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Founded in 1964 to promote and assist the teaching of mathematics at all levels.

Membership is open to all those interested in mathematics and mathematics education.

The IMTA is represented on all NCCA mathematics course committees.

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Editorial

Welcome to the Autumn 2009 edition of the IMTA Newsletter.

Quite a lot has changed since this time last year. We are now in the advent of Project Maths as a practical reality in all our schools. In anticipation of this the IMTA was delighted to hear the thoughts and experiences of Professor Erkki Pehkonen from the University of Helsinki, Finland, who delivered the Fr. Ingram Memorial Lecture at the AGM last November. As has been the custom over the last number of years the script of the Fr. Ingram Lecture is reproduced here. Professor Pehkonen's conclusions are based on extensive research which is well referenced in this article.

Other articles in this publication also show the results of erudite research. From NUI, Maynooth, Professor Anthony G. O'Farrell details the philosophical basis of the new Geometry syllabus which forms part of Strand 2 in Project Maths. This has been accepted by the syllabus committees and Board of Studies for Maths.

Also from NUI, Maynooth, Dr. Fiacre Ó Cairbre presents a nuanced account of one of the most important aspects of mathematics—its connection with the concept of 'beauty'. It is, perhaps, a necessary antidote to the more prevalent utilitarian view of mathematics that many imagine may underpin the new approach to mathematics teaching.

Beauty and the appreciation of beauty in mathematics are driving forces much more potent than is often realised. Through the article on wallpaper groups, yours truly, Neil Hallinan, attempts to indicate how the beauty of mathematical constructs which was utilised by the Moorish residents in the Alhambra over seven hundred years ago may be made even more accessible through mathematical analysis.

The flavour of Project Maths is continued with a relevant update on GeoGebra by David Hobson who reports on the most recent GeoGebra Conference in Hagenburg, Austria—the first such International Conference. GeoGebra is a dynamic mathematics software programme (freely distributed) which will, no doubt, gain in significance in the near future. Familiarity with such packages is a stated objective of the new Geometry syllabus.

Gordon Lessells from the University of Limerick annotates the history of the International Mathematical Olympiad (IMO) in Ireland and the laudable achievements of its Irish participants.

In this edition we are also celebrating the work of some younger contributors who presented their work at the Young Scientist Exhibition 2009.

Special thanks to Maurice O'Driscoll for the Solutions to the Leaving Certificate 2009 Higher Level Mathematics questions. We constantly appreciate the contribution of this valuable resource.

Congratulations to the winners of Team Maths, 2009, from Middleton, Co. Cork and to the runners-up from Marists, Athlone.

A hearty congratulations, too, to all recently retired members but especially to our past Chairman, Liam Sayers, whose encouragement and wisdom were an invaluable input into the production of this Newsletter. Health and long years of enjoyment be yours!

With thanks to all contributors. Enjoy the read!

All contributions are welcome.
Send by e-mail to : hallinann@gmail.com
or St. Mary's, Holy Faith, Glasnevin, Dublin 11.

Neil Hallinan

Addendum: This Newsletter is not funded by the Teacher Education Section (TES) of the DES (Department of Education and Science.

Branches : Contacts

Cork (Sec.): *Brendan O'Sullivan*, bos4@esatclear.ie
Donegal (Sec.): *Joe English,* mathsjc@eircom.net
Dublin (Sec.): *Barbara Grace,* barbaragrace@eircom.net
Galway (Tr.): *Mary McMullin*, mcmullin.mary@googlemail.com
Kerry (Tr.): *John O'Flaherty,* flahjohn@eircom.net
Limerick (Chair): *Gary Ryan,* theboyryan@hotmail.com
Mayo (Sec.): *Lauranne Kelly*, lakelly@stgeraldscollege.com
Midlands (Sec.): *Teresa Cushen,* bandtcushen@eircom.net
Tipperary (Chair): *Donal Coughlan,* donal.coughlan@esatclear.ie
Wexford : (Rep.): *Sean MacCormaic*, sean_maccormaic@yahoo.co.uk

Michael Brennan (Waterford) is still available for contact at mbrennan@wit.ie

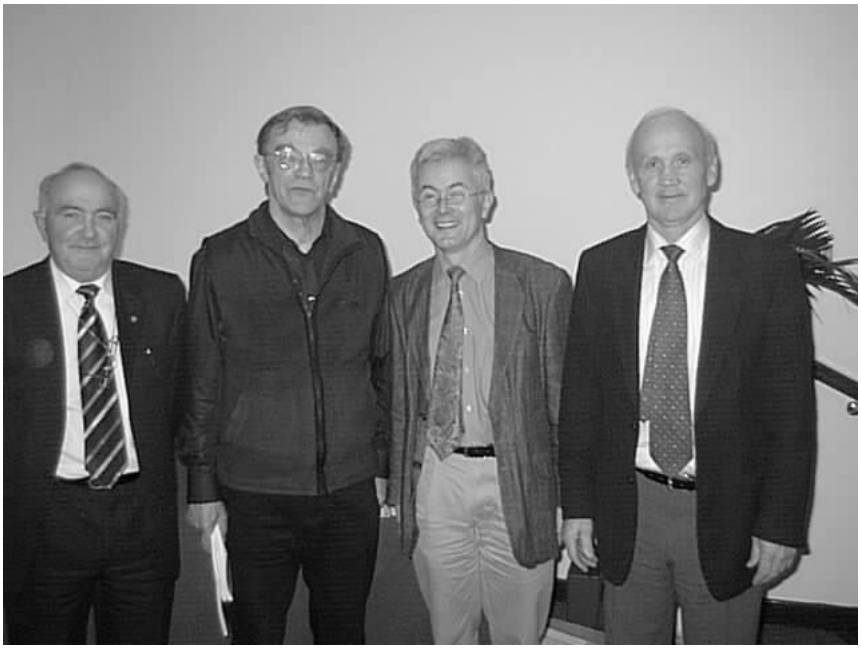
We note with regret the passing of Victor Bond, one of the "greats" of the IMTA, who contributed much in particular to its foundation and early years. We offer our sincere condolences to his family.

The views expressed in this Newsletter are those of the individual authors and do not necessarily reflect the position of the IMTA. While every care has been taken to ensure that the information in this publication is up-to-date and correct no responsibility will be taken by the IMTA for any errors that might occur.

Problem Solving in Second Level Mathematics —The Finnish Experience

Fr. Ingram Memorial Lecture - 28th November, 2008

Erkki Pehkonen
University of Helsinki



Liam Sayers, IMTA Chair; Dr. Erkki Pehkonen; Dr. Maurice O'Reilly, Seminar Chair; Maurice O'Driscoll

Abstract

Since problem solving is also internationally recognized to be a fuzzy concept, the presentation begins with a working characterization for a problem and some examples. The main content of the paper is to describe problem solving in Finnish school mathematics. This begins with considering Finnish mathematics curricula with the focus on the role of problem solving. Furthermore, different manifestations problem solving in mathematics textbooks are discussed. And it will be described how Finnish teachers implement problem solving in mathematics lessons. Additionally the way teachers use problem solving in assessment is discussed briefly. At the end of the paper a solution for teaching reform within the curriculum is dealt with; the reform is based on the use of open problems. Finally some evaluation on the success of teaching reform in problem solving is given.

The purpose of school education in each country is, more or less, to develop independent, self-confident, critically thinking, motivated and multitalented individuals who will manage in societal settings that they will encounter later on in their life. The key question is what kind of school instruction is optimal for this goal.

PROBLEM SOLVING IN SCHOOL MATHEMATICS

Problem solving has generally been accepted as a means for advancing thinking skills (e.g. Schoenfeld 1985). For example, in the NCTM Standards it is stated: *Solving problems is not only a goal of learning mathematics but also a major means of doing so. ... In everyday life and in the workplace, being a good problem solver can lead to great advantages. ... Problem solving is an integral part of all mathematics learning.* (NCTM, 2000, 52)

Yet, the basic concepts 'problem' and 'problem solving' seem still to be rather ambiguous in mathematics education. Sometimes a simple arithmetic task is understood as a 'problem' although it can be solved in a routine way, whereas at other times only more complex situations are considered as problems. The conception of mathematics underlying the definition is probably essential, but this conception is unfortunately seldom explicitly expressed. The theories of learning behind the practices of teaching can also be very different. Therefore, there might be big differences in the conceptions of problems and problem solving between different countries, and perhaps, also within a country. For example, some distinct differences in emphasis have been singled out between the American and the British conceptions (Lingard & al. 1985). Thus, for a paper to describe the implementation of problem solving in one particular country, it is important to explain the conception of problem solving within that country or at least its author's conception.

Characterization of a problem

Here we will adopt the following characterization of a problem, which is widely used in the literature (e.g. Kantowski 1980): A task is said to be a *problem* if its solution requires that an individual combines previously known data in a way that is new (to him). If he can immediately recognize the measures that are needed to complete the task, it is a *routine task* (or a standard task or an exercise) for him. If one is not willing to distinguish between a problem and a routine task, one can use the word "task". In this context, *problem solving* can be understood as "a process where previously acquired data are used in a new and unknown situation" (NCSM 1989).

In the following, we use an example to further illustrate our view of the concept of problem. The following task is modified from a problem of the book of Müller & Wittmann (1978, 43).

Example 1 (Construct a wall). Add always two neighbouring numbers and put the sum above them (Figure 1). What might be the top number?

Example 1 is a routine task for everybody who can calculate with whole numbers. But if we ask the solver to work backwards, i.e. giving only some numbers from between, this might turn out to be a problem

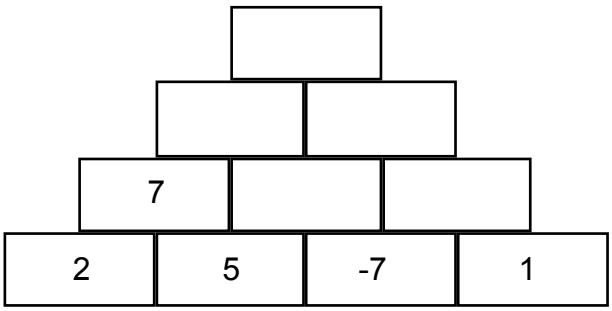


Figure 1. Construct a wall.

It is worthwhile noting that the concept of problem is relative to the individual and to time. A task that is a problem for one person, might be a routine task for another. A problem of today might become a routine task later.

Often in the beginning of problem solving, pupils deal with problems where they need only to have one insight in order to find the solution. Usually the key point is to perceive the problem situation in a new way. These problems are called *one-step problems* or mathematically simple problems; the wording *mathematical puzzle* is also used. For example usually matchstick problems are such.

In the 1970's the term 'investigation' was introduced in England to mean an extended problem situation. In *investigations* the starting situation is usually given and, perhaps, the first problems, too, in order to show some possible ways ahead. Pupils are expected to choose their problems and route. For example, the first example above might give a rise for an investigation, as follows: "What problems can you develop from the situation of Figure 1?"

One may separate structured and unstructured investigations. The latter one is the English version. In the continent, the concept of structured investigations was developed (e.g. Pehkonen 1995). He called them *problem fields*, and characteristic for them was that a teacher has a collection of different connected problems, and he/she may select the way of continuation. For example, in Figure 1 the teacher might ask "What will happen to the top number (-3), if you change some of the starting numbers (2, 5, -7, 1)? Can you predict?" or "Can you construct a wall, where the top number is 100 (or -10,5 or →5)?"

In the following, the use of problem solving in Finnish mathematics instruction is described. Special attention is given to the way the textbooks deal with problem solving, how problem solving is implemented in school, and in what way problem solving and pupils' problem solving skills are assessed.

Problem solving in the Finnish curriculum

Curriculum development in Finland has reflected the international trends – usually with a delay

of about 10 years. After the “new math” movement in the late 1960’s and early 1970’s, there was a shift ‘back to basics’ towards the end of the decade. Since the 1980’s a lot of emphasis has been given for problem solving. (Kupari 1999). For more than twenty years, problem solving has been one of the general overall goals in the Finnish curricula (NBE 1985, 1994, 2004). Its implementation has been in the focus of teacher pre-service and in-service education since the end of the 1980’s.

In 1986 the National Board of Education made systematic efforts to promote problem solving in school mathematics. It organized a two-part seminar 1986 and 1987 in problem solving for teacher educators. There were lectures and demonstrations, also in a school class, on the use of different problems. The participants were urged to apply these problems in their own teaching and to reflect upon them in the second part of the seminar a year later. In the seminar, the participants' conceptions of problem solving were charted with a questionnaire, and reported later on (Pehkonen 1993).

The main results of the study were, as follows: The teacher educators saw problem solving as important, since it helps the fostering of the pupils’ cognitive skills, and helps pupils to use the mathematics they have learned. Giving appropriate tasks was considered as the most important approach to teaching problem solving. Teaching problem solving should be carried out in a creative, flexible and approving manner, and open discussions were perceived a necessity. The teacher should involve pupils in problem solving through letting them solve their own problems and through dealing with problems from their familiar surroundings. Pupils’ willingness to study problem solving was considered the most important prerequisite for teaching problem solving. (Pehkonen 1993)

For example, the national curriculum for the comprehensive school (NBE 1994) provided rather general guidelines, and local schools were supposed to plan their more detailed curriculum documents within this framework (cf. Pehkonen & al. 2007). The importance of problem solving is clearly acknowledged in the curricular documents (NBE 1985, 1994, 2004).

Problems in textbooks

Before the problem-solving seminars in 1986–87, problem tasks were rather rare in Finnish mathematics textbooks. After the seminar almost every printing house published a set of problems, either as a booklet or as a deck of cards, and with time some problems were taken into the textbooks. But a study (Kari 1991) shows that in the Finnish textbooks for grade 7, the proportion of problem tasks was about 11 % of all tasks. Further non-systematic investigations of Finnish mathematics textbooks by teacher students show that the number of problem tasks has not increased in the last decade.

The 1990’s was a very fruitful decade in Finnish mathematics education. The National Board of

Education published a guide book (Seppälä 1994) to help teachers when implementing the curricular framework (NBE 1994). Furthermore, new textbooks (usually three or four competing series) were elaborated and published according to the curricular framework. This time both in the elementary level and in upper level of the comprehensive school, there was a book series that was devoted to train pupils' problem solving skills.

For example, the mathematics book series "Mieti ja laske" [Think and Calculate] (Vähäpassi & al. 1997) provides grade 3 pupils (and their teachers) with one page of exercises, one page of word problems, one page of problems and one page of investigations to choose from for each topic. A mathematics book for grades 7–9 of the comprehensive school "Matka matematiikkaan" [A Journey to Mathematics] (Espo & Rossi 1996) was launched. The focus of this textbook was teaching mathematics via problem solving, i.e. almost all contents were introduced via proper problem situations. On one side the use of the book demanded much preparatory work from teachers' side, but on the other hand it made mathematics teaching more interesting and for pupils an adventure.

But the time seems not to be ripe for such radical textbooks, since teachers were not willing accept them but to stick in traditional mathematics books. It seems that the elementary school book (Vähäpassi & al. 1997) is used without thinking or solving problems, only calculating. And the upper school text book (Espo & Rossi 1996) was selected only by a few teachers, less than 10 %. But the influence of these books can be seen in the next generation of the mathematics books from other printing houses.

Use of problems in mathematics lessons

In the 1980's, there was much teacher in-service training for teachers of comprehensive school on activating teaching methods and problem solving. These components could be seen also in teachers' beliefs. Both elementary teachers and mathematics teachers regard problem solving as an important aspect of mathematics teaching (Kupari, 1999). However, results after twenty years show that only some of the teachers have changed their teaching style (ibid). Even teachers who express beliefs favorable to problem solving, often fail to implement it in their own teaching (Perkkilä, 2002). This phenomenon of unsuccessful teacher change has been dealt with in a recently published paper (Pehkonen 2006). A recent study on primary teachers' conceptions on problem solving (Sivunen & Pehkonen 2008) shows that teachers' conceptions on problem solving have not much changed within twenty years (cf. Pehkonen 1993).

Although the development in mathematics teaching has not been as rapid as expected, there are some changes to be observed. The use of problem solving tasks is quite popular today in mathematics lessons, but mainly in the form of mathematical puzzles. If we use the language introduced by Schroeder & Lester (1989), we might say that only few teachers are teaching *via* problem solving, while most of them teach something *about* problem solving. The latter means

that they might use some mathematical puzzles in their teaching or have a problem box in their class or something similar. And the former states that these teachers use problem solving as a teaching method, and that is very rare.

AN OPPORTUNITY FOR CHANGE: USING OPEN-ENDED PROBLEMS

In the attempts to find a new teaching method that might meet the challenges set by constructivism, the so-called open approach was developed in the 1970's in Japan (e.g. Becker & Shimada 1997, Nohda 2000). Internationally it is accepted that open-ended problems form a useful tool in the development of mathematics teaching in schools, in a way that emphasizes understanding and creativity (e.g. Nohda 1991, Silver 1993, Stacey 1995). Discussion papers from a sizable group of international specialists have been collected and published in a special report (Pehkonen 1997).

What are open problems?

Tasks are said to be *open*, if their starting or goal situation is not exactly given (cf. Pehkonen 1995). In open tasks, pupils are given freedom, possibly even in the posing of the question, but at least in the solving of the task. In practice this means that they may end up with different, but equally correct solutions, depending on the additional choices made and the emphases placed during their solution processes. Therefore, open tasks usually have several correct answers. When open tasks are used in mathematics teaching, pupils have an opportunity to act like creative mathematicians (cf. Brown 1997).

In Finland, these ideas have been spread out in teacher in-service courses, in teachers' journals, and in teacher pre-service education for more than twenty years. The leading idea has been to increase openness and creativity in mathematics teaching.

Open problems encompass several types of problems (cf. Pehkonen 1995): *investigations* (a starting point is given), *problem posing* (or problem finding or problem formulating), *real-life situations* (which have their roots in everyday life), *projects* (larger study entities, requiring independent work), *problem fields* (or problem sequences or problem domains; collections of contextually connected problems), *problems without a question*, and *problem variations* (the "what-if"-method). Several examples of different types of open problems can be found e.g. in the papers by Nohda (1991), Silver (1995), Stacey (1995), Schupp (2002) and Pehkonen (1997, 2004).

Some problem fields as an example

As an example of open-ended problems, we will consider a couple of problem fields. These represent a type of problems that the author has developed for Finnish heterogeneous classes of comprehensive schools. In each problem field, the difficulty of the problems ranges from very

simple ones that can be solved by the whole class, to harder problems that only the more advanced students might be able to solve.

One key characteristic of problem fields is that they are not bound to a fixed grade or age, but are suitable for mathematics teaching from the primary level to teacher in-service education. The particular role of the easier problems in problem fields is to reinforce the problem solving persistence of the pupils. A very important technical aspect of these problems is the way in which they are introduced to a class: The problem field ought to be given gradually to the pupils, and the continuation should be related to the pupils' solutions. The mental processes involved in the problem solving are of paramount importance, since the role of the answers and results is played down. One aspect of this is using the pupils' own creative power. Thus, in which direction and to what scope the teacher expands a problem field depends on pupils' answers. We will consider a couple of examples.

Example 2. Number triangle.
There is a triangle where the corners are free and certain numbers are fixed on the sides (Fig. 2). What numbers should be placed in the blank circles of the triangle, in order the sum of the three numbers on each side is the same? (Pehkonen 1988)

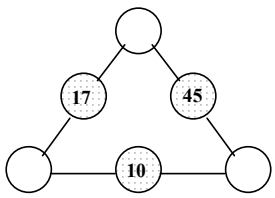


Figure 2. Number Triangle problem.

Since there are many solutions for this problem, we can continue e.g. with following questions:
- Can you find another solution?
- How many different solutions could there be?
- Is it possible to use negative numbers in the circles?
- Can you find a solution where the triangle's side sum (i.e. the sum of the numbers on the same side) will be 80?

Example 3. Polygons with matchsticks
Twelve matchsticks (or cocktail-sticks, etc.) will be needed to concretize the problems. The starting situation is the following:

> *With twelve matchsticks one can make a square (Fig. 3) the area of which is 9 au (au = area units).*

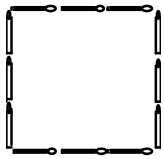


Figure 3. A square with 12 matches.

From this situation a sequence of problems (a problem field) is developed. Firstly, we will choose another area, but have the perimeter of the polygon constant. Thus, in each problem the perimeter of the polygon should be made up of 12 matches.

- *Can you use twelve matches to make a polygon with an area of 5 au?*

If we are willing to give more thinking time to slower pupils, the faster ones can be asked to find another (perhaps also a third) solution. Usually pupils will find some solutions, and therefore, we might ponder as the next question the number of different solutions.

