
$$a = -\frac{g}{4} \quad \text{or} \quad -2.45$$

3 kg mass
$$v^2 = u^2 + 2as$$
$$0 = \left(\frac{3\sqrt{3g}}{8}\right)^2 + 2\left(-\frac{g}{4}\right)s$$
$$s = \frac{27}{32} \quad \text{or} \quad 0.84$$

As $s < 1 \Rightarrow$ 3 kg mass will not reach the table

5. **(a)** Three identical smooth spheres P, Q and R, lie at rest on a smooth horizontal table with their centres in a straight line. Q is between P and R.
Sphere P is projected towards Q with speed 2 m/s. Sphere P collides directly with Q and then Q collides directly with R.
The coefficient of restitution for all of the collisions is $\frac{3}{4}$.

Show that P strikes Q a second time.

PQ PCM $m(2)+m(0) = mv_1 + mv_2$

NEL $v_1 - v_2 = -\frac{3}{4}(2-0)$

$$v_1 = \frac{1}{4}$$
$$v_2 = \frac{7}{4}$$

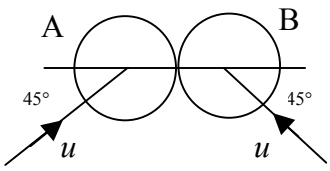
QR PCM $m\left(\frac{7}{4}\right)+m(0) = mv_1 + mv_2$

NEL $v_1 - v_2 = -\frac{3}{4}\left(\frac{7}{4}-0\right)$

$$v_1 = \frac{7}{32}$$

As $\frac{1}{4} > \frac{7}{32}$ P strikes Q a second time

5 **(*b*)** A smooth sphere A, of mass m, moving with speed u, collides with an identical smooth sphere B moving with speed u.



The direction of motion of A, before impact, makes an angle $45°$ with the line of centres at impact.
The direction of motion of B, before impact, makes an angle $45°$ with the line of centres at impact.
The coefficient of restitution between the spheres is e.

(i) Find, in terms of e and u, the speed of each sphere after the collision.

(ii) If $e = \frac{1}{2}$, show that after the collision the angle between the directions of motion of the two spheres is $\tan^{-1}\left(\frac{4}{3}\right)$.

(i) PCM $\quad m\frac{u}{\sqrt{2}} - m\frac{u}{\sqrt{2}} = mv_1 + mv_2$

NEL $\quad v_1 - v_2 = -e\left(\frac{u}{\sqrt{2}} + \frac{u}{\sqrt{2}}\right)$

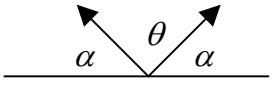
$\Rightarrow \quad v_1 = -\frac{eu}{\sqrt{2}}$ and $v_2 = \frac{eu}{\sqrt{2}}$

$$\text{Speed of A} = \sqrt{\left(-\frac{eu}{\sqrt{2}}\right)^2 + \left(\frac{u}{\sqrt{2}}\right)^2} = \frac{u}{\sqrt{2}}\sqrt{1+e^2}$$

$$\text{Speed of B} = \frac{u}{\sqrt{2}}\sqrt{1+e^2}$$

(ii) $e = \frac{1}{2} \quad \Rightarrow \quad v_1 = -\frac{u}{2\sqrt{2}}$ and $v_2 = \frac{u}{2\sqrt{2}}$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2(2)}{1-4} = -\frac{4}{3}$$

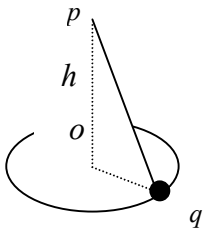


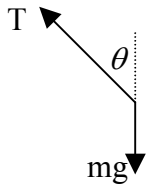
$$\theta = 180 - 2\alpha$$

$$\tan\theta = \frac{\tan 180 - \tan 2\alpha}{1 + \tan 180 \tan 2\alpha}$$

$$\tan\theta = \frac{0 - \left(-\frac{4}{3}\right)}{1+0} = \frac{4}{3}$$

6. **(a)** A conical pendulum consists of a light inelastic string [*pq*], fixed at the end *p,* with a particle attached to the other end *q* .
The particle moves uniformly in a horizontal circle whose centre *o* is vertically below *p*.
If $|po| = h$, find the period of the motion in terms of *h*.