- *How many different polygons of 5 au can you make with twelve matches? Can there be more than ten different solutions?*

The pupils will probably find many of the solutions. But there are still some complicated solutions that they might not find. The following step might be the comparison of the different solutions found by pupils. How many of all the different solutions can be found when the whole class is working together? Another easier direction to vary the problem is to change the area again.

- *Is it possible to use twelve matches to make a 6 au (or 7 au, 8 au) polygon?*

Some solutions can be found easily. But are there any other solutions in each case? And how many different ones?

The method of cutting out a corner from a rectangular polygon, as earlier, is successful down to area 5 *au*. But the question of smaller areas is more complicated, since we are compelled to change the method.

- *Is it possible to use twelve matches to make a 1 au (or 2 au, 3 au, 4 au) polygon?*

With the aid of the Pythagorean theorem, one can construct polygons with areas of 4 *au* and 3 *au*. It should also be possible to find a general solution: the parallelogram. But the question of different solutions and their number in the case of area 2 *au* (or 1 *au*) is according to our experience a really hard, but possible one. Still one extension of the problem field is to ask for polygons with areas greater than 9 *au*:

- *Using twelve matches is it possible to make a polygon whose area is greater than 9 au?*

This seems to be a tough one, since in teacher pre-service and in-service courses the problem has so far not been solved.

Teaching experiences. During the last fifteen years, the author has worked through the problem field “Polygons with matchsticks” with many groups of teachers at pre-service and in-service

courses, as well as with some school classes in different countries. The main reasons for introducing problem fields in teacher education have been to describe to teacher students how a problem field can be used and to give them an idea how pupils feel when they work with them. Usually, the duration of work with the problem field has taken about 30 minutes, until the ideas of the group have been used up. In the teacher groups, many polygons with areas of 3, 4, 5, 6, 7, 8 *au* have been found. But those polygons with areas smaller than 3 au seem to be very complicated to construct. Only in a couple of groups somebody has produced the parallelogram as a general solution.

CONCLUDING NOTES

Summarizing the Finnish experiences of problem solving in mathematics education, we could state that teachers in Finland are changing in the direction of a more favorable attitude to problem solving. But the use of problem solving in teaching demands much from the teacher, and, therefore, they find excuses why not to use a problem-solving approach. The younger generation of teachers seem to be more self-confident and open to changes.

The positive experiences of the use of problem fields are similar to the ones reported e.g. by Liljedahl (2005). As part of a compulsory mathematics course he presented a group of pre-service elementary teachers a set of mathematical problems to solve. Some of the tasks allowed a form of mathematical discovery that he called a 'chain of discovery'. They facilitated a state of sustained engagement and even helped to change the student teachers' negative beliefs and attitudes. Problem fields, as described here, also allow such chains of discovery and sustained engagement.

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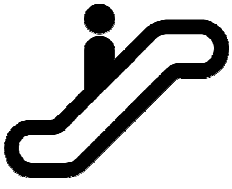
A PRACTICAL PROBLEM?

A person walks up an escalator every day, taking the steps at a rate of 1 step per second.

Doing this, they count 20 steps.

One day, when in a rush, they take the steps 2 per second. This time the count comes to 32 steps.

How many steps are there on the escalator?



Groups and the Mosaics of the Alhambra

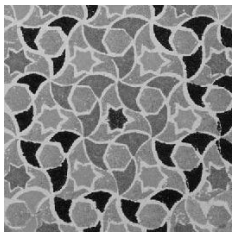
The Alhambra

The sun lazily flicked at my eye-lids as the bus came to a stop at a roadside café. It was 8 o'clock and time for a wake-up coffee. I came to life and looked out at the rolling countryside of Southern Spain. Almost in Granada.

Some time later, the bus brought us to our destination under the foot of the Alhambra itself. I had heard so much about that famous city, especially its Islamic origin and its fascinating mosaics. Now I would see for myself how its religious artists tried to capture intimations of the infinite through their replicating patterns.

A courtyard in the Alhambra

A close-up of one of the patterns of mosaics looks like this



Besides being an object of beauty, what does a mathematician see in this pattern? Which of the familiar isometric transformations of the plane can be distinguished here? The replication of pattern indicates that there must be translational symmetry. Indeed, there must be at least two translations in different directions in order to tessellate the plane – which, after all, is the object of the construction.

In the example pattern shown above there is no line of reflection – there are no reflection symmetries but there are other symmetries. Closer inspection of a primitive cell for the pattern above will reveal that rotations of 120^o (order 3) may be performed without distorting the pattern – the pattern has rotational symmetry.

> Note: The smallest section which may be translated en bloc and thereby tessellate the plane is known as a *primitive cell (or translation region)*. It is the smallest design object which can tessellate the plane by composition of translations *ad infinitum* – hence the cosmologi-

cal and theological connection. A section which may be reflected, glide reflected, or rotated to generate the primitive cell is known as a *fundamental region*. This is the smallest design object which may tessellate the plane by composition of all the symmetries involved in its pattern.

The Group Structure

Mathematically speaking the collection of symmetries described above form a group under composition of transformations. i.e. the set consisting of the identity translation, two translations (at least) in different directions and a rotation of order 3 satisfies all the group properties under composition of transformations

- there is an identity (the zero translation);
- each operation has an inverse;
- the set is closed under composition of these operations;
- composition of symmetries obeys the associative law.

Because of the connection with the tessellation of the plane this group is known as a 'wallpaper' group. In crystallography it is identified as the 'p3' group – 'p' for 'primitive cell', '3' for 'rotation order 3' (This is the notation used by the International Union of Crystallographers since 1952).

How many symmetries?

There are other patterns to be found in the Alhambra. Do we know what symmetries are involved in each of the different patterns? The answer is that in order to tessellate the plane only these following symmetries are possible – translations, rotations of order 2 (180^o), rotations of order 3 (120^o), rotations of order 4 (90^o), rotations of order 6 (60^o), reflections and glide reflections. This last symmetry is unusual in that it involves a combination of a reflection followed by a translation – first a reflection across an axis, then followed by a translation parallel to the line of reflection in a 'following-footprint' pattern.

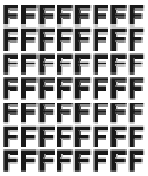
How many groups?

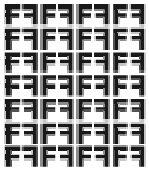
The next interesting question: how many different sets of symmetries (and thus distinctive tessellating structures) can there be and does each set form a group?

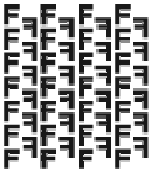
This question was first answered in 1891 when it was proved (independently by Fedorov (Russia), Schoenflies (Germany), and Barlow (England)) that there are 17 different sets of symmetries or distinct wallpaper groups. George Polya rediscovered and popularised the idea in 1924.

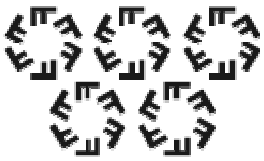
Patterns which embody the different group structures

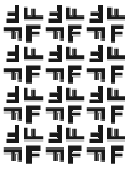
Examples of patterns which contain sets of symmetries of the plane:


Translations


Reflections


Glide reflections


Rotations of order 2,3,6


Rotations of order 2,4

These illustrations may be further investigated at http://mathforum.org/geometry/rugs/symmetry/fp.html

Back to the Alhambra

A long-running challenge has been to identify how many of these distinct wallpaper group patterns may be found in the Alhambra?

During Maths Week 2008 Professor Jose Maria Montesinos from the University of Madrid outlined how he took on this task and found representatives of each of the 17 wallpaper group patterns in the Alhambra. The talk was delivered, appropriately, at the Chester Beatty Library in Dublin which houses many examples of Islamic art.

Theory of Orbifolds

The identification of the patterns is also done under the theory of Orbifolds invented by Conway (1992). The codes for this method are given , as well as the classical system of identification, on the chart opposite.

In orbifold notation

- Numbers indicate the order of rotation involved,
- * (asterisk) refers to reflections,
- x (miracle) refers to glide translations, and
- o (wonder) only contains translations.

See: http://jwilson.coe.uga.edu/emt668/EMAT6680.F99/McCallum/WALLPA~1/SEVENT~1.HTM

Under the theory of Orbifolds each symmetry is assigned a number or fractional value.
The total sum of these features values must be 2 in order that the orbifold Euler characteristic sum is zero. [In three dimensions, V - E + F = 2 for a convex polyhedron (vertices, edges, faces) is known as the Euler Formula. The Euler characteristic for planar graphs is also 2.]

ie Euler characteristic = 2 - features sum = 0 for a wallpaper group.

o has value 2
* has value 1
x has value 1
a digit before a * counts as (n-1)/n
a digit after a * counts as (n-1)/2n

There can only be 17 permutations of the plane symmetries whose feature values sum to 2.

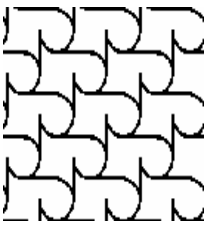
o 2	** 1+1	xx 1+1	*x 1+1	
2222 1/2 + 1/2 + 1/2 + 1/2	*2222 1 + 1/4 + 1/4 + 1/4 + 1/4	22x 1/2 +1/2 + 1	22* 1/2 +1/2 + 1	2*22 1/2 + 1 + 1/4 + 1/4
4*2 3/4 + 1 + 1/4	442 3/4 + 3/4 + 1/2	*442 1 + 3/8 + 3/8 + 1/4		
3*3 2/3 + 1 + 2/6	333 2/3 + 2/3 + 2/3	*333 1 + 2/6 + 2/6 + 2/6		
632 5/6 + 2/3 + 1/2	*632 1 + 5/12 + 2/6 + 1/4			

Table showing the 17 permutations of features and their values under orbifold theory

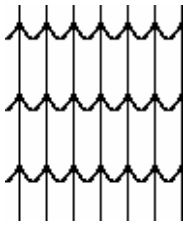
Further investigations and references:

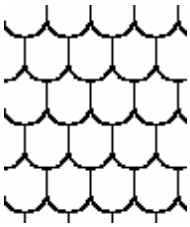
http://www.jcrystal.com/steffenweber/JAVA/jwallpaper/J2DSPG.html (Interactive)
http://www.scienceu.com/geometry/articles/tiling/symmetry/p4g.html (Animations).
http://clowder.net/hop/17walppr/17walppr.html (Animations)
du Sautoy, Marcus. 2008; *Finding Moonshine;* The Fourth Estate, London. This book gives an interesting account of the symmetries found in the Alhambra.

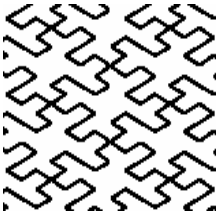
The 17 patterns and their identification codes

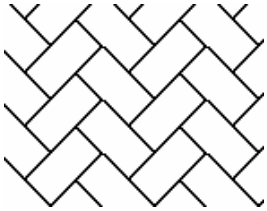

p1 o

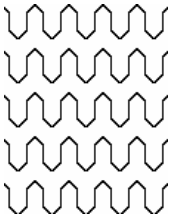
pg xx

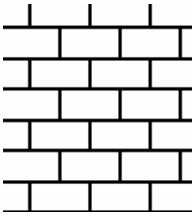

pm **

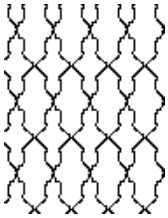

cm x*

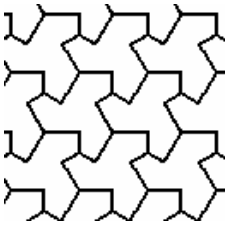

p2 2222

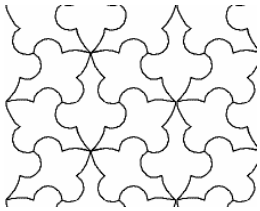

pgg 22x

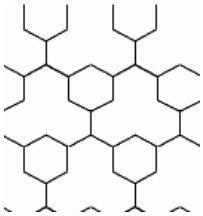

pmg 22*


cmm 2*22

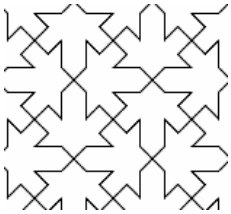

pmm *2222

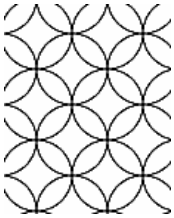

p3 333


p31m 3*3

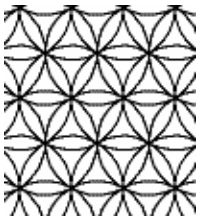

p3m1 *333

p4 442

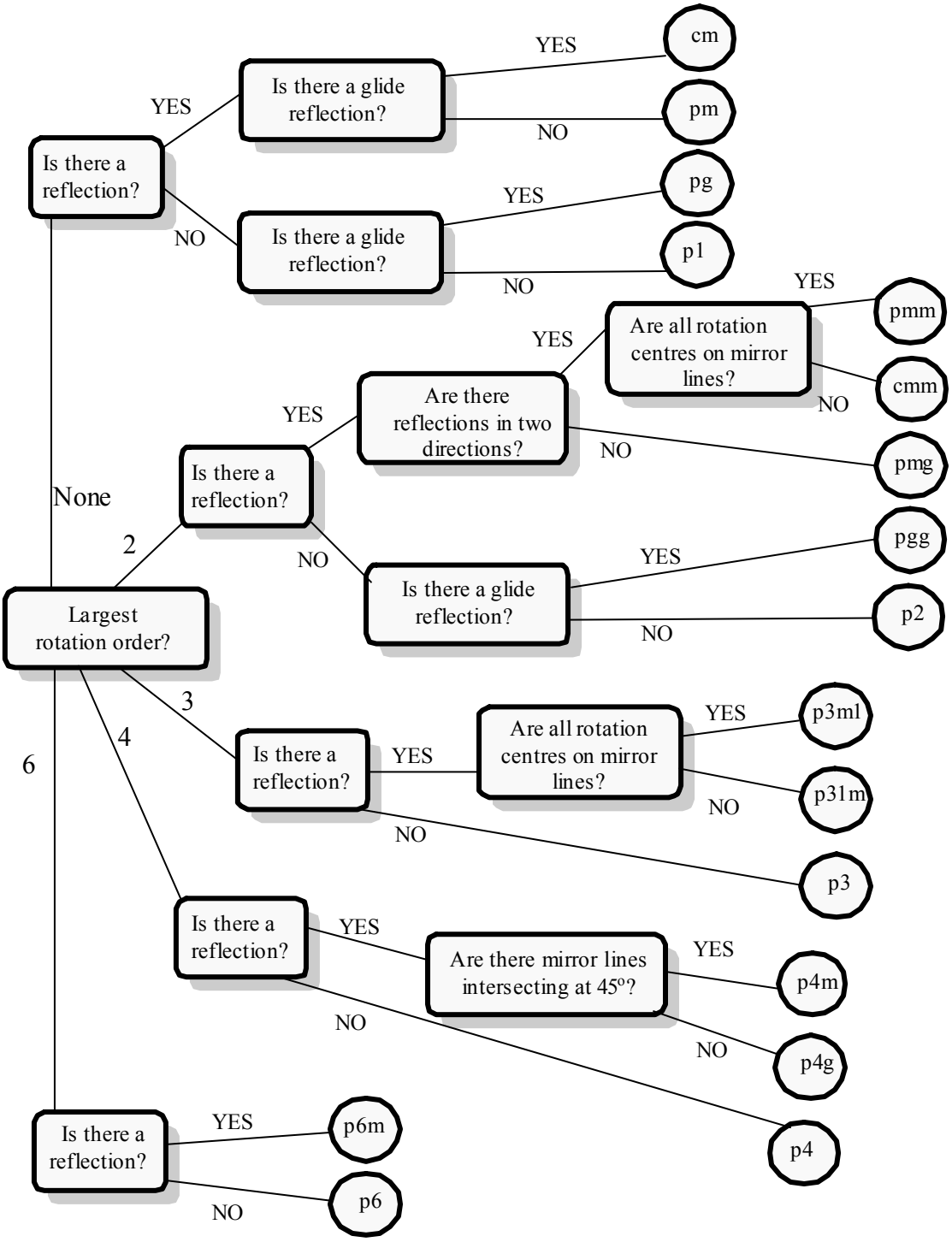

p4g 4*2


p4m *442

p6 632


p6m *632

Designs from : http://clowder.net/hop/17walppr/17walppr.html



Decision chart for identification of a wallpaper group

This flow-chart guides through the process of deciding which pattern belongs to which group.
[Adapted from http://euler.slu.edu/escher/index.php/Wallpaper_Patterns]

Neil Hallinan, Dublin

Comórtas Sóisearach Matamaitice Éireann 2010
(Irish Junior Mathematics Competition 2010)

This competition is organised by the Irish Mathematics Teachers Association (I.M.T.A.)

Eligibility First Year Students 2009/10

Format One set of question papers and answer key will be posted to each participating school some days before the competition date.

Each school will be responsible for photocopying the question paper and administering the First Round.

First Round Wednesday, March 10th, 2010 Time : 40 minutes

Final May 2010

The top students from the First Round may be invited to compete in the Final at venues to be arranged, **provided a certain standard is reached.**

Entrance fee €30 **per school** (cheques payable to I.M.T.A)

If you wish your school to participate please return the completed Registration Form with the fee **no later than November 27th 2009**

Applications received after this date may not be accepted.

Applications should be sent to:

Michael D. Moynihan (Mícheál D. Ó Muimhneacháin)
Coláiste an Spioraid Naoimh,
Bishopstown,
Cork.

Phone : 087-2860666 / 021- 4870362 (Evenings)
email mmoynihan@eircom.net

Fax : 021 - 4543625

Registration Form 2010

Name of Teacher:_____

School:_____

School address_____

Phone number_____(Home)_____(School)

School Fax number_____ email address_____

Approximate number of students participating_____

School Geometry

Anthony G. O'Farrell
Mathematics Department
NUI, Maynooth
Co. Kildare
email: admin@maths.nuim.ie

April 24, 2008

1 Levels of Sophistication

In the following, Geometry refers to plane geometry.

In designing the programme in Geometry for schools, it is essential to ensure the logical coherence of the system that is used. This is not something that can be taken for granted, and in fact, things have gone wrong in the past in Ireland[8].

We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many children from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

In schools, one should begin with Level 3, developing the pupils' geometrical intuition and teaching them geometrical language, such as the names of various shapes. One should progress to geometrical problem-solving, addressing concrete questions. At an appropriate point (Secondary Year 3 seems to be favoured), one should introduce the semiformal Level 2, involving such things as definitions, theorems, and proofs. At no stage should the material

of the fully-rigorous Level 1 be used in school. Its purpose is to ensure that the semiformal level is gounded on a sound system, and to act as a guide to the construction of Level 2.

2 Relationship of the Levels

2.1 Levels 2 & 3

The undefined terms, defined terms, and axioms of Level 2 should be based on experience at Level 3. Nothing should be asserted at Level 3 that is not (semi-formally) provable at Level 2.

2.2 Levels 1 & 2

Level 2 should be constructed from Level 1 by simplifications, such as the elimination of technical terminology and notation needed for fine distinctions and a generally more relaxed approach to the detail of proofs.

Every statement and proof in Level 2 should have a completely-rigorous counterpart in Level 1.

3 Alternatives for Level 1

There are many possible versions of Level 1. All versions should, in the end, provide derivations of precisely the same list of geometrical statements. But different choices will affect the logical sequence in Level 2, so it is essential to pick one specific system, as long as we are operating a uniform national system of examination and assessment.

To elaborate, let A and B be two versions of Level 1. Each will have undefined terms (i.e. undefined words), defined terms, definitions, axioms and theorems. Let

$$\begin{aligned} A_u &= \text{the set of undefined terms of A,} \\ A_w &= \text{the set of defined terms of A,} \\ A_d &= \text{the set of definions A,} \\ A_a &= \text{the set of axioms of A,} \\ A_t &= \text{the set of theorems of A.} \end{aligned}$$

Note that A_u and A_w are sets of words, whereas A_d, A_a and A_t are sets of statements. Let B_u, B_w, B_d, B_a and B_t be, respectively, the sets of undefined terms, defined terms, definitions, axioms and theorems of system B. Then, whereas the entire collection of terms will be the same in both theories:

$$A_u \cup A_w = B_u \cup B_w,$$

(apart from trivial changes involving replacement of a word by a synonym, and assuming definitions are added for any terms of A that do not appear at all in B, and vice-versa.) it may happen that terms that are undefined in A are defined in B, and vice-versa. In symbols, we may have

$$A_u \cap B_w \neq \emptyset \neq B_u \cap A_w.$$

Even more significantly, although the entire collection of statements

$$A_d \cup A_a \cup A_t = B_d \cup B_a \cup B_t$$

will be the same in both, it may happen that assumptions (axioms) in A become theorems in B, and vice-versa, so we may have

$$A_a \cap B_t \neq \emptyset.$$

For instance, in A one might have Playfair's Axiom of parallels

(Through any point in the plane, there is at most one straight line parallel to a given straight line.)

as an axiom, and get the statement that the angle-sum in each triangle is equal to two right angles as a theorem, and in B their rôles could be reversed. Obviously, a Level 2 programme based on B must differ markedly from one based on A.

4 How to Choose?

At the NCCA Junior Cycle Course Committee for Mathematics, I was asked to recommend a suitable Level 1, on which to base our programmes. How does one address this question?

4.1 Target

Let us denote the union of all the provable propositions[1] in system A by

$$A_p = A_d \cup A_a \cup A_t.$$

Considering the entire body of (Hilbert-)Euclidean synthetic Geometry, and coordinate geometry, the chosen system must deliver an A_p that includes the standard theorems of Geometry and the standard properties of the real number system.

4.2 Types of System

In the literature, there are two broad categories of geometrical system. The first treats "purely synthetic geometry" — Geometry without real numbers — separately, and either requires a separate, self-contained theory of the real numbers, or actually constructs the real numbers from geometrical objects. We will refer to all such systems as systems of *Hilbert type*. the second type begins with (or takes for granted) a self-contained theory of the real numbers, and uses real numbers in the axiomatization of Geometry. We shall refer to such systems as of *Birkhoff type*.

The first successful example of a system of Hilbert type was Hilbert's [6]. Hilbert added precision and additional axioms to Euclid's system, in order to repair its deficiencies and allow the rigorous deduction of all Euclid's theorems. (The main lacunae in Euclid had to do with congruence, betweenness, and completeness.) Hilbert's system has two axioms of connection, five axioms of order, an axiom of parallels

(Through any point in the plane, there is one and only one straight line parallel to a given straight line.)

six axioms of congruence, an Archimedean axiom, and an axiom of completeness.

Variations on Hilbert's system may be found described in many books designed for use in university-level courses for prospective teachers, such as Greenberg [5] or Cederberg [4]. Typically, one has about 15 axioms.

[1] Purists will have noted that, logically, the axioms and definitions are theorems with very, very simple proofs, so that $A_p = A_t$. I have been quietly assuming that a theorem is a statement provable in the theory *that is not already a definition or axiom.*

The first successful example of Birkhoff type was David Birkhoff's [2]. Taking the system of real numbers as given, he gave just four axioms. These axioms are about line measure, incidence, angle measure, and similar triangles. (In fairness to Hilbert, some of Birkhoff's axioms assert more than one simple statement.) The axioms of line measure and angle measure are usually called The Ruler and Protractor Axioms. This system, described fully in his paper in the Annals of Mathematics [2], was translated to Level 2 in the textbook of Birkhoff and Beatley [3], resulting in an account with five axioms. Variations on this are in common use in popular texts in the USA, such as [7].

4.3 Comparison

In discussions between the Heads of the National University of Ireland Mathematics schools, it was agreed that a system of the Birkhoff type has significant advantages.

- One advantage is that a Level 1 of Birkhoff type buries the difficulties associated to completeness in the underlying system of real numbers, and there is no explicit reference to completeness in the geometrical axioms. (Without completeness, a geometrical theory will not be *categorical*, i.e. there will be several different Euclidean geometries, in some of which, for instance, a line through the centre of a circle may fail to meet the circumference, and various constructions in Euclid may fail to attain their object.)

- Another advantage is that the use of real numbers makes the geometry simpler.

- Pupils may not, in fact, really understand much about the real number system, but they are familiar with it, on an intuitive level.

- The transition to coordinate geometry[2] is simpler.

Among systems of Birkhoff type, Birkhoff's own does not use an axiom of parallels. As a result of attempts to prove Euclid's Fifth Postulate, beginning seriously with the work of Saccheri (1667-1733), we know that there are

[2]By the way, it does not appear to be widely recognized that traditional coordinate geometry rests on the parallel postulate, via Pythagoras' Theorem.

various other plausible-looking axioms which, taken together with "uncontroversial" axioms of incidence, betweenness, and congruence, may be used to prove the axiom of parallels as a theorem. For instance, it is enough to assume that there exists some rectangle (i.e. a quadrilateral having four right angles). Another possibility is to assume that, given a triangle, it is possible to find a similar triangle having any given segment as one side. This is the axiom that Birkhoff uses.

Given the fact that many teachers are used to something along the lines of Playfair's or Hilbert's version, it is probably preferable to use a system that sticks more closely to Euclid on this one.

4.4 Conclusion

So one is looking for a Level 1 description of Birkhoff type, that employs an axiom of parallels. Ideally, this description should be already available in print. Necessarily, it should be thoroughly professional and complete. In particular, the account should include fully-articulated and rigorous proofs of all the theorems we might include in Junior or Senior Certificate programmes now or in the future. Without this, people who wish to construct Level 2 textbooks will have insufficient guidance.

5 Recommendation

There do not appear to be many choices available, in fact. Accounts written by experts tend to omit the detailed working-through of the theorems (because it is well-known that these can be proven once a small subset is proven), and so do not provide a reliable guide to those who will compose a simplified version at Level 2 for school use.

One book, by an Irish author, stands out. It is P.D. Barry's *Geometry and Trigonometry* [1]. Chapters 2 to 5 present a careful and complete Level 1 account, including rigorous proofs of all the theorems currently listed in the programme. There is a good deal more in the book, that builds on this foundation, and could inform Level 2 accounts of other areas of the programme.

Barry's system has the primitive undefined terms *plane, point, line,* $<_l$ *(precedes on a line), (open) half-plane, distance, and degree-measure*, and seven axioms:

A_1: about incidence,

A_2: about order,

A_3: about plane separation,

A_4: about distance,

A_5: about degree measure,

A_6: about congruence of triangles (SAS),

A_7: about parallels (Playfair, in fact).

I recommend the adoption of Barry's system as the Level 1 basis for Geometry in our schools.

It is to be emphasized again that this is *not* a recommendation to teach the exact content of Chapters 2–5 of Barry's book in schools, nor is it even a recommendation that all Maths teachers should read Barry's book. The book is aimed at a readership of university-level students. It must be converted to a Level 2 account by suitable simplification, before school use. Any proposed conversion should be vetted by experts, to validate its conformity with the Level 1 version. Also, before any Level 2 account is used with students, they should first have a thorough preparation[3] at the informal Level 3.

The simplifications required to convert Barry's account to a Level 2 account would certainly include informal, as opposed to formal, treatment of the order of points on lines, and the separation of the plane into two pieces by lines, so that the axioms A_2 and A_3 would have no counterparts at Level 2.

References

[1] Patrick D. Barry. *Geometry with Trigonometry.* Horwood. Chichester. 2001. ISBN 1-898563-69-1.

[3]Barry's Chapter 1 (especially Section 1.5) provides some useful guidance as to which concepts need to be familiar to students at an informal level, before they proceed to Level 2.

[2] George D. Birkhoff. *A set of postulates for plane geometry, based on scale and protractor.* Annals of Math. **3**3 (1932) 329-45.

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[4] Judith N. Cederberg. *A Course in Modern Geometries.* Springer. New York. 1989.

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[6] David Hilbert. *Foundations of Geometry, 3rd Ed.* (translated by E.J. Townsend). Open Court. La Salle, Illinois. 1938

[7] Harold R. Jacobs. *Geometry: Seeing, Doing, Understanding, 3rd Ed.* W.H. Freeman. New York. 2003.

[8] Susan M.C. Mac Donald. *The Paradigm Shift from Euclid to a composite System of Geometry in Intermediate Certificate Mathematics in Ireland, 1966-1973.* (Two volumes, 355+333 pp.) Ph.D. Thesis, NUI, Maynooth. 2007.

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MATHS WEEK 2009

Maths Week Ireland, October 10th to 17th, 2009 is the 4th annual Maths Week. It is an all-island celebration of mathematics. The website http://www.mathsweek.ie/index.html has all the details of public events. Hopefully, your school also participated through raised awareness in the classroom or some more formal event.

Dublin: The maths of social networking; iPods; Hamilton Walk; Maths Trails;...
Maynooth:Math Circus, ... **Dundalk**: Turing Machines, ... **Cork**: Mathemagic, …
Galway: Recreational maths from Spain, ...
Coleraine: Dr. Maths, … **Jordanstown:** Maths presentation, … **Belfast**: Dr. Maths, ...
Limerick: Math Morning at NCE-MSTL;... **Waterford**: Geometer's Sketchpad, Probability, …
There are also many Radio and TV shows with mathematical content surrounding this event.

THE IMPORTANCE OF BEING BEAUTIFUL IN MATHEMATICS

Fiacre Ó Cairbre,
Department of Mathematics,
NUI, Maynooth

1. Introduction

In this article I will discuss beauty in mathematics and I will present a case for why I consider beauty to be arguably the most important feature of mathematics. However, I will first make some general comments about mathematics that are relevant to my discussion.

Mathematics essentially comprises an abundance of ideas. Number, triangle and limit are just some examples of the myriad ideas in mathematics. I find from experience in teaching mathematics and promoting mathematics among the general public that it's a big surprise for many people when they hear that number is an idea that cannot be sensed with our five physical senses. Numbers are indispensable in today's society and appear practically everywhere from football scores to phone numbers to the time of day. One of my favourite football scores, which I refer to in some talks, is the ‘celebrated’ result:

Louth 1-9 v 1-7 Cork in 1957.

I will return to this football score later.

The reason number appears practically everywhere is because a number is actually an idea and not something physical. Many people think that they can physically see the number two when it's written on the blackboard but this is not so. The number two cannot be physically sensed because it's an idea.

Mathematical ideas like number can only be really ‘seen’ with the ‘eyes of the mind’ because that is how one ‘sees’ ideas. Think of a sheet of music which is important and useful but it is nowhere near as interesting, beautiful or powerful as the music it represents. One can appreciate music without reading the sheet of music. Similarly, mathematical notation and symbols on a blackboard are just like the sheet of music; they are important and useful but they are nowhere near as interesting, beautiful or powerful as the actual mathematics (ideas) they represent. The number 2 on the blackboard is purely a symbol to represent the idea we call two. Many people claim they

do not see mathematics in the physical world and this is because they are looking with the wrong eyes. These people are not looking with the eyes of their mind. For example if you look at a car with your physical eyes you do not really see mathematics, but if you look with the eyes of your mind you may see an abundance of mathematical ideas that are crucial for the design and operation of the car.

So what is this idea we call two? If one looks at the history of number one sees that the powerful idea of number did not come about overnight. As with most potent mathematical ideas, its creation involved much imagination and creativity and it took a long time for the idea to evolve into something close to its current state around 2500 BC. Here is one way to think of what the number two is:

> *Think of all pairs of objects that exist; they all have something in common and this common thing is the idea we call two.*

One can think of any positive whole number in a similar way. Note that this idea of two is different from two sheep, two cars etc. The seemingly simple statement that

$$20+31=51$$

is actually an abstract statement, since it deals with ideas rather than concrete objects, and solves infinitely many problems (since you can pick any object you want to count) in one go. This illustrates the incredible practical power of abstraction and many people do not realise that they use abstraction all the time, e.g. when adding. Note that it's not physically possible to solve infinitely many different problems and yet, Hey Presto! it can be done in the abstract in one go. It borders on magic that it can be done.

Abstraction essentially means that we work with ideas and also try to deal with many seemingly different problems/situations in one go, in the abstract, by discarding superfluous information and retaining the important common features, which will be ideas. Many people tend to think of abstraction as the antithesis of practicality but as the above example of addition shows, abstraction can be the most powerful way to solve practical problems because it essentially means you try to solve many seemingly different problems in one go, in the abstract, as opposed to solving all the different problems separately. The latter approach of solving the different problems separately is what people did as relatively recently as less than five thousand years ago by using different physical tokens for counting different objects. For example, they used circular tokens for counting sheep and cylindrical tokens for counting jars of oil etc. Nowadays, of course, thanks to abstraction, we just do it in one go as 20+31=51 and it doesn't matter whether we are counting sheep or jars of oil. Clearly, there are much more advanced examples of abstraction but the 20+31=51 example captures the essential feature of abstraction. See [1] for more on abstraction.

These surprises (that number is an idea and addition is an example of abstraction) can actually be

very positive experiences for some people and these surprises don't confuse them; in fact it can change their perception of mathematics for the better and make them more comfortable with other more complicated ideas because they are now already comfortable with one abstract mathematical idea, i.e. number. These surprises also enhance the understanding, awareness and appreciation of mathematics for many people. Some people also find it fascinating to know that the idea of number was not always known to humans and was actually created by somebody around 2500 BC. As I said above, before 2500 BC the idea of number had not been created and people used different physical tokens to count different objects.

Now, back to that pleasing football score:

Louth 1-9 v 1-7 Cork in 1957

Sometimes I use this result, and other examples, to illustrate how number is an idea and why it is so prevalent in today's society. I comment on how the same symbol 9 is used in two different places to indicate two different things. One refers to 9 very satisfying points scored by Louth, while the other refers to 9 hundreds of years. The reason for this is that 9 is just a symbol to represent an idea and that idea can slot into infinitely many different situations. This is one reason why mathematical ideas and abstraction are so powerful and ubiquitous in society today.

Some other important features of the above scoreline are that it was the last time that Louth won the All Ireland senior football title, it shows the smallest county defeating the largest county and I could go on!

I will now move on to the main topic of this article.

2. Beauty in mathematics

The beauty in mathematics typically lies in the beauty of ideas because, as already discussed, mathematics consists of an abundance of ideas. Our notion of beauty usually relates to our five senses, like a beautiful vision or a beautiful sound etc. The notion of beauty in relation to our five senses clearly plays a very important and fundamental role in our society. However, I believe that ideas (which may be unrelated to our five senses) may also have beauty and this is where you will typically find the beauty in mathematics. Thus, in order to experience beauty in mathematics, you typically need to look, not with your physical eyes, but with the ‘eyes of your mind’ because that is how you ‘see’ ideas.

From my experience in the teaching of mathematics and the promotion of mathematics among the general public, I have found that the concept of beauty in mathematics shocks many people. However, after a quick example (like the big sum for a little boy below) or two and a little chat

the very same people have changed their perception of mathematics for the better and agree that beauty is a feature of mathematics. One of the reasons why many people are shocked when I mention beauty in mathematics is because they expect the usual notion of beauty in relation to our five senses but as I said above the beauty in mathematics typically cannot be sensed with our five senses.

Around 2,500 years ago the Classical Greeks reckoned there were three ingredients in beauty and these were:

lucidity, simplicity and restraint.

Note that ‘simplicity’ above typically means ‘simplicity in hindsight’, because it may not be easy to come up with the idea initially. On the contrary, it may require much creativity and imagination to come up with the idea initially. These three ingredients above might not necessarily give a complete recipe for beauty for everybody, or maybe a recipe for beauty doesn't even exist. However, it can be interesting to have these ingredients in the back of your mind when you encounter beauty in mathematics. Also, for the Classical Greeks, the three ingredients applied to beauty, not just in mathematics, but in many of their interests like literature, art, sculpture, music, architecture etc.

3. Some examples of beauty in mathematics

Example 1. Big sum for a little boy

Here is a simple example of what I consider to be beauty in mathematics. A German boy, Karl Friedrich Gauss (1777-1855), was in his first arithmetic class in the late 18th century and the teacher had to leave for about 15 minutes. The teacher asked the pupils to add up all the numbers from 1 to 100 assuming that would keep them busy while he was gone. Gauss put up his hand before the teacher left the room. Gauss had the answer and his solution exhibits both beauty and practical power. Gauss observed that:

$$1+100=101,$$
$$2+99=101,$$
$$3+98=101,$$
$$\dots$$
$$\dots$$
$$\dots$$
$$50+51=101$$

and so the sum of all the numbers from 1 to 100 is 50 times 101 which is 5050. Notice how Gauss' solution exploits the symmetry in the problem and flows very smoothly. Compare it to the direct brute force approach of 1+2+3+4.... which is very cumbersome and would take a long time. Both approaches will give the same answer but Gauss' solution is elegant and the other is tedious. Gauss' approach is also much more powerful than the 1+2+3... approach because his idea can be

generalised to solve more complicated problems, but you cannot really do much more with the 1+2+3... approach. This power of the beauty in mathematics happens frequently. For those people who are shocked by the notion of beauty in mathematics, this example from Gauss usually changes their perception of mathematics very quickly for the better and they then agree that beauty can be a feature of mathematics.

Example 2. The Seven bridges of Königsberg

This is the famous Seven bridges of Königsberg puzzle. Königsberg, which is now called Kaliningrad in Russia, was a city in East Prussia during the eighteenth century. The city was on the banks of the River Pregel and the four parts of the city, denoted by A, B, C, and D, were linked by seven bridges. See Figure 1

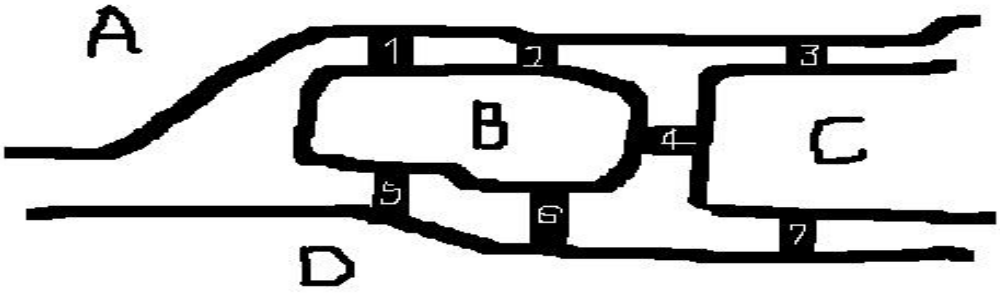


Figure 1

On Sundays people liked to walk around the city and the following question arose:

> Is it possible for one to return to their starting point, anywhere in the city, by crossing each bridge exactly once?

It's a bit like the Dublin puzzle which asks:
"Can you walk from one side of Dublin to the other without passing a pub?
I suppose you could call it 'The infinite pubs of Dublin' puzzle! Anyway, nobody could solve the Königsberg puzzle until the famous Swiss mathematician, Euler (1707-1783), heard about the puzzle and solved it in 1736.

Euler proved that it's impossible to return to your starting point by crossing each bridge exactly once. So, how did Euler's proof go? Well, suppose for convenience that your starting point is in A. The same argument will work for B, C and D. Now, Euler observed that a necessary condition

for being able to return to your starting point after crossing each bridge exactly once is that there must be an even number of bridges linked to A. The reason for this is that you must leave A on some bridge and then come back to A on a different bridge, then leave and come back again etc. If you think about it you will see that if there was an odd number of bridges linked to A, then you would have no last bridge to come back on. Now, one can see that there is actually an odd number of bridges linked to A and so you cannot return to your starting point after crossing each bridge exactly once. I think Euler's proof is ingenious and is the epitome of elegance. The Classical Greeks would consider it beautiful too because it certainly has those three ingredients of lucidity, simplicity (in hindsight) and restraint.

Euler's solution is a famous example of elegance in the history of mathematics. Furthermore, his solution of a seemingly trivial puzzle led to a whole new area in mathematics called network theory (or graph theory) which is now indispensable for understanding and designing telecommunication networks, computer circuits, complicated timetables (like our university timetable here in Maynooth) and much more. This is a great example of how an elegant solution of a seemingly innocent puzzle can lead to a major breakthrough in mathematics which in turn can produce very powerful solutions to all sorts of important problems in engineering, science and many other areas.

Notice that nowadays one could just throw this puzzle at a computer and the computer would just check all the millions of possible routes and conclude that it's impossible to return to your starting point by crossing each bridge exactly once. However, I don't see any elegance in a computer churning out the word ‘Impossible’. The computer approach provides no insight into why it's impossible and furthermore doesn't give you any new ideas that could be applied elsewhere. However, Euler's approach provides insight into why it's impossible and his idea, as I said above, led to a whole new area in mathematics that is now indispensable for solving many important problems in engineering, science and many other areas. So, maybe it's just as well there were no computers in Euler's time! Also, it's interesting to note that there are no longer seven bridges in Königsberg because the city was bombed heavily during the second world war. Only three of the original bridges are left and two of the others have been rebuilt. Apparently, it is now possible to return to your starting point by crossing each bridge exactly once, unlike back in Euler's time in 1736!