$$T\cos\theta = mg$$

$$T\sin\theta = mr\omega^2$$

$$\tan\theta = \frac{mr\omega^2}{mg}$$

$$\frac{r}{h} = \frac{mr\omega^2}{mg}$$

$$\frac{1}{\omega} = \sqrt{\frac{h}{g}}$$

$$\text{Period} = \frac{2\pi}{\omega}$$

$$= 2\pi\sqrt{\frac{h}{g}}$$

6 **(*b*)** A light elastic string of natural length a and elastic constant k is fixed at one end to a point o on a smooth horizontal table. A particle of mass m is attached to the other end of the string.
Initially the particle is held at rest on the table at a distance $2a$ from o, and is then released.

Show that the time taken for the particle to reach o is $\sqrt{\frac{m}{k}}\left(1+\frac{\pi}{2}\right)$.

$$\begin{aligned} \text{Force} &= -T \\ &= -kx \\ \text{acceleration} &= -\frac{k}{m}x \end{aligned}$$

$$\text{SHM with } \omega = \sqrt{\frac{k}{m}}$$

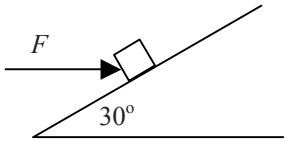
$$\begin{aligned} \text{Amplitude} &= a \\ \text{Velocity} &= \omega a = a\sqrt{\frac{k}{m}} \end{aligned}$$

$$\text{First stage} \quad \text{time} = \tfrac{1}{4}T = \frac{\pi}{2}\sqrt{\frac{m}{k}}$$

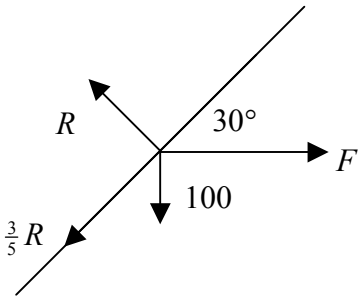
$$\begin{aligned} \text{Second stage} \quad \text{s} &= ut + \tfrac{1}{2}at^2 \\ a &= a\sqrt{\frac{k}{m}}\ t + 0 \\ t &= \sqrt{\frac{m}{k}} \end{aligned}$$

$$\text{Total time} = \sqrt{\frac{m}{k}}\left(1+\frac{\pi}{2}\right)$$

7. **(a)** A particle of weight 100 N lies on a plane. The plane is inclined at 30° to the horizontal. A horizontal force F is applied to the particle. The coefficient of friction between the particle and the inclined plane is $\frac{3}{5}$.



Find the least value of F that will move the particle up the plane.



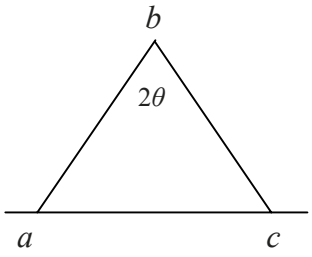
$$R = 100\cos 30 + F\sin 30$$

$$F\cos 30 = 100\sin 30 + \tfrac{3}{5}R$$

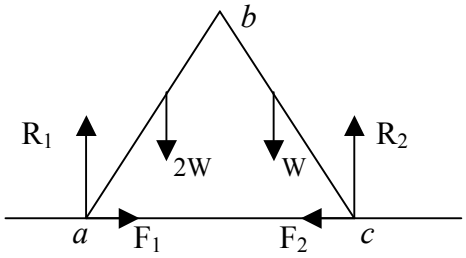
$$F\cos 30 = 100\sin 30 + \tfrac{3}{5}\left(100\cos 30 + F\sin 30\right)$$

$$F = 180.1$$

7 **(*b*)** Two uniform rods, [*ab*] and [*bc*], of equal length, are smoothly jointed at *b*.
They rest in a vertical plane with *a* and *c* on rough horizontal ground and $|\angle abc| = 2\theta$.
The weight of the rod [*ab*] is $2W$ and the weight of the rod [*bc*] is W.
The coefficient of friction at both *a* and *c* is μ.



Find the least value of μ , in terms of θ , necessary for equilibrium.