Euler was the most prolific mathematician ever, in terms of number of publications, until the Hungarian mathematician, Erdös (1913-1996), passed him out recently. Erdös was so prolific that apparently, on a long train journey once, he ended up chatting with the train conductor, who was not a mathematician, and between the two of them they solved a previously unsolved problem and published it later! Euler was not only prolific in mathematics; he also had thirteen children. Actually, he once said that some of his greatest mathematical ideas came to him while he had a sleeping baby on his lap. Note that one could base an outdoor mathematical activity on the

Königsberg puzzle by finding a place near your school, like an appropriate variety of paths in a park, and ask a similar question as in the Königsberg puzzle above.

Example 3. The walk along a mountain path

This strange looking puzzle seems to have nothing to do with mathematics and yet it is one of my favourite examples of beauty in mathematical thinking. The puzzle goes as follows:

> *Deirdre starts walking along a mountain path from her house to Ciara's house at 9 a.m. on Saturday morning. Deirdre stays overnight at Ciara's house and starts walking back along the same mountain path at 9 a.m. on Sunday morning. Is there a point on the mountain path where Deirdre passes at the same time on both days?*

Notice that there are no assumptions made about Deirdre's speed on either day. She may walk faster on one day than the other; we don't know and it doesn't matter. The solution to this puzzle involves thinking outside the box in a big way. Here is the solution: Imagine the Saturday walk and the Sunday walk starting simultaneously and you will see that the two walks must intersect at some point X. This point X is a point on the path where Deirdre passes at the same time on both days. That's it! Now, that solution has beauty.

Example 4. An extraordinary equation

The following equation is widely regarded to be the most beautiful equation in mathematics:

$$e^{i\pi} + 1 = 0 \qquad (*)$$

Why? Well, essentially because it embraces the five most important numbers in mathematics and it does so in quite a lucid and relatively simple way. The five numbers all have very different origins and yet it's quite extraordinary that one relatively 'simple' relationship embraces them all. Each of the five numbers

$$0,\ 1,\ \pi,\ e,\ i$$

has a fascinating history. The most interesting book title related to these numbers is undoubtedly '*Zero: The biography of a dangerous idea*' by Charles Seife. Read it and you will see why zero was and still is a dangerous idea. Second in the league of interesting book titles for these numbers is '*An imaginary tale: the story of* $\sqrt{-1}$ ' by Paul Nahin.

Equation (*) above can be proved by setting $\theta = \pi$ in the equation: $e^{i\theta} = \cos\theta + i\sin\theta$.

Equation (*) is not only very aesthetically pleasing, but it also is very useful in a practical way. For example, it plays a fundamental role in helping us understand how things change periodically

in time. In fact, the electricity supply industry, which utilises alternating current to provide electricity, uses equation (*) and its consequences every time it designs and operates a power station. So, quite literally, in this case mathematical beauty definitely has practical power!

Example 5. Magic

This example provides a taste of the magic in mathematics. I will present this example in the form of a trick below. Tricks can often be a good way to stimulate students. They can also provide an intriguing setting for the discussion of mathematics. The trick below has many important applications to science. For example, the trick relates to why students can listen to their favourite music on a CD and why a CD supposedly has no flaws/scratches etc. like the old LPs. The trick also relates to why we can view images from Mars!

Here is the trick:

Create an audience of students. Ask a volunteer to set up a square with five rows and five columns of cards (or anything that has a front and a back side that are different), with a random number of cards face up and face down. Ask the volunteer to turn one of the cards over while you are not looking. The trick is that you will be able to say which card was turned over. However, just before the student turns the card over, you suggest adding in one card to each of the five original rows and one card to each of the five original columns in order to make your problem more difficult. This action is crucial to the trick but you don't let the audience know this. You carefully, yet seemingly carelessly, append a new card to each of the five rows and a new card to each of the five columns such that the number of cards face up in each of the first five new rows is even and the number of cards face up in each of the first five new columns is even.

You then look away and let the volunteer turn one card over. You look back at the cards (and wave your magic wand!) and simply silently count the number of cards face up in each of the first five new rows and each of the first five new columns and note where you get an odd answer. This will tell you where the overturned card lies. It will seem like magic.

The above trick can also be performed by using zeros and ones on the blackboard instead of cards face up and face down.

I have performed this trick many times in my public promotion of mathematics in schools and the general public and the trick definitely makes a big impression on people. I feel the idea behind this trick has a certain beauty to it. Where there is beauty in mathematics, practical power will often follow, and so it's not surprising that this idea also has important practical applications. How does the above trick relate to applications in science? Well, in the trick you are using a basic version of a technique that is fundamental in the powerful practical area of ‘error correction in codes’. This is the technique where information is appended to the code (message) by the transmitter, in order that the receiver of the message will be able to detect a possible error, due to

physical interference etc, and hopefully correct the error. The analogue of the error in the above trick is the overturned card and you were able to detect where the error lies essentially by appending extra information before the card was turned over.

Error correction in codes is crucial in the performance of compact discs. Take a CD from your music collection. The sound is digitally stored on the CD. This digital information can be thought of as a code (message) consisting of zeros and ones, just like the face-up cards and face-down cards in the trick above. Extra information is also appended to the CD as in the trick above to give the total code on the CD. A laser beam in your CD player transmits this total code to a decoder. The decoder receives the total code and attempts to detect any errors which may have been caused by dirt or a scratch etc. This detection process is an advanced version of the method used in the trick above. When an error is detected it can then be corrected so that the sound emanating from your CD player is correct. This is a far cry from the needle on the turntable!

Error correction in codes is also fundamental in analysing information transmitted from spacecraft. For example, when a spacecraft takes photos of Mars, the information is digitally stored like in the CD above. This information (and the extra appended information like above) is transmitted to earth. Any errors caused along the way, like radio interference etc, can be detected and corrected like above. We can then see the correct images of Mars.

Example 6. It's a knock out

Recall that I said I could go on, in relation to the Louth v Cork scoreline. Well, this example is related to that scoreline. How many games took place in the 1957 All-Ireland senior football championship before the Louth captain, Dermot O'Brien, lifted the Sam Maguire to the cheers of all the jubilant Louth fans at Croke Park? It's not obvious, is it?

Here is a similar, yet seemingly more difficult problem: Pick any knock-out tournament you want; it could be football, tennis etc. Suppose there are 127 teams involved and that each game produces one winner who proceeds to the next round and one loser that cannot return to the tournament later on. So, there are no draws, replays or GAA-backdoor-like features. How many games must be played before the champion lifts the trophy? Generalise this to the case where you replace 127 by any positive number n.

This looks like quite a complicated problem because you don't know if some teams have byes into later rounds and you don't have any information on the structure of the tournament other than what is mentioned above which doesn't seem like enough information. Nevertheless, there will be a beautiful two-line solution to this problem. This is a good example of how, by looking at the problem in a completely different way, the solution just simply pops out. Another feature of this problem is the following metaphor which I sometimes mention in my promotion/teaching of mathematics:

You might feel like you are banging your head against a brick wall and there is no way through to the other side. However, maybe there is an unlocked door somewhere in the brick wall and you just need to gently push it open and there you are, on the other side.

Instead of concentrating on the start of the tournament and looking forward in time, like most people do, we will go to the end of the tournament to produce the elegant two-line solution:

> The 'champion lifting the trophy' is equivalent to 'exactly 126 losers' which is equivalent to 'exactly 126 games played'! Consequently, the answer is 126.

Similarly, the general solution to the n-team problem is n-1.

It's interesting to note that, in the Junior Certificate Syllabus, one of the general objectives in mathematics education is that the students should appreciate mathematics as a result of being able to acknowledge the beauty. There are many other examples of beauty in mathematics. Some require more advanced material and some don't. Here are just two more examples, of many: Euclid's elegant proof that there are infinitely many primes and the aesthetically pleasing proof by Hippasus that $\sqrt{2}$ is irrational.

4. Why beauty is arguably the most important feature of mathematics

From my experience teaching a course on the history of mathematics I feel that beauty in mathematics is arguably the most important feature of mathematics. I will present a case for this opinion shortly. Five other important features of mathematics are:

a) Deductive reasoning. See Reason (iii) below for more on this.

b) Abstraction. See section 1 above for more on this.

c) The practical power of mathematics, i.e. the powerful applications of mathematics to science, engineering, navigation, meteorology, finance and many other areas.

d) Research. Historically, research in mathematics has been very vibrant with mathematicians trying to solve many unsolved problems and also developing new theories. The motivation for mathematical research can come from a problem in the physical world or just from pure human imagination. One can play 'Who wants to be a Millionaire?' in mathematical research! How? Well, go to www.claymath.org and check out the Clay Mathematics Institute's Millennium Problems. There is a million dollars prize money for solving

any of the seven Millennium Problems. Let me know if you solve any and I would be happy to be your agent! One of the unsolved Millennium problems relates to the Navier-Stokes equation, which is partially named after an Irish mathematician. George Stokes (1819-1903) was born in Skreen, Co. Sligo. The Navier-Stokes equation is important in many practical problems including the stability of ships and can also be used to model ocean currents. Coincidentally Stokes did a lot of work on fluid dynamics, related to waves and ocean currents, and now Skreen is close to some of the best waves for surfing in Europe (e.g. Easkey).

e) Freedom. The notion of freedom in mathematics shocks many people. However, as Cantor (1845-1918) once said,

"The essence of mathematics lies in its freedom".

The reason freedom is an important feature of mathematics is because one is free to conceive of any ideas one wants in mathematics. Whether or not these ideas will lead to anything interesting or useful is another matter. Historically, the major breakthroughs in mathematics have typically happened because the great mathematicians were free to conceive of any ideas they wanted even if they broke with conventions and seemed bizarre to other mathematicians and the general public. Three examples, of many, are the discovery that $\sqrt{2}$ was irrational by Hippasus in Ancient Greece, the discovery of Non-Euclidean Geometry in the 19th century which liberated geometry and the creation of Quaternions by Hamilton on the banks of the Royal Canal in Dublin in 1843 which liberated algebra from arithmetic. See section 6 for more on Quaternions.

Mathematics is so much more than mere numbers, techniques and formulas. Techniques on their own are usually devoid of stimulation and beauty. The art of doing mathematics may involve any of the following: creativity, imagination, inspiration, ingenuity, surprise, mystery, beauty, intuition, insight, subtlety, fun, a wild thought, wonder, symmetry, harmony, aesthetic pleasure, originality, a great sense of achievement, a profound idea, a simple and yet powerful idea, deep concentration and hard work.

As I will outline below, features (a), (b), (c), and (d) above are all intimately related to beauty in mathematics.

I will now present a case for why I believe that beauty is, arguably, the most important feature of mathematics. I will give four reasons.

Reason (i) *The quest for beauty has often been the motivation for why the great mathematicians do research in mathematics.*

Intellectual curiosity, the quest for beauty and the need to understand and solve important practi-

cal problems (in science and many other areas) are some of the motivating elements for doing mathematics. From my experience in teaching a course on the history of mathematics, I feel that the search for beauty has often been the motivation for why the great mathematicians do research in mathematics. I will let some of these mathematicians speak for themselves:

Ireland's greatest mathematician, William Rowan Hamilton (1805-1865), was also a poet and regarded *"Mathematics as an aesthetic creation, akin to poetry, with its own mysteries and moments of profound revelation"*.

He also wrote: *"For mathematics, as well as poetry, has its own enthusiasm and holds its own communion, with the sublimity and beauty of the universe"*.

Hardy (1877-1947), once wrote:

> *"The mathematician's patterns, like the painter's or poet's, must be beautiful, the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics. It may be hard to define mathematical beauty, but that is just as true of beauty of any kind - we may not quite know what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it"*.

The great French mathematician, Poincare (1854-1912), said:

> *"The mathematician does not do mathematics because it's useful, he studies it because he delights in it and he delights in it because it's beautiful"*.

Somebody once wrote: *"Many mathematicians do research out of a desire for mathematical elegance and the thrill of exploring the unknown*".

Archimedes (287-212 BC) is widely regarded as one of the three greatest mathematicians of all time. The historian, Plutarch, once wrote about Archimedes:

> *"He, i.e. Archimedes, regarded the business of engineering, and indeed of every art which ministers to the material needs of life, as an ignoble and sordid activity, and he concentrated his ambition exclusively upon those speculations whose beauty and subtlety are untainted by the claims of necessity. These studies, he believed, are incomparably superior to any others, since here the grandeur and beauty of the subject matter vie for our admiration with the cogency and precision of the methods of proof"*.

This is quite a remarkable statement when one considers that Archimedes' mathematics was, and still is, exceptionally powerful when applied to the areas of engineering, science and many other areas.

Archimedes and Hamilton are great examples of people who pursued mathematics for its aes-

thetic qualities and yet their mathematics has turned out to be incredibly powerful when applied to science, engineering and many other important practical areas. There are many other examples of such people. The moral of the story here is that the practical power of mathematics can be an offspring of the search for beauty. This leads us on to reason (ii) below.

Reason (ii) *The practical power of mathematics is often an offspring of the search for beauty in mathematics.*

See examples 1, 2, 4 and 5 in section 3 where one can see the practical power of the beauty in mathematics. The quest for beauty in mathematics is what has motivated many of the great mathematicians and yet their mathematics has turned out to be incredibly powerful in science and many other areas. Very often this search for beauty in mathematics has led to new ideas and discoveries of new theories that have fundamentally changed our understanding of the physical world and are now indispensable in the physical world. It's clear from the history of mathematics that the practical power of mathematics is often an offspring of the quest for beauty in mathematics. For example, in the sixteenth century the Polish mathematician, Copernicus, was convinced that the universe was a systematic harmonious structure framed on the basis of mathematical principles, designed by God. This pursuit for an aesthetic harmonious mathematical structure led Copernicus to his famous heliocentric theory which stated that the earth and the planets revolved around the sun as opposed to the earlier belief that the earth was the centre of the universe with the sun revolving around the earth. Copernicus had no experimental evidence for his theory. The motivation for his theory was purely aesthetic because the mathematics describing the sun-centred universe was more aesthetically pleasing than the mathematics describing the earth-centred universe. Galileo and Kepler would later pursue Copernicus' ideas and provide experimental evidence that the earth revolved around the sun. This shocked the world and revolutionised science and society.

As we know, Hamilton's motivation for doing research in mathematics was the search for beauty and yet his mathematics has turned out to be incredibly powerful when applied to science and many other areas. For example, his fundamental theory of dynamics was indispensable for the creation of Quantum Mechanics which is how we now understand the physical world at the microscopic level. Also, his famous Hamiltonian function is fundamental to many aspects of physics. Here is what Hamilton wrote about his new 'General method of dynamics' in 1834:

> *"The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another; and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function, the unfolding of one central relation..."*

It's clear that Hamilton didn't care if his new theory had practical applications. The important

point for him is that it had ‘intellectual pleasure’, i.e. aesthetic pleasure. However, his new theory did turn out to have many powerful practical applications later on, e.g. in Quantum Mechanics as mentioned above. Again, here we have practical power being an offspring of the search for beauty in mathematics. In section 6 I also show some of the many powerful applications of Hamilton's Quaternions.

The Classical Greeks did mathematics for aesthetic pleasure, as we will see below in reason (iii). However, their mathematics has turned put to be exceptionally powerful in the practical world. Two examples, of many, are the ellipse and the parabola. They studied the abstract ellipse intensively for aesthetic pleasure and their results were exactly what Kepler needed two thousand years later to show that the orbits of the planets were ellipses with the sun at one of the foci. They also investigated the abstract parabola for aesthetic pleasure and their results later helped Galileo show that projectiles from the surface of the earth followed a parabolic trajectory. This solved a very important practical problem in the seventeenth century, around two thousand years after the Classical Greeks.

It's important to realise that in the applications of mathematics to the physical world, and elsewhere, mathematics does a lot more than solve problems: it can analyse, predict and prescribe: it can provide deeper insight and it can generate and explore new ideas. Mathematics has a rich history and it has played a very significant role in our civilisation.

Reason (iii) *Mathematics, as we know it today, was essentially born out of a pursuit for aesthetic pleasure and beauty by the Classical Greeks around 600 BC.*

The two main pillars of mathematics are Deductive Reasoning and Abstraction. Around 600 BC the Classical Greeks essentially created mathematics, as we know it today, based on these two pillars. Also, these two pillars, Deductive Reasoning and Abstraction, appealed to the Greeks for aesthetic reasons, as we will see below.

Deductive reasoning works as follows:
We start with premises (which are accepted facts) and then we make conclusions with certainty.
It's this word certainty that makes deductive reasoning very special and distinguishes it from all other forms of reasoning. Deductive reasoning lies at the heart of a mathematical proof and means that a proof, once done correctly, is eternal. This made deductive reasoning very appealing to the Classical Greeks for aesthetic reasons because they were deep philosophers and found the quest for eternal truths to be aesthetically pleasing. Deductive reasoning is also one of the reasons why mathematics underpins so much of science. Notice that in some cases we don't care whether the premises mentioned above are true or not (e.g. in the ‘proof by contradiction’ approach we want to prove our premises are actually false). Note that ‘potentially uncertain’ things like intuition, conjectures, etc. also play an important role in the art of deductive reasoning because they

can give you powerful insight, targets for what to prove etc. Much creativity and imagination can also be involved in the art of deductive reasoning because there is no guaranteed approach that will always work. An example of an old proof that is eternal is Euclid's proof that there are infinitely many primes. Euclid's proof is as valid today as when it was first done around 2,300 years ago. Deductive reasoning has been called a celebration of the power of pure reason. See [1] for more on deductive reasoning.

See section 1 for a discussion about abstraction. The Classical Greeks were greatly attracted to abstract concepts and ideas which they considered to be eternal, perfect and aesthetically pleasing. Concrete physical things were, in their opinion, ephemeral and imperfect. They studied the abstract circle, i.e. the mathematical circle rather than a particular physical circle. However, many of their results about abstract concepts turned out to be fundamental in solving practical problems as we saw above in reason (ii). Note that the abstract circle has no thickness, colour or molecular structure whereas a physical circle does. Even though the abstract circle may be suggested by the physical circle, the Greeks emphasised that the abstract circle and the physical circle were two totally different creatures.

Deductive reasoning and abstraction are always around whenever mathematics appears. It's very revealing when you realise that mathematics, as we know it today, was born out of a pursuit of aesthetic pleasure and beauty by the Classical Greeks around 600 BC.

Reason (iv) *As Keats wrote: Beauty is truth, truth beauty.*

I feel that this reason is not necessarily as strong as the previous three reasons but I think it's worth mentioning. Deductive reasoning in mathematics is like finding truths, in a certain sense, that follow from accepted facts. In this way a big part of mathematics relates to searching for truths, in a certain sense, and therefore searching for beauty.

5. Some consequences of mathematical beauty elsewhere

Some beautiful visions and sounds can be a consequence of beauty in mathematics. For example, a physically beautiful piece of architecture may be based on the famous number called the Golden Ratio or a beautiful piece of Bach's music may be underpinned by the Fibonacci numbers. Also, certain aesthetically pleasing symmetries in mathematics may produce visually beautiful pieces of art. There are many other examples where beauty, related to our five physical senses, can be a consequence of beauty in mathematics.

6. The importance of being...

What the hell! Since this is a piece about beauty in mathematics, I might as well end with a bit of symmetry by circling back to the title of this article. Hamilton created a strange new number system called Quaternions on October 16, 1843 at Broombridge on the banks of the Royal Canal in Dublin. He has been called the Liberator of Algebra because his Quaternions did not satisfy the commutative rule of multiplication (ab=ba) in arithmetic and so they liberated algebra from the shackles of arithmetic. In Quaternions the order in which the numbers appear is important and this did not bother the creative Hamilton because this is usually what happens in the physical world. For example, consider an empty swimming pool and the two operations of diving in head-first and turning the water on. The order in which the operations take place is quite important! See [2] and [3] for more on Hamilton's life and works.

Quaternions now play a prominent role in many areas. One example, of many, is that they are heavily used in the computer animation industry and in special effects in movies. An example of this that always appeals to journalists, radio hosts and students of course, is that Lara Croft of Tomb Raider was created using Quaternions! Also, Quaternions were involved, through the Irish company Havok, in creating the renowned new special effects in the film, The Matrix Reloaded, and in the recent Bond film, Quantum of Solace. Havok won an Emmy award in the US in 2008 for pioneering new levels of realism and interactivity in movies and games. Havok were involved in creating the special effects for the movie, Poseidon, which was nominated for an OSCAR for its visual effects in 2007. Hamilton's WILDE thought on that famous day, October 16, 1843, with his creation of these strange four dimensional numbers, Quaternions, shocked the mathematical community and changed the whole landscape of mathematics forever because soon after the event many other mathematicians followed in Hamilton's footsteps and felt free to conceive of all sorts of seemingly strange number systems or algebraic structures (e.g. matrices) that did not satisfy the commutative rule (ab=ba) from arithmetic.

Quaternions also played a significant role in Maxwell's mathematical theory and prediction of electromagnetic waves in 1864. Thus, the inventions of radio, television, radar, X-rays and many other important products of our times are directly related to Hamilton.

I organise the annual Hamilton walk on October 16 where people retrace Hamilton's steps from Dunsink to Broombridge. It's an ideal event for Transition Year groups and many from the general public also participate. Typically, around 200 people come on the walk. Contact me if you are interested in bringing a school group. Many famous people including Fields Medallists and Nobel Prize winners have participated in the walk, which will be celebrating its 20th anniversary this year in 2009.

In 1855 Hamilton received a very unusual request. He refers to the event as follows:

"A very odd and original lady has also had a baby; such things, as you know will happen... Recently, when I met her for the first time in my life, she told me of this young Pagan as she called him. And she asked me to be a godfather, perhaps because she is an admirer of Wordsworth. However, I declined".

The young pagan above was none other than Oscar Wilde. Lady Wilde soon became a close friend of Hamilton. Wordsworth is mentioned above because Hamilton and Wordsworth were good friends. In fact, Hamilton was the godfather of Wordsworth's son William. On a visit to the Dublin Writers' Museum in Parnell Street some years back, I noticed that Oscar Wilde was born on October 16. Who knows what Hamilton's reply to Lady Wilde might have been if he had known this?

References

[1] A Resource for Transition Year Mathematics Teachers, by F. Ó Cairbre, R. Watson and J. McKeon, published by the Department of Education and Science, 2006.

Note: Two free copies of this Resource Book were sent to every Second Level school by the Department of Education and Science in 2006.

[2] William Rowan Hamilton (1805-1865), Ireland's greatest mathematician, by F. Ó Cairbre, Ríocht na Midhe, 2000, XI, 124-150.

[3] William Rowan Hamilton and the Quaternion Walk, by Fiacre Ó Cairbre, Irish Mathematics Teachers' Association Newsletter 105, 2006, 3-13.

Multiplication in Algebra Revisited

Here is a multiplication of two algebraic expressions using the method of 'long multiplication'.

Expand and simplify: $(3x^2+2x-1)(x+2)$

$3x^2+2x-1$

$x+2$ Start the multiplication with the LEFTMOST term

$3x^3+2x^2-1x$

$+6x^2+4x-2$ Align the 'like terms' in the products

$3x^3+8x^2+3x-2$ **Result**

From a discussion with John Courlander, Dublin

A Summary Of Efficient Numerical Tests Of Robin's Reformulation Of The Riemann Hypothesis – A Young Scientist Project

The Riemann Hypothesis is a highly significant and important unsolved problem in mathematics. It originates from a German mathematician named Bernhard Riemann (1859). Riemann's Hypothesis, despite numerous efforts, still remains unsolved.

The proofs of many important theorems rely on the truth of his hypothesis. Most people will remember being asked to find the roots of certain functions at school. These are the values of x which make the function equal to zero.

The Riemann hypothesis is concerned with the roots of a certain mathematical function called the Riemann zeta function. Riemann made a very specific statement about these roots and what form they would have to take. This became known as the Riemann hypothesis. Because of the difficulties faced with proving the Riemann hypothesis mathematicians have put forward various reformulations of the Riemann hypothesis, that is, statements that are equivalent to it so that if the reformulations could be proved then the proof of the Riemann hypothesis would follow.

Our project is a study of one of these reformulations – the one due to G. Robin (1984). It appears that very little work has been done on this reformulation. The only study we found was by a man named Keith Briggs (2006). The particular version of Robin's reformulation that we are interested in involves a very special type of integer called a Colossally abundant number. In a very broad sense, without being totally accurate mathematically, a Colossally abundant number is an integer which has many divisors. The precise definition is rather complex but the description we have given conveys the essential idea.

In our project we introduce new mathematical functions and we restate Robin's reformulation in terms of this function as follows: if the function is > 0 for a Colossally abundant number > 5040 then the Riemann hypothesis is true. We plotted a graph related to this function for the first 10 000 Colossally abundant numbers. After an initial wobble this graph has an extraordinary simple pattern – it is very nearly a straight line.

We therefore have compelling numerical evidence that our function will remain positive for all Colossally abundant numbers even those beyond 10 000. The very nearly straight line pattern points to some, as yet, undiscovered properties of the Colossally abundant numbers which, if understood, might lead to a mathematical proof that our function remains positive for all Colossally abundant numbers > 5040 and therefore the proof of the Riemann hypothesis.

Gary Carr; Graham McGrath; Darragh Moriarty—C.B.S., Synge St., Dublin

SIMPLE CABLES
- A YOUNG SCIENTIST PROJECT

I carried out an investigation into the mathematics of "Simple Cables" and the relationship between the formulae in the chosen subcategories. The mathematics involved included a large amount of calculus and a small amount of geometry.

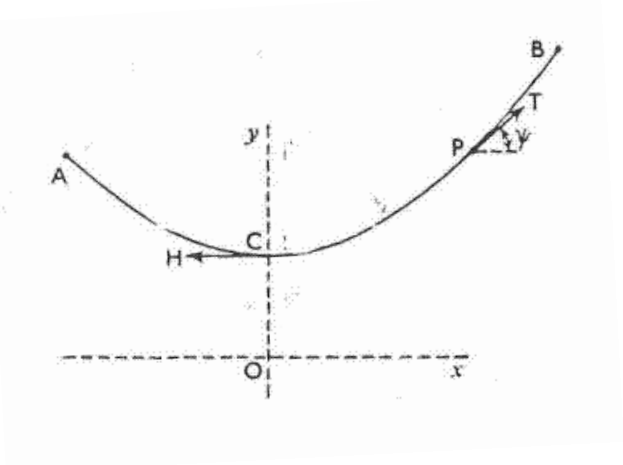
My chosen subcategories are

The Common Catenary,
The Parabolic cable,
The Heterogeneous cable,
The Comparison of Cable Shapes, and
The Extensions and Deformations of the Cable.

I drew up the following table:

Subject	The Common Catenary	The Parabolic Cable	The Heterogeneous Cable
Stable Equilibrium	$T\cos\varphi = H$	$T\cos\varphi = H$	$T\cos\varphi = H$
Unstable Equilibrium	$T\sin\varphi = w's$	$T\sin\varphi = wx$	$T\sin\varphi = \int_0^s w'ds$
Slope of the Cable at "P"	$Tan\varphi = \frac{w's}{w'c}$	$Tan\varphi = \frac{wx}{H}$	$HTan\varphi = \int_0^s w'ds$

which allowed me to make a straight comparison and conclude that $T\cos\varphi = H$ was present in all cables for all in stable equilibrium. Similarly $T\sin\varphi$ was present in all for unstable equilibrium and $Tan\varphi$ for all slopes of the cable at any point "p"



Helen Iliff

Eureka Secondary School, Kells, Co. Meath

The First International GeoGebra Conference

Introduction

GeoGebra, the award winning, free dynamic maths software was developed in 2002 by Markus Hohenwarter [1]. Since then the programme has been continuously improved and has increased in popularity. One of the features of GeoGebra is that a community of ***users*** has grown up around the programme. In fact the number of core developers is small but a large group of people are active in the forum and through the Wiki. There are also many sites on the internet where teachers at all levels have uploaded content created with GeoGebra [2].

In July this year I attended the First International GeoGebra Conference in Hagenburg, near Linz in Austria. The main aim of the conference was to discuss which directions the GeoGebra community should take in the future. Over 100 delegates from America, Asia and Europe, including five from Ireland attended. Delegates divided into five groups for workshops. These were interspersed with plenary sessions for all and plenty of coffee breaks!

New Version and Future developments

The latest version of GeoGebra, 3.2, was released just before the conference and new features were demonstrated at the conference. These include animation, dynamic colours and layers. A spreadsheet and statistics commands from the Input window have also been added. It is planned to make GeoGebra easier to use by adding new toolbars for the statistics commands and to allow dragging and dropping of panes. Many delegates demonstrated ways in which they have simulated 3D effects in GeoGebra. The good news is that a 3D version is being developed. This will work in parallel with the current 2D version. It is also planned to add a CAS (computer algebra system) feature that will integrate with GeoGebra. Also, a book, "Model-centered Learning with GeoGebra: Theory and Practice in Mathematics Education" is in the process of being written.

With the increase in popularity some aspects of the GeoGebra project have become unwieldy. In particular the Wiki and the sharing of applets [1]. It is planned to reorganise the Wiki and to look at ways in which applets can be evaluated and tagged. This will make it easier for teachers to search for and find good relevant material.

Best Educational Practice

One of the most interesting aspects of the conference for me was the amount of thought that has gone into finding the most effective ways to use GeoGebra in the teaching and learning of maths. I attended the workshops on "The Creation of Instructional Materials with GeoGebra". There were many demonstrations of using GeoGebra that were very impressive [3].

Judith Hohenwarter led a discussion on the principles of good design for applets created with dynamic geometry software. This is an area that she is actively working in and has already published guidelines which are useful for any teacher creating or evaluating applets.[4]

Since 2003 laptops have been provided for students in the state of Maine. Pamela Buffington gave a presentation on her work for "EDC In Maine". [5] This involves supporting students and improving teacher practice. Extensive use is made of GeoGebra. Many of the resources can be accessed from their Moodle site, the links at the end of the page can be used without logging in [6]. One idea being developed is the use of probes. These are questions designed to identify common misconceptions. The plan is to directly link the students response to appropriate material that will help the student to understand and correct the mistake. There are some examples in the Moodle with probes and separate associated resources.

The coffee and lunch breaks provide opportunities to meet and talk to teachers from other countries. Access to computers is still an issue for many, though they generally have better facilities than we do. Dynamic geometry software has been used in some countries for many years. It was interesting to see that where there has been recent curricular change the use of GeoGebra has been incorporated into the syllabus and /or text books. Norway is a good example of this [7]. I also met two Flemish teachers who have written dozens of maths text books. They showed me some of their books which make extensive use of dynamic geometry software. The books have links to a website containing the applets. The books are quite different to ours with a lot of pictures and more emphasis on real maths problems.

An Irish Version?

Over one hundred volunteers from around the world have been involved in translating GeoGebra into at least 46 languages including Welsh but not Irish. The Wiki contains instructions on how to translate the different parts of GeoGebra [1]. Some of these are not too difficult, in particular the GUI (Menus, buttons etc.) which if translated would allow teachers and students to use an Irish version of GeoGebra. It is possible for a group of people to collaborate and divide the work up between them.

The two day conference was a great experience and I am still sifting through the information gathered at the conference. GeoGebra is an impressive piece of software and the community around it makes it even more useful. Being free means that every teacher and student with access to a computer can use it. Hopefully we can develop a community in Ireland which will develop and share resources created with GeoGebra. There are already some GeoGebra resources on the Project Maths website. Ideally there would be a Wiki similar to the GeoGebra one but just for resources for the maths curriculum in Ireland. There could also be a forum for Irish teachers where people could ask questions and share ideas. The conference clearly showed that the use of

GeoGebra is most successful where training is provided to teachers (both pre and CPD) and when resources are shared.

Notes: (This notes page is also available at http://tinyurl.com/imta-dh)

[1] The main page is http://www.geogebra.org and it contains links to download the program, to the Wiki with resources and information on translation (click on Credits).

[2] For an Irish example see http://www.ul.ie/cemtl/resources.htm which has two pages of applets as well as notes on using GeoGebra. My own examples are at http://www.loretothegreen.scoilnet.ie/GreenMaths/

[3] For just one example see the links in the presentation by Andreas Lindner http://ggbconference2009.pbworks.com/f/Using+Images+in+GeoGebra+for+Teaching+Mathematics.doc

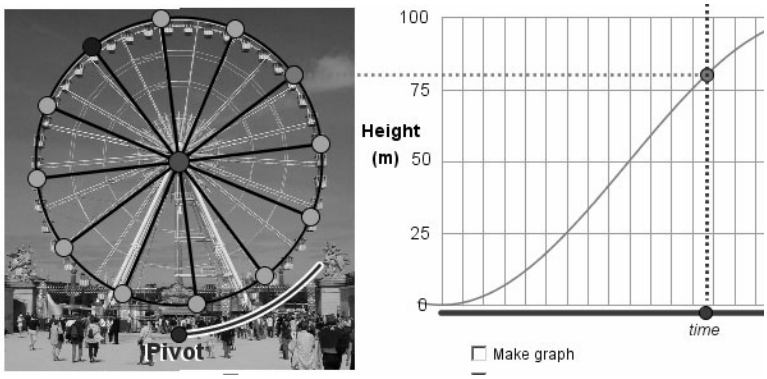
[4] http://ggbconference2009.pbworks.com/f/Hohenwarter_DynWSDesignGuidelines.zip

[5] http://www.edc.org/newsroom/articles/maine_learning_technology_initiative and http://www.edc.org/about the Education Development Centre is a not for profit global organisation.

[6] the Moodle for Maine is at http://www.maine.edc.org/ check the four links at the end of the page, the first one is a Zip file for downloading all the applets and worksheets.

[7] http://ggbconference2009.pbworks.com/f/SigbjornHals_ConferenceNote.pdf and http://ggbconference2009.pbworks.com/f/SigbjornHals-ProjectDdescription.pdf

An example of an applet using images by Andreas Lindner



An example from EDC in Maine – this one is quite complex

David Hobson
dh@loretothegreen.ie Dublin

JUST FOR FUN

Take a chess board of 64 squares. Place one coin on the first square, 2 on the second, 4 on the third, 8 on the fourth...how many on the 64th? As you know it is 2^{63} but you don't often see it written out in full.

So here it is: the number of coins on the 64th square is 9,223,372,036,854,775,808 !!

But more interesting is the total number of coins now on the chess board which is 18,446,744,073,709,551,615 . That's a lot of coins and quite a big chess board.

John Courlander, Dublin

Ireland and the IMO (1988-2009)

1 The Origins

The International Mathematical Olympiad(IMO) celebrated its 50th Birthday in Bremen, Germany in July, 2009. The first IMO was held in Romania in 1959 with seven countries taking part. Ireland's first participation in this annual event was in 1988 when the IMO was held in Australia on the occasion of their bicentennial. Ireland has now taken part for twenty two consecutive years. In 1988, 49 countries participated. This number has more than doubled with 104 countries participating in 2009. Benin, Mauritania, Syria and Zimbabwe participated for the first time in 2009. The distribution of participating countries by continent was Europe 40, Asia 34, Americas 20, Africa 8 and Australasia 2. Egypt and Ethiopia are the largest countries which do not participate in the IMO. Since 1988, an Irish team of six secondary school students have travelled nine times to Europe, eight times to Asia, four times to the Americas and once to Australasia.
The IMO students sit two exams on successive days. Each exam has three questions and the students have four and a half hours to answer or attempt to answer them. The remainder of the time is taken up with sightseeing and social activities. Medals are awarded to the top half of the participants in a ratio of 1:2:3 for Gold Silver and Bronze.

2 Preparation

In recent years, training has been provided for mathematically talented students at five centres throughout Ireland.

1. UCD (Saturday mornings)
2. UCC (Saturday mornings)
3. NUIM (Saturday mornings)
4. NUIG (Saturday mornings)
5. UL (Thursday evenings).

The training runs for up to four months beginning in December or January.In the past, MICE and Dundalk RTC provided training for a number of years. The target pupils are those that have completed the Junior Certificate but age is not a barrier. Certain notable students such as Raja Mukherji, Fiachra Knox and Galin Ganchev were spotted earlier. One of the world's leading Mathematicians Terence Tao, a winner of the Fields medal in 2006, represented Australia at the tender age of 10 in IMO 1986.
The Irish Mathematical Olympiad(IrMO) is held at the beginning of May. This consists of two papers of duration three hours. Each paper has five questions. This is used to select the top six pupils to represent Ireland at the

IMO in July. The top performer in the IrMO is awarded the Fergus Gaines Cup.
Since 1993, a training camp has been held for team members at the University of Limerick in the time between the end of the Leaving Certificate and the IMO. Usually a few other promising students are invited to attend and several of these have made the team in subsequent years. In 2002, teams from Luxembourg and New Zealand took part in the training camp.
Close to 100 mathematicians have been involved in providing training at the various centres over the years including some former Irish Olympiad students. Those involved in the training at Maynooth put together in 2002 some of their teaching material into a volume Maynooth Mathematical Olympiad Manual published by Logic Press. However, many books have been published in the last ten years aimed at Olympiad and aspiring Olympiad students. There are also a number of Websites such as The Art of Problem Solving and Mathlinks which provide a forum for Olympiad activities.

3 Irish Olympiad Students

During the twenty two years, 95 students have represented Ireland at an IMO. The record participation stands with Galin Ganchev who participated four times. Raja Mukherji and Fiachra Knox participated three times. Thirty students have taken part twice and 62 have participated once.
Professor Tom Laffey and Professor Finbarr Holland were the two key figures in Ireland's first involvement in the IMO. Since 1988, thirteen Irish mathematicians have attended the IMO as Leader, Deputy Leader or Observer. Dr Fergus Gaines was leader at eight Olympiads and Professor Tom Laffey at six Olympiads. Gordon Lessells has been Deputy Leader at 15 Olympiads. There are some people in the IMO movement who have been involved as a student or leader for more than 40 of the 50 Olympiads. The most notable is Dr Józef Pelikán from Hungary a student from 1963-66 and Leader for most of the years since.
Ireland's top student is Fiachra Knox who won a Bronze Medal in 2004 and a Silver Medal in 2005. Six other Bronze medals were won by Andrew McMurry (1990), Kevin McGerty (1993), Raja Mukherji (1998 and 1999), Marty O'Leary (2001) and Stephen Dolan (2007). In addition, 19 Honourable mentions have been won. These are awarded to students who have received full marks on one of the six questions and has not won a medal.
The highest score by an Irish student is 34 by Fiachra Knox in 2005 which ranked him 43rd in the world. The highest total by an Irish team was 68 in 1994 in Hong Kong. The highest ranking of an Irish team was in 1990 when Ireland came 40th of the 54 countries participating. The highest relative ranking was 44% in 2005 mainly thanks to Fiachra's top score.
Of the 95 students, only six have been girls. Three of these Jillian Delaney, Deirdre O'Brien and Kate McCutcheon participated twice. This reflects a tendency in other countries. In the 2009 Olympiad, of the 565 contestants only 59 were female. At 10.4% this was in fact the largest percentage of female students in the last 40 years. The third student overall in this year's

IMO was Lisa Sauermann who now has 2 Gold and one Silver Medal. She is young enough to participate in two more Olympiads and could become the all time best student if she wins two more Golds.
There are two pairs of brothers who have represented Ireland, Gavin and Hugh Hurley from Cork, and Greg and James Wall from Carlow. There have been six students called David and five called Stephen. Murphys beat Ryans by four to three.
The official IMO Website http://www.imo-official.org/ has a wonderful collection of data arranged by countries for the fifty IMOs to date.

4 Schools

Sixty five schools throughout the country have had students on the Irish IMO team. Sixteen have had two or more representatives. Coláiste an Spioraid Naomh and Blackrock College have each had five representatives. Dublin has 32 representatives and Cork 26. The 65 schools are distributed throughout 18 counties. The eight counties with no representative are Galway, Kilkenny, Leitrim, Laois, Offaly, Roscommon, Sligo and Wexford. Kilkenny, Leitrim, Sligo and Wexford have the disadvantage of being at least two counties away from any of the training centres. The three representatives from the counties of Donegal (Stephen Dolan), Cavan (Raymond Kinnane) and Monaghan (John Holland) did not find distance a barrier and each attended two Olympiads.

5 Where are they now?

Of the 95 students on Irish IMO teams, most have continued their education in Ireland but about one sixth have gone to Oxford or Cambridge as undergraduates 11 to Cambridge and 5 to Oxford. Not all have studied Mathematics. Of those not studying mathematics, physics and engineering have been the most popular alternatives but others have studied Medicine, Languages and Business. At least 16 have completed a Ph.D. and have academic positions in Universities around the world. I have picked out nine students from the early years to display the wide variety of careers that they have followed. Quite a few have studied abroad, some staying abroad and others returning to Ireland.