Vert $R_1 + R_2 = 3W$

Take moments about *a* for system :

$$2W(1) + W(3) = R_2(4)$$

$$R_2 = \frac{5W}{4}$$

$$R_1 = \frac{7W}{4}$$

Take moments about *b* for *bc* :

$$F_2(\ell\cos\theta) + \mathrm{W}\left(\tfrac{1}{2}\ell\sin\theta\right) = R_2(\ell\sin\theta)$$

$$F_2 + \mathrm{W}\left(\tfrac{1}{2}\tan\theta\right) = \frac{5W}{4}(\tan\theta)$$

$$F_2 = \frac{3W}{4}(\tan\theta)$$

$$R_2 < R_1$$

$$\Rightarrow F_2 = \mu R_2$$

$$\frac{3W}{4}(\tan\theta) = \mu\frac{5W}{4}$$

$$\Rightarrow \mu = \frac{3}{5}\tan\theta$$

8. **(a)** Prove that the moment of inertia of a uniform rod of mass *m* and length 2*l* about an axis through its centre perpendicular to the rod is $\frac{1}{3}ml^2$.

$$\text{Let} \quad \mathrm{M} = \text{mass per unit length}$$

$$\text{mass of element} = \mathrm{M}\{\mathrm{dx}\}$$

$$\text{moment of inertia of the element} = \mathrm{M}\{\mathrm{dx}\}\mathrm{x}^2$$

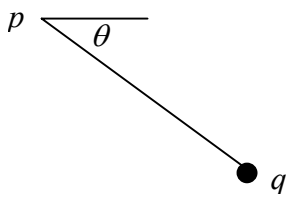
$$\text{moment of inertia of the rod} = \mathrm{M} \int_{-\ell}^{\ell} \mathrm{x}^2 \, \mathrm{dx}$$

$$= \mathrm{M}\left[\frac{\mathrm{x}^3}{3}\right]_{-\ell}^{\ell}$$

$$= \tfrac{2}{3}\,\mathrm{M}\,\ell^3$$

$$= \tfrac{1}{3}\,\mathrm{m}\,\ell^2$$

8 **(*b*)** A uniform rod [*pq*], of mass $9m$ and length $2l$, has a particle of mass $2m$ attached at q. The system is free to rotate about a smooth horizontal axis through p. The rod is held in a horizontal position and is then given an initial angular velocity $\sqrt{\frac{3g}{2l}}$ downwards.



The diagram shows the rod [*pq*] when it makes an angle θ with the horizontal.

(i) Show that when the rod makes an angle θ below its initial horizontal position, its angular velocity is

$$\sqrt{\frac{g(15+13\sin\theta)}{10l}}\ .$$

(ii) Hence, or otherwise, show that the rod performs complete revolutions about p.

(i) moment of inertia $= \frac{4}{3}(9m)\ell^2 + (2m)(2\ell)^2$

$= 20m\ell^2$

Gain in KE = Loss in PE

$$\tfrac{1}{2}I\omega_2^{\,2} - \tfrac{1}{2}I\omega_1^{\,2} = mgh$$

$$\tfrac{1}{2}(20m\ell^2)\omega_2^{\,2} - \tfrac{1}{2}(20m\ell^2)\left(\frac{3g}{2\ell}\right) = 9mg(\ell\sin\theta) + 2mg(2\ell\sin\theta)$$

$$\omega = \sqrt{\frac{g(15+13\sin\theta)}{10\ell}}$$

(ii) $\theta = \dfrac{3\pi}{2}$

$$\omega = \sqrt{\frac{g(15-13)}{10\ell}}$$

> 0

$\Rightarrow$ rod performs complete revolutions

9. **(a)** An alloy is made of iron and aluminium.
A piece of the alloy has a mass of 0.441 kg and a volume of 75 cm^3.
The relative density of iron is 8 and the relative density of aluminium is 2.7.

Find

(i) the volume of iron in the piece of alloy

(ii) the mass of aluminium in the piece of alloy.

(i) $\text{Mass of iron} = 8000\,(V)\,10^{-6}$

$\text{Mass of aluminium} = 2700\,(75-V)\,10^{-6}$

$\text{Mass of alloy} = \text{Mass of iron} + \text{Mass of alum.}$

$0.441 = 8000(V)10^{-6} + 2700(75-V)10^{-6}$

$441 = 8\,(V) + 2.7\,(75-V)$

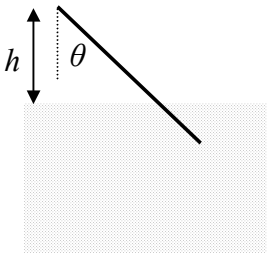
$238.5 = 5.3V$

$V = 45\ \text{cm}^3$

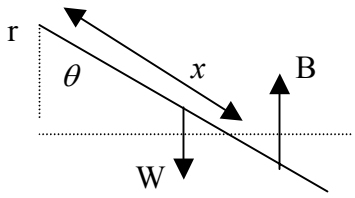
(ii) $\text{Mass of alluminium} = 2700\,(75-V)\,10^{-6}$

$= 0.081\ \text{kg}$

9 **(*b*)** A uniform rod of length ℓ and relative density s can turn smoothly about its upper end which is fixed at a height h above the surface of water. The rod is inclined at an angle θ to the vertical and is partially immersed in the water. The rod is at rest.



Show that $\cos\theta = \dfrac{h}{\ell\sqrt{1-s}}$, where $s < 1$.



$$\text{Buoyancy B} = \frac{W_I\, s_L}{s}$$

$$= \frac{\dfrac{(\ell - x)}{\ell} W(1)}{s} \quad \text{or} \quad \frac{W(\ell - x)}{\ell s}$$

Take moments about r :

$$B\left(\ell - \frac{(\ell - x)}{2}\right)\sin\theta = W\left(\frac{1}{2}\ell\right)\sin\theta$$

$$B\left(\frac{(\ell + x)}{2}\right) = W\left(\frac{1}{2}\ell\right)$$

$$\left(\frac{W(\ell - x)}{\ell s}\right)\left(\frac{(\ell + x)}{2}\right) = W\left(\frac{1}{2}\ell\right)$$

$$\ell^2 - x^2 = \ell^2 s$$

$$x^2 = \ell^2(1 - s)$$

$$x = \ell\sqrt{(1 - s)}$$

$$\cos\theta = \frac{h}{x}$$

$$= \frac{h}{\ell\sqrt{(1 - s)}}$$

10. **(a)** Solve the differential equation

$$x\frac{dy}{dx} - xy - y = 0$$

given that $y = 1$ when $x = 1$.

$$x\frac{dy}{dx} - xy - y = 0$$

$$\frac{dy}{dx} = \frac{y(x+1)}{x}$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x}\right)dx$$

$$\ln y = x + \ln x + C$$

$$y = 1,\ x = 1 \Rightarrow C = -1$$

$$y = e^{x+\ln x-1} \quad \text{or} \quad xe^{x-1}$$

10 **(*b*)** A mass of 9 kg is suspended at the lower end of a light vertical rope. Initially the mass is at rest. The mass is pulled up vertically with an initial pull on the rope of 137.2 N.
The pull diminishes uniformly at the rate of 1 N for each metre through which the mass is raised.

(i) Show that the resultant upward force on the mass when it is x metres above its initial position is

$$49 - x\,.$$

(ii) Find the speed of the mass when it has been raised 15 metres.

(iii) Find the work done by the pull on the rope when the mass has been raised by 15 m.

(i) $$\begin{aligned} \text{Force} &= 137.2 - x - 9g \\ &= 49 - x \end{aligned}$$

(ii) $$\begin{aligned} 9v\frac{dv}{dx} &= 49 - x \\ 9\int_0^v v\,dv &= \int_0^{15} (49 - x)\,dx \\ \left[\frac{9}{2}v^2\right]_0^v &= \left[49x - \frac{x^2}{2}\right]_0^{15} \\ \frac{9}{2}v^2 &= 735 - 112.5 \\ v &= 11.76 \end{aligned}$$

(iii) $$\begin{aligned} \text{Work done} &= \text{Change in K.E.} \\ &= \tfrac{1}{2}(9)v^2 - 0 \\ &= 622.5 \end{aligned}$$

John Maher
Dublin