5.1 Luis O'Shea: Canberra 1988

Luis studied Mathematical Sciences at UCD and obtained a PhD in mathematics from Cornell University in 1999. He worked as a management consultant for McKinsey & Co in Madrid, then returned to the US and worked at a number of internet start-ups. Currently he works in finance as a quantitative researcher (a "quant") at RiskMetrics Group in New York. He is married with one daughter.

5.2 Eamonn Maher: Canberra 1988

Eamonn studied mathematics at Trinity College, Dublin where he became a Scholar in 1991. Since graduating with a B.A. he has mainly worked in the software industry. At Baltimore Technologies he worked as a Java engineer where he was involved in developing internet security products including cryptographic toolkits, security protocol libraries and wireless technology products. He currently works as a Senior Technical Consultant for NewBay Software who provide digital lifestyle services for mobile operators. He helps design, deploy and integrate these services. He is married to Ruth Parkin and they are expecting their first child.

5.3 Cormac O'Sullivan : Canberra 1988

Getting on the Olympiad team did help direct Cormac towards pure maths. He is still friends with Eamonn Maher from the 1988 team and they both went to TCD to do maths degrees. After that Cormac came to New York to do a Ph.D. at Columbia University in Number Theory. Now he is an associate mathematics professor for the City University of New York, one of the largest public universities in the U.S. with 21 campuses in all five boroughs. He is based at Bronx Community College in the northern Bronx and at the Graduate Center in midtown Manhattan. This spring he taught graduate algebra to the first year Ph.D. students - was a lot of work and a lot of fun. Since getting his doctorate he has been publishing about one research paper a year. For him, research is more of a hobby than work. He does not have any involvement with the IMO these days except for trying his hand at the latest problems occasionally. Actually two good friends of his in New York were on the Iranian olympiad teams in 1988 and 1989. He remembers the Irish team hung out with and played cards against the Iranian team in 1988.

5.4 Jillian Delaney : Canberra 1988, Braunschweig 1989

Jillian studied maths and philosophy at U.C.C. graduating with a B.A. in 1993. She worked in the arts for a couple of years (worked for a theatre company, taught music and art etc) before changing career and working in IT from 1999 to 2007 - mostly database programming and administration. She changed career again in 2007 and has been working as a statistician at the Central Statistics Office from then to the present. She is responsible for business demography data, which involves collecting and disseminating information about changes to the population of businesses in Ireland over time. She has been based in Cork all along, plays music in her spare time and has one daughter.

5.5 Conrad Hughes : Braunschweig 1989

Conrad co-wrote a few computer games, got a BA in Mathematics at Trinity, worked for Baltimore Technologies implementing cryptographic software for a few years (when they were still fewer than ten people), contributed security chapters to a well-regarded book on the Java programming language, moved to Scotland, bit of environmental studies and social activism, continued to work in software with various companies, and has been a researcher at the University of Edinburgh and an occasional software consultant and software testing lecturer for the last few years. Work at Edinburgh has included visualisation for epidemic management for the World Health Organisation, among many other things.

5.6 Andrew McMurry, First Irish medallist: Braunschweig 1989, Beijing 1990

Andrew went to Oxford to study Physics. He went on to complete a D. Phil. From 2000 he has worked in Norway in Astrophysics studying the atmospheres of our sun and other near stars. He has completed five years at the Centre of Mathematics for Applications (CMA) in Oslo where he has been working on developing a high level language for numerical analysis. In his spare time he is a member of the Tolkien Society. He is married with one daughter.

5.7 Cian Dorr : Beijing 1990

Cian did a B.A. and M.A. at U.C.C., and then a Ph.D. in Philosophy at Princeton University. His first job was as an assistant professor in New York University, for three years; he then spent five years in the University of Pittsburgh, before moving to Oxford, where he is currently Senior Faculty Lecturer in Philosophy.

5.8 Cathal Walsh : Sigtuna 1991

Cathal studied Theoretical Physics and Mathematics at undergraduate level before completing a PhD in Bayesian statistical modelling. He worked in the financial services industry for a short period before taking up an academic post at Trinity College Dublin. At present he is a Senior Lecturer in Statistics and is an active researcher in the field of Biostatistics, with a particular interest in computationally intensive methods and techniques for evidence synthesis. He has published over 30 peer reviewed articles in journals in the statistics, computer science and medical literatures. He is an active contributor to the work of the Royal Statistical Society and the International Society for Bayesian Analysis. Through research collaborations and sabbatical leave he has had the opportunity for prolonged visits to Australia and the United States. He lives in Dublin and is married with four children.

5.9 Michael Sharpe : Moscow 1992

Michael is currently working as the Life Product actuary for Hibernian Aviva. He is the manager of a team that is responsible for all the life assurance product development and pricing work. The work is very challenging and there is continually new challenges and work cropping up in a very competitive industry. After leaving school he went on to do a degree in Pure Maths at Trinity College Dublin (part of the reason he chose this degree course was due to the maths olympiad training that he did in UL). He would recommend it to anyone who finds the olympiad problems challenging and enjoyable. After college he travelled for 2 years before coming back to Ireland and starting work as a trainee actuary with New Ireland assurance in 1998. He found the actuarial exams enjoyable having had a background in maths - he also found the whole world of finance and insurance to be an enjoyable one. He worked in IT, pensions and product development while working for New Ireland assurance/Bank of Ireland Life. In 2007 he qualified as an actuary and in the same year joined Hibernian. He thinks that a career in actuary is ideal for some mathematics students who would have a business/economics bent.

5.10 Kevin McGerty : Moscow 1992 Bronze in Istanbul 1993

Kevin studied mathematics in Cambridge, and then went to M.I.T. to do a Ph.D. in geometric representation theory. He was then a member of the Institute for Advanced Study and a L.E. Dickson Instructor at the University of Chicago, before taking up a Royal Society University Research Fellowship which he currently holds at Imperial College London. In his spare time he dodges death cycling to and from work in London, and has less time to read and play the cello than he would like.

5.11 Peter McNamara : Toronto 1995

After the IMO, Peter studied maths at Trinity, followed by a postgrad at the Massachusetts Institute of Technology. He then had two years of postdoctoral research in Montreal and one year in Lisbon. In 2006, Peter began a "real job" at Bucknell University, a small private university in rural Pennsylvania. He lives there with his American wife, Emily Dryden, who is also a maths faculty member.

5.12 Gavin Hurley : Toronto 1995 and Mumbai 1996

Gavin studied Maths at Cambridge and then did a PhD in Operations Research at Cornell University. Following this, he spent three years as a quantitative analyst in interest rate products at Goldman Sachs in London. He has now returned to Cork where he works in the Energy Trading division of Bord Gais.

6 What are Irish students best at?

At the IMO, the students have two examinations on successive days. The duration of each examination is four and a half hours and on each day there are three questions. On each day the questions are usually ordered by difficulty with Q1 and Q4 regarded as *easy* and Q3 and Q6 regarded as extremely hard. Over the years questions 3 and 6 have definitely got harder. This is reflected in the following counts of the number of complete solutions by Irish students over 22 years.

Problem	Correct Irish solutions
Question 1	11
Question 2	6
Question 3	2
Question 4	10
Question 5	3
Question 6	1

Each question is marked out of 7. The two students who achieved a 7 on Q3 were Eoghan Flanagan in 1994 and Fiachra Knox in 2005. Andrew McMurry achieved a 7 on Q6 in 1989.

The problems used at the IMO are selected by the team leaders one week before the contest takes place. A month beforehand a panel of a dozen mathematicians have been drawing up a short-list from problems submitted by the participating countries. In this short-list the problems are divided into four categories : Geometry, Algebra, Combinatorics and Number Theory. Each paper usually has two geometry problems and four from the other three categories. Ireland's 33 perfect answers can be categorised as 18 in Geometry, 6 in Number Theory, 5 in Combinatorics and 4 in Algebra. Many of the perfect geometry solutions have come from good use of coordinate geometry (the Irish Method) rather than elegant geometrical arguments. In fact the two questions best answered by Irish students are Q2 in 1994 bringing 30 marks including three 7s and Q4 in 2007 bringing 32 marks and four 7s. Both of these are geometry questions. One might conclude that Irish students are better at geometry than the other topics. However, in most years a geometry question is chosen as one of the easier questions (1 and 4). Among the Irish IMO students who have completed a PhD more have studied combinatorics than other areas. David Conlon, Mark Dukes, Peter McNamara and David Galvin have completed PhDs in this area and Fiachra Knox is about to start a PhD in Oxford in this area with Tim Gowers a former IMO Gold medallist and Fields medallist.

7 Irish Problems

Since getting involved in the IMO, Ireland has regularly contributed problems to the Problem Selection Committee and on four occasions an Irish problem has been selected. Finbarr Holland created three of these 1988 Q4, 2003 Q5 and 2006 Q3. Tom Laffey created the fourth 1993 Q1. Over the years other problems have been shortlisted but not selected. Four out of 132 problems in

22 years is well above the average. Finbarr's problem from 1988 is regarded by many as one of the most elegant problems to have been selected over the years.

7.1 1988 Question 4

Show that the set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.
One strategy which is common in problem solving is the following: If you cannot solve the problem, try to solve an easier problem.
Here is an easier version. Show that the set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{2} \frac{k}{x-k} \geq \frac{5}{2}$$

is a union of two intervals, the sum of whose lengths is $\frac{6}{5}$.
This version can be solved by anyone who can solve quadratic equations.

8 Other Olympiads

There are now a large number of Science Olympiads. Over the years, several Maths Olympiad students also took part in these. Oliver Nash attended the Physics Olympiad, Raja Mukherji won a Silver Medal at the Chemistry Olympiad, and Brian Jones, Stephen Dolan and David McCarthy have won medals at the Informatics Olympiad.

9 Conclusion

Over the 22 years the IMO has provided a wonderful experience for the students chosen on Irish teams. In many cases, they have travelled to a new country and have met students from many different countries. These have not always been students from English-speaking countries. Over the years, friendships have developed with students from Bosnia, Georgia, Iceland, Israel, Mexico, Norway and Russia to name a few. In some cases, the experience was important in career choice. The Department of Education has been the main provider of money to enable Ireland to participate in the IMO. However, it is maths teachers throughout the country who provide the inspiration to enable students with talent in mathematics to emerge.

Gordon Lessells,
Department of Mathematics and Statistics,
University of Limerick.
e-mail: gordon.lessells@ul.ie

SOLUTIONS

Leaving Certificate Mathematics
Higher Level 2009—Paper 1

1. (a) Find the value of $\frac{x}{y}$ when $\frac{2x+3y}{x+6y}=\frac{4}{5}$.

$$\frac{2x+3y}{x+6y}=\frac{4}{5} \Rightarrow 5(2x+3y)=4(x+6y)$$
$$\Rightarrow 10x+15y=4x+24y$$
$$\Rightarrow 6x=9y$$
$$\Rightarrow \frac{6x}{y}=9$$
$$\Rightarrow \frac{x}{y}=\frac{9}{6}=\frac{3}{2}$$
$$\Rightarrow \underline{\underline{\frac{x}{y}=\frac{3}{2}}}$$

(b) Let $f(x) = x^2 - 7x + 12$.

(i) Show that if $f(x + 1) \neq 0$, then $\frac{f(x)}{f(x+1)}$ simplifies to $\frac{x-4}{x-2}$.

(ii) Find the range of values of x for which $\frac{f(x)}{f(x+1)}>3$.

(i) $f(x) = x^2 - 7x + 12$

$f(x + 1) = (x + 1)^2 - 7(x + 1) + 12$
$= x^2 + 2x + 1 - 7x - 7 + 12$
$= x^2 - 5x + 6$

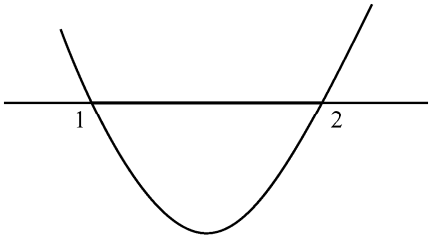
$$\frac{f(x)}{f(x+1)}=\frac{x^2-7x+12}{x^2-5x+6}=\frac{(x-3)(x-4)}{(x-2)(x-3)}=\underline{\underline{\frac{x-4}{x-2}}}$$

(ii) $\frac{f(x)}{f(x+1)}>3 \Rightarrow \frac{x-4}{x-2}>3$

Multiply both sides by $(x-2)^2$ .... $(x-2)^2 > 0$

$$\Rightarrow (x-4)(x-2)>3(x-2)^2$$
$$\Rightarrow x^2-6x+8>3(x^2-4x+4)$$
$$\Rightarrow x^2-6x+8>3x^2-12x+12$$
$$\Rightarrow 0>2x^2-6x+4$$
$$\Rightarrow 2x^2-6x+4>0$$
$$\Rightarrow x^2-3x+2>0$$

Now $x^2 - 3x + 2 = 0$
$\Rightarrow (x-1)(x-2) = 0$
$\Rightarrow x = 1 \quad \text{or} \quad x = 2$



From the graph,
$x^2 - 3x + 2 > 0$
$\Rightarrow x > 1$ and $x < 2$
i.e. $\underline{\underline{1 < x < 2}}$

(c) Given that $x - c + 1$ is a factor of $x^2 - 5x + 5cx - 6b^2$, express c in terms of b.

$$x - c + 1 \overline{)\, x^2 - 5x + 5cx - 6b^2 \;\; (x + (6c-6)}$$
$$\underline{x^2 - cx + x}$$
$$6cx - 6x - 6b^2$$
$$= \underline{(6c-6)x - 6b^2}$$
$$(6c-6)x - c(6c-6) + (6c-6)$$

Remainder: $-6b^2 + c(6c-6) - (6c-6)$

Since $(x-c+1)$ is a factor of $f(x) \Rightarrow$ Remainder $= 0$
$\Rightarrow -6b^2 + c(6c-6) - (6c-6) = 0$
$\Rightarrow -6b^2 + 6c^2 - 6c - 6c + 6 = 0$
$\Rightarrow 6c^2 - 12c + 6 = 6b^2$
$\Rightarrow c^2 - 2c + 1 = b^2$
$\Rightarrow (c-1)^2 = (b)^2$
$\Rightarrow c - 1 = \pm b$

(i) $c - 1 = b$
$\Rightarrow c = 1 + b$

(ii) $c - 1 = -b$
$\Rightarrow c = 1 - b$

$\therefore \underline{\underline{c = 1 \pm b}}$

OR Since $(x-c+1)$ is a factor of $f(x)$,
then $x - c + 1 = 0$
$\Rightarrow x = c - 1$
i.e. $(c-1)$ is a root of $f(x)$

$f(x) = x^2 - 5x + 5cx - 6b^2$

$f(c-1) = (c-1)^2 - 5(c-1) + 5c(c-1) - 6b^2 = 0$
$\Rightarrow c^2 - 2c + 1 - 5c + 5 + 5c^2 - 5c = 6b^2$
$\Rightarrow 6c^2 - 12c + 6 = 6b^2$
$\Rightarrow c^2 - 2c + 1 = b^2$
$\Rightarrow (c-1)^2 = (b)^2$
$\Rightarrow (c-1) = \pm(b)$

(i) $c - 1 = b$
$\Rightarrow c = 1 + b$

(ii) $c - 1 = -b$
$\Rightarrow c = 1 - b$

$\therefore \underline{\underline{c = 1 \pm b}}$

2. (a) Solve the simultaneous equations

$x - y + 8 = 0$

$x^2 + xy + 8 = 0.$

(i) $x - y + 8 = 0$ (i) $x = y - 8$

(ii) $x^2 + xy + 8 = 0$

(ii) $(y - 8)^2 + y(y - 8) + 8 = 0$

$y^2 - 16y + 64 + y^2 - 8y + 8 = 0$

$2y^2 - 24y + 72 = 0$

$y^2 - 12y + 36 = 0$

$(y - 6)(y - 6) = 0$

$\Rightarrow y = 6$

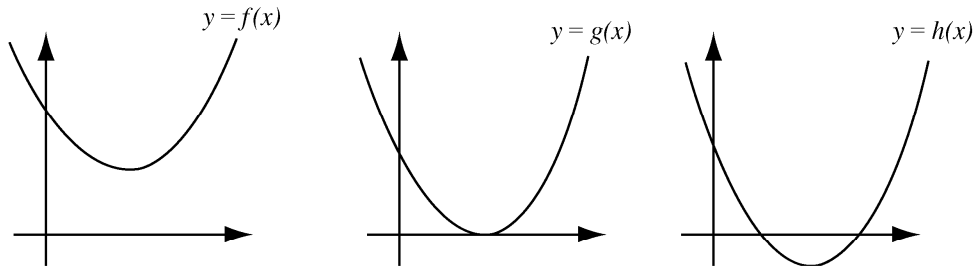
(i) $x = y - 8$

$\Rightarrow x = 6 - 8$

$\Rightarrow x = -2$

$\therefore$ the solution is $\underline{\underline{x = -2, y = 6}}$

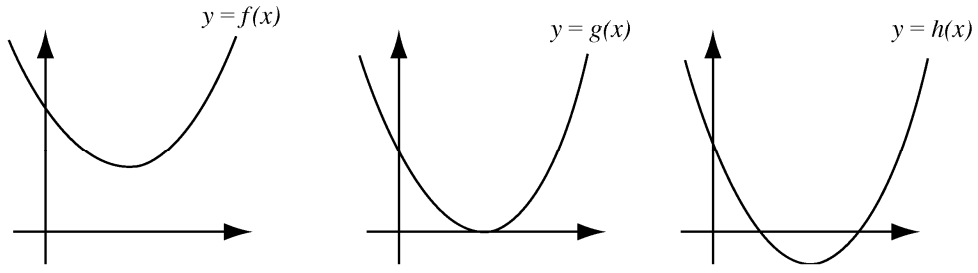
(b) (i) The graphs of three quadratic functions, f, g and h, are shown



In each case, state the nature of the roots of the function.

(ii) The equation $kx^2 + (1 - k)x + k = 0$ has equal real roots.
Find the possible values of k.

(i)



2 different Complex Roots | 2 equal Real Roots | 2 distinct Real Roots

(ii) $kx^2 + (1 - k)x + k = 0$

For real equal roots: $b^2 - 4ac = 0$

$\Rightarrow (1 - k)^2 - 4(k)(k) = 0$

$\Rightarrow 1 - 2k + k^2 - 4k^2 = 0$

$\Rightarrow -3k^2 - 2k + 1 = 0$

$\Rightarrow 3k^2 + 2k - 1 = 0$

$\Rightarrow \quad (3k-1)(k+1)=0$

$\Rightarrow \quad 3k-1=0 \quad \text{or} \quad k+1=0$

$3k-1=0 \quad \text{or} \quad k+1=0$

$\Rightarrow k=\frac{1}{3} \qquad \Rightarrow k=-1$

$\therefore k=-1 \quad \text{or} \quad k=\frac{1}{3}$

(c) (i) One of the roots of $px^2 + qx + r = 0$ is n times the other root.
Express r in terms of p, q and n.

(ii) One of the roots of $x^2 + qx + r = 0$ is five times the other.
If q and r are positive integers, determine the set of possible values of q.

(i) $px^2 + qx + r = 0$

Let α and $n\alpha$ be the roots.

$\text{Sum of Roots} = -\frac{q}{p} \Rightarrow \alpha + n\alpha = -\frac{q}{p}$

$\text{Product of Roots} = \frac{r}{p} \Rightarrow n\alpha^2 = \frac{r}{p}$

(a) Now $\alpha + n\alpha = \frac{-q}{p}$

$\Rightarrow \alpha(1+n) = -\frac{q}{p} \Rightarrow \alpha = \frac{-q}{p(1+n)}$

(b) $n\alpha^2 = \frac{r}{p}$

$\Rightarrow n\left[\frac{-q}{p(1+n)}\right]^2 = \frac{r}{p}$

$\Rightarrow \quad \frac{nq^2}{p^2(1+n)^2} = \frac{r}{p}$

Multiply both sides by p.

$\Rightarrow \quad \frac{nq^2}{p(1+n)^2} = r$

$\Rightarrow r = \frac{nq^2}{p(1+n)^2}$

(ii) $r = \frac{nq^2}{p(1+n)^2}$

When $p = 1$ and $n = 5$

$r = \frac{5q^2}{36}$

For r to be a positive integer, q^2 must be divisible by 36.

$\Rightarrow q = \{6, 12, 18, ...\}$

OR (c) (ii) Equation: $x^2 + qx + r = 0$

$$x^2 - (-q)x + (r) = 0$$

Roots: $\alpha, 5\alpha \Rightarrow x^2 - x(\alpha + 5\alpha) + 5\alpha \cdot \alpha = 0$

$$\Rightarrow x^2 - (6\alpha)x + (5\alpha^2) = 0$$

Equating coefficients:

(i) $-q = 6\alpha \Rightarrow \alpha = -\dfrac{q}{6}$

(ii) $r = 5\alpha^2 = 5\left(-\dfrac{q}{6}\right)^2 = \dfrac{5q^2}{36}$

For r to be a positive integer, q^2 must be divisible by 36, and so q is divisible by 6

$\Rightarrow \underline{\underline{q = \{6, 12, 18, ...\}}}$

3. (a) $z_1 = a + bi$ and $z_2 = c + di$, where $i^2 = -1$.
Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, where $\overline{z}$ is the complex conjugate of z.

$z_1 = a + bi \Rightarrow \overline{z_1} = a - bi$

$z_2 = c + di \Rightarrow \overline{z_2} = c - di$

$\overline{z_1} + \overline{z_2} = (a + c) - (b + d)i$

$z_1 + z_2 = (a + c) + (b + d)i$

$\overline{z_1 + z_2} = (a + c) - (b + d)i$

$\Rightarrow \underline{\underline{\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}}}$

(b) Let $A = \dfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

(i) Express A^3 in the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{Z}$.

(ii) Hence, or otherwise, find A^{17}.

(i) $A^2 = \dfrac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \dfrac{1}{4}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$.

$\therefore A^3 = \dfrac{1}{8}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = \dfrac{1}{8}\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}}$.

(ii) $A^{17} = (A^3)^5 A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^5 \frac{1}{4}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{pmatrix}$

$= \underline{\underline{\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}}}.$

(c) (i) Use De Moivre's theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(ii) Hence, find $\int \sin^3\theta\, d\theta$.

(i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$.

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$.

$= \cos^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta$.

$\therefore \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$

$= \underline{\underline{3\sin\theta - 4\sin^3\theta}}$.

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$.

$\therefore \int \sin^3\theta\, d\theta = \frac{1}{4}\int (3\sin\theta - \sin 3\theta)\, d\theta = \underline{\underline{\frac{1}{4}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right] + C}}.$

4. (a) Three consecutive terms of an arithmetic series are $4x + 11$, $2x + 11$, and $3x + 17$. Find the value of x.

(a) $(2x + 11) - (4x + 11) = (3x + 17) - (2x + 11)$ … $[T_2 - T_1 = T_3 - T_2]$

$-2x = x + 6 \Rightarrow x = -2$.

Series is 3, 7, 11, ...

(b) (i) Show that $\frac{2}{r^2 - 1} = \frac{1}{r-1} - \frac{1}{r+1}$, where $r \neq \pm 1$.

(ii) Hence, find $\sum_{r=2}^{n} \frac{2}{r^2 - 1}$.

(iii) Hence, evaluate $\sum_{r=2}^{\infty} \frac{2}{r^2 - 1}$.

(i) $\frac{1}{r-1} - \frac{1}{r+1} = \frac{r+1-r+1}{(r-1)(r+1)} = \frac{2}{r^2 - 1}$.

OR Let $\frac{2}{r^2 - 1} = \frac{a}{r-1} - \frac{b}{r+1}$

$2 = q(r + 1) - b(r - 1)$

$(0)r + (2) = (a - b)r + (a + b)$

Equating Coefficients: (i) : $a - b = 0$

(ii) : $a + b = 2$

(i) : $a - b = 0$

(ii) : $\underline{a + b = 2}$

$2a = 2$

$a = 1$

(i) $a - b = 0 \Rightarrow a = b \Rightarrow a = b = 1$

$$\underline{\underline{\frac{2}{r^2 - 1} = \frac{1}{r-1} - \frac{1}{r+1}}}$$

(ii) $u_n = \dfrac{1}{n^2 - 1} = \dfrac{1}{n-1} - \dfrac{1}{n+1}$

$u_{n-1} = \dfrac{1}{n-2} - \dfrac{1}{n}$

$u_{n-2} = \dfrac{1}{n-3} - \dfrac{1}{n-1}$

.....

.....

$u_4 = \dfrac{1}{3} - \dfrac{1}{5}$

$u_3 = \dfrac{1}{2} - \dfrac{1}{4}$

$u_2 = \dfrac{1}{1} - \dfrac{1}{3}$

$S_n = 1 + \dfrac{1}{2} - \dfrac{1}{n} - \dfrac{1}{n-1}$

$$\underline{\underline{S_n = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}}}$$

(iii) $\displaystyle\sum_{r=2}^{\infty} \frac{2}{r^2 - 1} = \lim_{n \to \infty} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \underline{\underline{\frac{3}{2}}}.$

(c) A finite geometric sequence has first term a and common ratio r.
The sequence has $2m + 1$ terms, where $m \in \mathbb{N}$.

(i) Write down the last term, in terms of a, r, and m.

(ii) Write down the middle term, in terms of a, r, and m.

(iii) Show that the product of all of the terms of the sequence is equal to the middle term raised to the power of the number of terms.

(i) $u_1 = a$; ratio $= r$; number of terms, $n = (2m + 1)$

$u_n = ar^{n-1}$

$u_{2m+1} = ar^{(2m+1)-1}$

$= \underline{\underline{ar^{2m}}}$

(ii) $(2m + 1)$ terms $\Rightarrow (m + 1)$ is the middle term

u_{m+1} = middle term $= ar^{(m+1)-1} = \underline{\underline{ar^m}}$

(iii) Product of terms $= a \times ar \times ar^2 \times .... \times ar^{2m}$

$= a^{2m+1} \times r^{0 + 1 + 2 + 3 + \ldots + 2m}$

$[0 + 1 + 2 + \ldots + 2m]$ is an A.P. with $(2m + 1)$ terms

Here $a = 1$, $d = 1$ and $n = 2m + 1$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{2m+1}{2}[0 + 2m(1)] = m(2m+1)$$

$$\therefore u_1 \cdot u_2 \cdot u_3 \ .... \ u_{2m+1} = a^{2m+1} \cdot r^{m(2m+1)}$$

$$= \underline{\underline{(ar^m)^{2m+1}}}$$

5. (a) Solve for x: $x - 2 = \sqrt{3x - 2}$.

$x - 2 = \sqrt{3x-2} \Rightarrow (x-2)^2 = 3x - 2.$

$x^2 - 4x + 4 = 3x - 2 \Rightarrow x^2 - 7x + 6 = 0.$

$(x-6)(x-1) = 0 \Rightarrow x = 6$ and $x = 1.$

Test: $x = 1$ LHS: $(x-2) = (1-2) = -1$

RHS: $\sqrt{3x-2} = \sqrt{1} = 1$

$x \neq 1$

$x = 6$ LHS: $x - 2 = 6 - 2 = 4$

RHS: $\sqrt{3x-2} = \sqrt{16} = 4$

Solution: $\underline{\underline{x = 6}}$

(b) Prove by induction that, for all positive integers n, 5 is a factor of $n^5 - n$.

Let $P(n)$ be the proposition that 5 is a factor of $n^5 - n$.

Test $P(1)$: $1 - 1 = 0$, which is divisible by 5.

Assume $P(k)$: $k^5 - k$ is divisible by 5.

Try to deduce $P(k + 1)$: that $(k + 1)^5 - (k + 1)$ is divisible by 5.

$(k + 1)^5 - (k + 1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1.$

$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$

which is divisible by 5 … (we assume $k^5 - k$ is divisible by 5)

So $P(k + 1)$ is true whenever $P(k)$ is true

Since $P(1)$ is true, then by induction $P(n)$ is true for all positive integers.

OR To prove: $(n^5 - n) \div 5$

$n = 1 : 1^5 - 1 = 0$, which is divisible by 5

$\Rightarrow$ true for $n = 1$

Assume true for $n = k \Rightarrow (k^5 - k) \div 5$

To prove: $\left[(k+1)^5 - (k+1)\right] \div 5$

$$p(k+1) - p(k)$$
$$= \left[(k+1)^5 - (k+1)\right] - \left[k^5 - k\right]$$

$= [k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1] - k^5 + k$

$= 5k^4 + 10k^3 + 10k^2 + 5k$

$= 5(k^4 + 2k^3 + 2k^2 + k)$ which is divisible by 5

$\Rightarrow$ formula true for $n = k + 1$

So $p(k + 1)$ true whenever $p(k)$ true

Since $p(1)$ true, then by induction $p(n)$ true for all positive integers

$[n \in N, n \geq 1]$

(c) Solve the simultaneous equations

$\log_3 x + \log_3 y = 2$

$\log_3(2y - 3) - 2\log_9 x = 1.$

$$\log_3 x + \log_3 y = 2$$
$$\log_3(xy) = 2$$
$$xy = 9$$
$$x = \frac{9}{y}$$

$$\log_3(2y-3) - 2\log_9 x = 1$$
$$\Rightarrow \log_3(2y-3) - 2\frac{\log_3 x}{\log_3 9} = 1$$
$$\Rightarrow \log_3(2y-3) - \cancel{2}\frac{\log_3 x}{\cancel{2}} = 1$$
$$\log_3\left(\frac{2y-3}{x}\right) = 1$$
$$\frac{2y-3}{x} = 3$$

$$(2y-3)\frac{y}{9} = 3$$
$$2y^2 - 3y - 27 = 0$$
$$(2y-9)(y+3) = 0$$

$y > 0 \Rightarrow y \neq -3$, so $y = \frac{9}{2}$, giving $x = 2$.

$$\therefore x = 2,\ y = \frac{9}{2}$$

6. (a) Differentiate $\sin(3x^2 - x)$ with respect to x.

Let $f(x) = \sin(3x^2 - x)$

$\Rightarrow f'(x) = \cos(3x^2 - x)(6x - 1)$

$= \underline{\underline{(6x-1)\cos(3x^2 - x)}}$

(b) (i) Differentiate $\sqrt{x}$ with respect to x, from first principles.

(ii) An object moves in a straight line such that its distance from a fixed point is given by $s = \sqrt{t^2 + 1}$, where s is in metres and t is in seconds.
Find the speed of the object when t = 5 seconds.

(i) $f(x) = \sqrt{x} \Rightarrow f(x) = \sqrt{x+h}$

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x} = \left(\sqrt{x+h} - \sqrt{x}\right) \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{x+h-x}{\sqrt{x+h} + \sqrt{x}} = \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \underline{\underline{\frac{1}{2\sqrt{x}}}}$$

(ii) $s = (t^2 + 1)^{\frac{1}{2}} \Rightarrow \frac{ds}{dt} = \frac{1}{2}(t^2 + 1)^{-\frac{1}{2}} \cdot 2t = \frac{t}{\sqrt{t^2 + 1}}.$

$\therefore$ At $t = 5$, $\frac{ds}{dt} = \frac{5}{\sqrt{26}}$ metres per second.

$\therefore$ Speed $= \underline{\underline{\frac{5}{\sqrt{26}}}}$ metres per second.

(c) The equation of a curve is $y = \frac{2}{x-3}$.

(i) Write down the equations of the asymptotes and hence sketch the curve.

(ii) Prove that no two tangents to the curve are perpendicular to each other.

(i) $y = \frac{2}{x-3}$

Vertical asymptote is $x - 3 = 0$

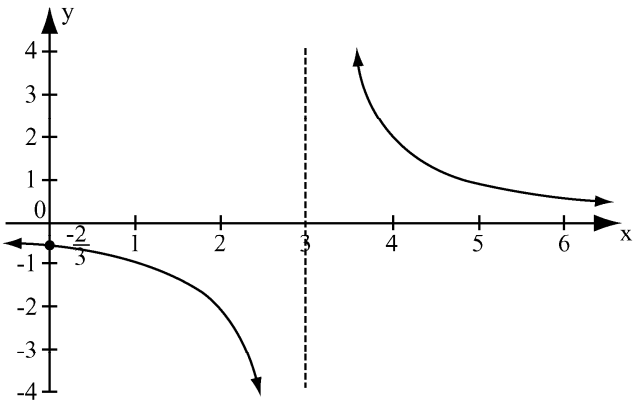
i.e. $\underline{\underline{x = 3}}$

Horizontal asymptote : $\lim_{x\to\infty} y = 0$

$$\lim_{x\to\infty} \frac{2}{x-3} = 0$$

$\Rightarrow \underline{\underline{y = 0}}$ is the horizontal asymptote.

Curve:



(ii) $y = \frac{2}{x-3} = 2(x-3)^{-1} \Rightarrow \frac{dy}{dx} = -2(x-3)^{-2} = \frac{-2}{(x-3)^2}$.

$\therefore$ Slope of tangent at (x, y) is $m = \frac{-2}{(x-3)^2}$.

But m will be negative for all values of $x \Rightarrow m_1 \cdot m_2 \neq -1$

$\therefore$ No two tangents are perpendicular to each other.

OR $y = 2(x-3)^{-1}$

$$m = \frac{dy}{dx} = \frac{-2}{(x-3)^2}$$

Let tangents at $x = a$ and $x = b$ be perpendicular.

At $x = a : m_1 = \frac{-2}{(a-3)^2}$

At $x = b : m_2 = \frac{-2}{(b-3)^2}$

$$(m_1)\cdot(m_2) = \frac{-2}{(a-3)^2} \cdot \frac{-2}{(b-3)^2} = \frac{4}{(a-3)^2 \cdot (b-3)^2} \neq -1$$

$\Rightarrow$ Tangents cannot be perpendicular ever.

7. (a) The equation of a curve is $x^2 - y^2 = 25$. Find $\frac{dy}{dx}$ in terms of x and y.

$$x^2 - y^2 = 25$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{2y} = \underline{\underline{\frac{x}{y}}}$$

OR $x^2 - y^2 = 25$

$$y^2 = x^2 - 25$$

$$y = \pm\sqrt{x^2 - 25}$$

$$y = \pm(x^2 - 25)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \pm\left[\frac{1}{2}(x^2 - 25)^{-\frac{1}{2}} \cdot 2x\right]$$

$$= \pm\left[\frac{x}{\sqrt{x^2 - 25}}\right] \ldots\ldots \left[x^2 - 25 = y^2\right]$$

$$= \frac{x}{\underline{\underline{y}}}$$

(b) A curve is defined by the parametric equations $x = \frac{3t}{t^2 - 2}$ and $y = \frac{6}{t^2 - 2}$, where $t \neq \pm\sqrt{2}$.

(i) Find $\frac{dy}{dx}$ in terms of t.

(ii) Find the equation of the tangent to the curve at the point given by $t = 2$.

(i) $x = \frac{3t}{t^2 - 2} \Rightarrow \frac{dy}{dt} = \frac{3(t^2 - 2) - 3t \cdot 2t}{(t^2 - 2)^2} = \frac{-3t^2 - 6}{(t^2 - 2)^2}.$

$$y = \frac{6}{t^2 - 2} = 6(t^2 - 2)^{-1} \Rightarrow \frac{dy}{dt} = -6(t^2 - 2)^{-2} \cdot 2t = \frac{-12t}{(t^2 - 2)^2}.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-12t}{(t^2 - 2)^2} \cdot \frac{(t^2 - 2)^2}{-3t^2 - 6} = \frac{12t}{3t^2 + 6} = \underline{\underline{\frac{4t}{t^2 + 2}}}.$$

(ii) Slope $= m = \frac{dy}{dx} = \frac{4t}{t^2 + 2}$

At $t = 2$: Slope $= m = \frac{4(2)}{2^2 + 2} = \frac{8}{6} = \frac{4}{3}$

At $t = 2$: $x = \frac{3t}{t^2 - 2} = \frac{6}{4 - 2} = 3$

At $t = 2$: $y = \frac{6}{t^2 - 2} = \frac{6}{4 - 2} = 3$

$\Rightarrow$ point, $p = (3,3)$

Equation of tangent at $p(3, 3)$ with slope $m = \frac{4}{3}$

$y - y_1 = m(x - x_1)$

$$y - 3 = \frac{4}{3}(x-3)$$

$$3y - 9 = 4x - 12$$

$$\Rightarrow \underline{\underline{4x - 3y - 3 = 0}} \text{ is the required equation.}$$

(c) The function $f(x) = x^3 - 3x^2 + 3x - 4$ has only one real root.

(i) Show that the root lies between 2 and 3.

Anne and Barry are each using the Newton-Raphson method to approximate the root. Anne is starting with 2 as a first approximation and Barry is starting with 3.

(ii) Show that Anne's starting approximation is closer to the root than Barry's. (That is, show that the root is less than 2.5.)

(iii) Show, however, that Barry's next approximation is closer to the root than Anne's.

(i) $f(x) = x^3 - 3x^2 + 3x - 4$.

$f(2) = 8 - 12 + 6 - 4 = -2 < 0$.

$f(3) = 27 - 27 + 9 - 4 = 5 > 0$.

$\therefore$ <u>root lies between 2 and 3.</u>

(ii) $f(2.5) = (2.5)^3 - 3(2.5)^2 + 3(2.5) - 4$

$= 15.625 - 18.75 + 7.5 - 4$

$= 0.375$

$f(2) < 0$ and $f(2.5) > 0$.

$\therefore$ root is between 2 and 2.5.

So the root is closer to 2 than to 3.

Therefore, Anne's approximation was closer.

(iii) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $f(x) = x^3 - 3x^2 + 3x - 4$ and

$$f'(x) = 3x^2 - 6x + 3$$

Anne: $f(2) = -2$ and $f'(2) = 3$. $x_2 = 2 - \dfrac{f(2)}{f'(2)} = 2 - \dfrac{-2}{3} = 2\dfrac{2}{3} = 2.666$

Barry: $f(3) = 5$ and $f'(3) = 12$. $x_2 = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{5}{12} = 2\dfrac{7}{12} = 2.583$

Both of these are above the root, so the lower one is closer (i.e. Barry's).

8. (a) Find $\displaystyle\int\left(6x + 3 + \frac{1}{x^2}\right)dx$.

$$\int\left(6x + 3 + \frac{1}{x^2}\right)dx$$

$$= \int(6x + 3 + x^{-2})dx$$

$$= \frac{6x^2}{2} + 3x + \frac{x^{-1}}{-1} + c$$

$$= \underline{\underline{3x^2 + 3x - \frac{1}{x} + c}}$$

(b) Evaluate (i) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x \, dx$ (ii) $\int_{\ln 3}^{\ln 8} e^x \sqrt{1+e^x}\, dx.$

(i) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x \, dx = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2x - \cos 4x)dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{4}\sin\pi\right) - \left(\frac{1}{2}\sin\left(\frac{-\pi}{2}\right) - \frac{1}{4}\sin(-\pi)\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - 0\right) - \left(-\frac{1}{2} - 0\right)\right]$$

$$= \frac{1}{2}[1] = \underline{\underline{\frac{1}{2}}}$$

(ii) $\int_{\ln 3}^{\ln 8} e^x \sqrt{1+e^x}\, dx$

$$= \int \sqrt{1+e^x} \cdot (e^x dx)$$

$$= \int u^{\frac{1}{2}} du$$

$$= \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^9$$

$$= \frac{2}{3}\left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}[27-8] = \frac{2}{3}[19] = \underline{\underline{\frac{38}{3}}}$$

Let $u = 1+e^x$

$\Rightarrow du = e^x dx$

New limits

$x = \ln 8 : u = 1+e^{\ln 8}$

$= 1+8$

$= 9$

$x = \ln 3 : u = 1+e^{\ln 3}$

$= 1+3$

$= 4$

Note: To show that $e^{\ln a} = a$:

Let $p = e^{\ln a}$

$\ln(p) = \ln(e^{\ln a})$

$\Rightarrow \ln(p) = (\ln a)\ln e$

Since $\ln e = 1 \Rightarrow \ln(p) = \ln(a)$

$\Rightarrow p = a$

$\Rightarrow e^{\ln a} = a$

OR Using old Limits

$$\int_{\ln 3}^{\ln 8} e^x\sqrt{1+e^x}\,dx$$

$$=\frac{2}{3}u^{\frac{3}{2}}$$

$$=\frac{2}{3}(1+e^x)^{\frac{3}{2}}\Big]_{\ln 3}^{\ln 8}$$

$$=\frac{2}{3}\left[(1+e^{\ln 8})^{\frac{3}{2}}-(1+e^{\ln 3})^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[(9)^{\frac{3}{2}}-(4)^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}[27-8]=\frac{2}{3}(19)=\frac{38}{3}$$

$$=\underline{\underline{\frac{38}{3}}}$$

OR $\int_{\ln 3}^{\ln 8} e^x\sqrt{1+e^x}\,dx$ Let $u=e^x$

$$=\int\sqrt{1+e^x}\cdot e^x\,dx \qquad \frac{du}{dx}=e^x$$

$$=\int(1+u)^{\frac{1}{2}}du \qquad du=e^x dx$$

$$=\frac{(1+u)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$=\frac{2}{3}(1+u)^{\frac{3}{2}}$$

$$=\frac{2}{3}(1+e^x)^{\frac{3}{2}}\Big]_{\ln 3}^{\ln 8}$$

$$=\frac{2}{3}\left[(1+e^{\ln 8})^{\frac{3}{2}}-(1+e^{\ln 3})^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[(1+8)^{\frac{3}{2}}-(1+3)^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[(9)^{\frac{3}{2}}-(4)^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}[27-8]=\frac{2}{3}(19)=\underline{\underline{\frac{38}{3}}}$$

(c) Use integration methods to establish the standard formula for the volume of a cone.

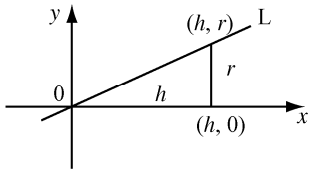
We rotate the line L about the x-axis from $x = 0$ to $x = h$.

Slope of L: $m=\frac{r-0}{h-0}=\frac{r}{h}$

Equation of L: $y = mx$

$$y=\frac{r}{h}x$$

$$\Rightarrow y=\frac{rx}{h}$$



$$\text{Volume}=\int_0^h \pi y^2\,dx$$

$$=\pi\int_0^h\left(\frac{rx}{h}\right)^2 dx$$

$$=\frac{\pi r^2}{h^2}\int x^2\,dx$$

$$=\frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$=\frac{\pi r^2}{3h^2}[h^3-0]=\frac{1}{3}\pi r^2 h$$

$$\therefore \text{Volume}=\underline{\underline{\frac{1}{3}\pi r^2 h}}$$

Leaving Certificate Mathematics
Higher Level 2009—Paper 2

1. (a) Show that, for all values of $t \in \mathbb{R}$, the point $\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ lies on the circle $x^2 + y^2 = 1$.

Substituting $\frac{2t}{1+t^2}$ for x and $\frac{1-t^2}{1+t^2}$ for y, we get

$$\left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2$$

$$= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2}$$

$$= \frac{t^4 + 2t^2 + 1}{(1+t^2)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1$$

Therefore $\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ lies on the circle $x^2 + y^2 = 1$ as the points satisfy the equation.

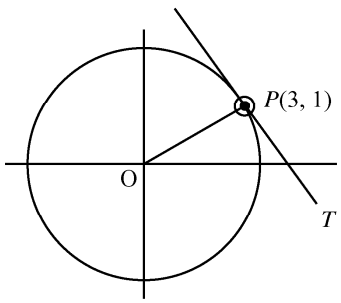
(b) (i) Find the equation of the tangent to the circle $x^2 + y^2 = 10$ at the point (3, 1).

(ii) Find the values of $k \in \mathbb{R}$ for which the line $x - y + k = 0$ is a tangent to the circle $(x - 3)^2 + (y + 4)^2 = 50$.

(i) Equation of tangent at (x_1, y_1) is

$$xx_1 + yy_1 = a^2$$

$$\Rightarrow \quad \underline{\underline{3x + y = 10 \text{ is the equation}}}$$



OR (i) Slope of OP $= \frac{1-0}{3-0}$

$$= \frac{1}{3}$$

$\Rightarrow$ slope of $T = -3$

Equation of T: $y - y_1 = m(x - x_1)$

$$y - 1 = -3(x - 3)$$

$$\Rightarrow \quad y - 1 = -3x + 9$$

$$\Rightarrow \underline{\underline{3x + y = 10}} \text{ is the equation of the tangent}$$

(ii) $(x-3)^2+(y+4)^2=50$

$\Rightarrow$ centre $=(3,-4)$ and radius $=\sqrt{50}$

If $x-y+k=0$ is a tangent to the circle, then the perpendicular distance from the centre $(3,-4)$ to the line is equal to the radius length.

$$\perp \text{ distance} = \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

$$= \left|\frac{3(1)+(-1)(-4)+k}{\sqrt{1^2+1^2}}\right| = \frac{7+k}{\sqrt{2}}$$

$$\Rightarrow \left|\frac{7+k}{\sqrt{2}}\right| = \sqrt{50}$$

$$\Rightarrow |7+k| = \sqrt{2}\cdot\sqrt{2}\cdot 5$$

$$= 10$$

$\Rightarrow 7+k=10$ or $-(7+k)=10$

$\Rightarrow k=3$ or $7+k=-10$

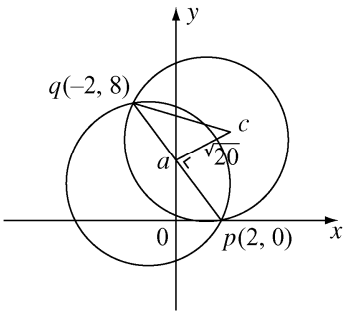
$\Rightarrow k=-17$

$\therefore k=3$ or $k=-17$

(c) Two circles intersect at $p(2, 0)$ and $q(-2, 8)$. The distance from the centre of each circle to the common chord $[pq]$ is $\sqrt{20}$.
Find the equations of the two circles.

$|ac|$ is the perpendicular distance from the centre of the circle to the common chord.

$\Rightarrow |ac| = \sqrt{20}$



$$|pq| = \sqrt{(2+2)^2+(0-8)^2}$$

$$= \sqrt{16+64}$$

$$= \sqrt{80}$$

$$= \sqrt{16}\ \sqrt{5} = 4\sqrt{5}$$

Since ca bisects $|pq| \Rightarrow |aq| = \frac{1}{2}\left[4\sqrt{5}\right] = 2\sqrt{5}$

Δacq is right-angled

$$\Rightarrow |cq|^2 = |aq|^2 + |ac|^2$$

$$= \left[2\sqrt{5}\right]^2 + \left[\sqrt{20}\right]^2$$

$$= 20+20 = 40$$

$$\Rightarrow |cq| = \sqrt{40} = \sqrt{4}\cdot\sqrt{10} = 2\sqrt{10}$$

$\Rightarrow$ the radius of the circle $= 2\sqrt{10}$

Let the equation of the circles be

$x^2 + y^2 + 2gx + 2fy + c = 0$

$(2, 0) \in$ circle $\Rightarrow 4 + 0 + 4g + 0 + c = 0$

$\Rightarrow 4g + c = -4$

$\Rightarrow c = -4 - 4g \ldots (1)$

$(-2, 8) \in$ circle $\Rightarrow 4 + 64 - 4g + 16f + c = 0$

$\Rightarrow -4g + 16f + c = -68 \ldots (2)$

$\Rightarrow -4g + 16f - 4 - 4g = -68$

$\Rightarrow -8g + 16f = -64$

$\Rightarrow -g + 2f = -8$

$\Rightarrow g = 2f + 8 \ldots (3)$

Radius of circle $= \sqrt{g^2 + f^2 - c} = 2\sqrt{10}$

$\Rightarrow \sqrt{(2f+8)^2 + f^2 - (-8f - 36)} = 2\sqrt{10}$

$\Rightarrow (2f + 8)^2 + f^2 + 8f + 36 = 4(10)$

$\Rightarrow 4f^2 + 32f + 64 + f^2 + 8f + 36 - 40 = 0$

$\Rightarrow 5f^2 + 40f + 60 = 0$

$\Rightarrow f^2 + 8f + 12 = 0$

$\Rightarrow (f + 2)(f + 6) = 0$

$\Rightarrow f = -2$ or $f = -6$

$\Rightarrow g = 2f + 8$ $\quad \Rightarrow g = 2f + 8$

$= -4 + 8$ $\quad = -12 + 8$

$= 4$ $\quad = -4$

$c = -4 - 4g$

$= -4 - 4(2f + 8)$

$= -4 - 8f - 32$

$c = -8f - 36$

$c = -8f - 36$

For $f = -2$, $c = -8(-2) - 36 = 16 - 36 = -20$

For $f = -6$, $c = -8(-6) - 36 = 48 - 36 = 12$

$\therefore$ the values of g, f and c are

(i) $f = -2$, $g = 4$ and $c = -20$

(ii) $f = -6$, $g = -4$ and $c = 12$

The two equations of the circles are

$\underline{\underline{x^2 + y^2 + 8x - 4y - 20 = 0}}$

and $\underline{\underline{x^2 + y^2 - 8x - 12y + 12 = 0}}$

2. (a) If $\vec{a} = 2\vec{i} + \vec{j}$, $\vec{b} = -\vec{i} + 5\vec{j}$, find the unit vector in the direction of $\overrightarrow{ab}$.

$\overrightarrow{ab} = \vec{b} - \vec{a}$

$= -\vec{i} + 5\vec{j} - (2\vec{i} + \vec{j})$

$= -\vec{i} + 5\vec{j} - 2\vec{i} - \vec{j}$

$= -3\vec{i} + 4\vec{j}$

The unit vector in the direction of $\overrightarrow{ab}$ is $\dfrac{\overrightarrow{ab}}{|ab|}$

$\therefore$ the unit vector is $\dfrac{-3\vec{i}+4\vec{j}}{\left|-3\vec{i}+4\vec{j}\right|}$

$$= \frac{-3\vec{i}+4\vec{j}}{\sqrt{9+16}} = \frac{-3\vec{i}+4\vec{j}}{5}$$

$$= \underline{\underline{-\frac{3}{5}\vec{i}+\frac{4}{5}\vec{j}}}$$

(b) In the triangle abc, p is a point on the side $[bc]$.
The point q lies outside the triangle such that $\overrightarrow{pq} = \overrightarrow{pb} + \overrightarrow{pc} - \overrightarrow{pa}$.

(i) Express $\vec{q}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

(ii) Hence show that $abqc$ is a parallelogram.

(i) $\overrightarrow{pq} = \overrightarrow{pb} + \overrightarrow{pc} - \overrightarrow{pa}$

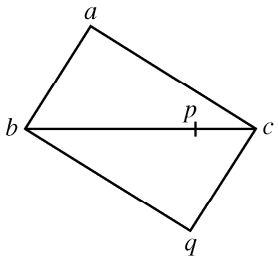
$$\Rightarrow \vec{q} - \vec{p} = \vec{b} - \vec{p} + \vec{c} - \vec{p} - \left(\vec{a} - \vec{p}\right)$$

$$= \vec{b} - \vec{p} + \vec{c} - \vec{p} - \vec{a} + \vec{p}$$

$$\Rightarrow \vec{q} - \vec{p} = \vec{b} + \vec{c} - \vec{a} + \vec{p}$$

$$\Rightarrow \quad \underline{\underline{\vec{q} = \vec{b} + \vec{c} - \vec{a}}}$$

(ii) To show that $abqc$ is a parallelogram, we must show that $[ac]$ is equal and parallel to $[bq]$.



$$\overrightarrow{ac} = \vec{c} - \vec{a}$$

$$\overrightarrow{bq} = \vec{q} - \vec{b}$$

$$= \vec{b} + \vec{c} - \vec{a} - \vec{b}$$

$$= \vec{c} - \vec{a}$$

Since $\overrightarrow{ac} = \overrightarrow{bq}$, it implies that $[ac]$ is equal and parallel to $[bq]$.

$\therefore$ $abqc$ is a parallelogram

(c) (i) $\vec{p} = 12\vec{i} + 5\vec{j}$ and $\vec{q} = 3\vec{i} + 4\vec{j}$.
Find the value of the scalar k such that
$k\left|\vec{p}^{\perp} - \vec{q}\right| = \left|\vec{p}^{\perp}\right| - \left|\vec{q}\right|$.

(ii) Prove that for all vectors $\vec{r}$ and $\vec{s}$
$\left(\vec{r} - \vec{s}\right)^{\perp} = \vec{r}^{\perp} - \vec{s}^{\perp}$.

(i) $\vec{p} = 12\vec{i} + 5\vec{j} \Rightarrow \vec{p}^{\perp} = -5\vec{i} + 12\vec{j}$

$$\vec{p}^{\perp} - \vec{q} = -5\vec{i} + 12\vec{j} - (3\vec{i} + 4\vec{j})$$
$$= -5\vec{i} + 12\vec{j} - 3\vec{i} - 4\vec{j}$$
$$= -8\vec{i} + 8\vec{j}$$
$$\Rightarrow \left|\vec{p}^{\perp} - \vec{q}\right| = \left|-8\vec{i} + 8\vec{j}\right|$$
$$= \sqrt{64+64} = \sqrt{128} = \sqrt{64}\cdot\sqrt{2} = 8\sqrt{2}$$
$$k\left|\vec{p}^{\perp} - \vec{q}\right| = k[8\sqrt{2}] = 8k\sqrt{2}$$

Now $\left|\vec{p}^{\perp}\right| = \left|-5\vec{i} + 12\vec{j}\right| = \sqrt{25+144} = \sqrt{169} = 13$

$$\left|\vec{q}\right| = \left|3\vec{i} + 4\vec{j}\right| = \sqrt{9+16} = \sqrt{25} = 5$$
$$k\left|\vec{p}^{\perp} - \vec{q}\right| = \left|\vec{p}^{\perp}\right| - \left|\vec{q}\right|$$
$$\Rightarrow 8k\sqrt{2} = 13 - 5$$
$$\Rightarrow 8k\sqrt{2} = 8$$
$$\Rightarrow \quad k = \frac{8}{8\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\therefore \quad \underline{\underline{k = \frac{\sqrt{2}}{2}}}$$

(ii) To prove $(\vec{r} - \vec{s})^{\perp} = \vec{r}^{\perp} - \vec{s}^{\perp}$

Let $\vec{r} = a\vec{i} + b\vec{j}$ and $\vec{s} = c\vec{i} + d\vec{j}$

$$\vec{r} - \vec{s} = a\vec{i} + b\vec{j} - (c\vec{i} + d\vec{j})$$
$$= a\vec{i} + b\vec{j} - c\vec{i} - d\vec{j}$$
$$= (a-c)\vec{i} + (b-d)\vec{j}$$
$$\Rightarrow (\vec{r} - \vec{s})^{\perp} = -(b-d)\vec{i} + (a-c)\vec{j} \quad \ldots \textcircled{1}$$
$$\vec{r}^{\perp} = -b\vec{i} + a\vec{j} \quad \text{and} \quad \vec{s}^{\perp} = -d\vec{i} + c\vec{j}$$
$$\Rightarrow \vec{r}^{\perp} - \vec{s}^{\perp} = -b\vec{i} + a\vec{j} - (-d\vec{i} + c\vec{j})$$
$$= -b\vec{i} + a\vec{j} + d\vec{i} - c\vec{j}$$
$$= -b\vec{i} + d\vec{i} + a\vec{j} - c\vec{j}$$
$$= -(b-d)\vec{i} + (a-c)\vec{j} \quad \ldots \textcircled{2}$$

Since $\textcircled{1} = \textcircled{2} \Rightarrow \underline{\underline{(\vec{r} - \vec{s})^{\perp} = \vec{r}^{\perp} - \vec{s}^{\perp} = -(b-d)\vec{i} + (a-c)\vec{j}}}$

3. (a) Find the equation of the line through the point (1, 0) that also passes through the point of intersection of the lines $2x - y + 6 = 0$ and $10x + 3y - 2 = 0$.

Required equation is $2x - y + 6 + \bullet(10x + 3y - 2) = 0$

$(1, 0) \in$ this line

$$\Rightarrow \quad 2(1) - 0 + 6 + \bullet(10 + 0 - 2) = 0$$
$$\Rightarrow \quad 2 + 6 + 8\bullet = 0$$
$$\Rightarrow \quad 8\bullet = -8$$
$$\Rightarrow \quad \bullet = -1$$

The equation is $2x - y + 6 - (10x + 3y - 2) = 0$

$$\Rightarrow \quad 2x - y + 6 - 10x - 3y + 2 = 0$$
$$\Rightarrow \quad -8x - 4y + 8 = 0$$
$$\Rightarrow \quad -2x - y + 2 = 0$$
$$\Rightarrow \quad \underline{\underline{2x + y - 2 = 0}} \text{ is the required equation}$$

(b) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by

$$\tan \bullet = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Find the equations of the two lines that pass through the point (6, 1) and make an angle of 45° with the line $x + 2y = 0$.

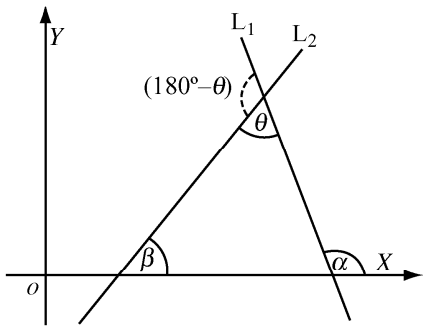
(i) The diagram on the right shows the lines

$L_1 : y = m_1 x + c_1$

$L_2 : y = m_2 x + c_2$

The slope of $L_1 = m_1$.

The slope of $L_2 = m_2$.



Proof:

Since the slope of a line is the tangent of the angle the line makes with the positive sense of the x-axis, then

$m_1 = \tan \bullet$ and $m_2 = \tan \bullet$.

From the diagram $\bullet = \bullet + \bullet$ ... exterior angle

$$\Rightarrow \quad \bullet = \bullet - \bullet$$
$$\tan \bullet = \tan(\bullet - \bullet)$$
$$= \frac{\tan \bullet - \tan \bullet}{1 + \tan \bullet \tan \bullet}$$
$$\Rightarrow \quad \underline{\underline{\tan \bullet = \frac{m_1 - m_2}{1 + m_1 m_2}}}$$

(ii) Let • be the angle between the given line and the required lines.

$$\tan \bullet = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan 45° = \pm \frac{-\frac{1}{2} - m_2}{1 + \left(-\frac{1}{2} m_2\right)} \quad \text{.... } m_1 \text{ is the slope of the given line } x + 2y = 0$$

$$= \pm \frac{-\frac{1}{2} - m_2}{1 - \frac{m_2}{2}} \qquad x + 2y = 0$$

$$\Rightarrow \quad 1 = \pm \frac{-1 - 2m_2}{2 - m_2} \qquad \Rightarrow y = -\frac{1}{2}x$$

$$\Rightarrow \quad \pm \left(\frac{-1 - 2m_2}{2 - m_2} \right) = 1 \qquad \Rightarrow m_1 = -\frac{1}{2}$$

$$\Rightarrow \quad \frac{-1 - 2m_2}{2 - m_2} = 1 \qquad \text{or} \qquad -\left(\frac{-1 - 2m_2}{2 - m_2} \right) = 1$$

$$\Rightarrow \quad -1 - 2m_2 = 2 - m_2 \qquad -1 - 2m_2 = -2 + m_2$$

$$\Rightarrow \quad -m_2 = 3 \qquad \Rightarrow \quad -3m_2 = -1$$

$$\Rightarrow \quad m_2 = -3 \qquad \Rightarrow \quad m_2 = \frac{1}{3}$$

The equations of the two required lines are

$$\text{point} = (6, 1) : \text{slope} = -3 \quad \text{or} \quad \text{point} = (6, 1) : \text{slope} = \frac{1}{3}$$

$$y - 1 = -3(x - 6) \qquad y - 1 = \frac{1}{3}(x - 6)$$

$$y - 1 = -3x + 18 \qquad 3y - 3 = x - 6$$

$$\Rightarrow 3x + y - 19 = 0 \qquad \Rightarrow x - 3y - 3 = 0$$

The equations are $3x + y - 19 = 0$ and $x - 3y - 3 = 0$

(c) f is the transformation $(x, y) \to (x', y')$, where $x' = -x + 2y$ and $y' = 2x - y$.

(i) L is the line $ax + by + c = 0$. Prove that $f(L)$ is a line.

(ii) The line $y = mx$ is its own image under f.
Find the two possible values of m.

(i) We first express x and y in terms of x' and y'.

$x' = -x + 2y$ ... ①

$y' = 2x - y$ .... ②

① × 2 : $2x' = -2x + 4y$

② : $y' = 2x - y$

$2x' + y' = 3y \quad \Rightarrow y = \dfrac{2x' + y'}{3}$

from ① : $2x' = -2x + 4\dfrac{(2x' + y')}{3}$

$\Rightarrow 6x' = -6x + 8x' + 4y'$
$\Rightarrow 6x = 2x' + 4y'$
$\Rightarrow \quad x = \dfrac{2x' + 4y'}{6} = \dfrac{x' + 2y'}{3}$

We find the equation of $f(L)$ by substituting these values for x and y into the equation $ax + by + c = 0$

This equation is : $a\left(\dfrac{x' + 2y'}{3}\right) + b\left(\dfrac{2x' + y'}{3}\right) + c = 0$

$\Rightarrow \quad a(x' + 2y') + b(2x' + y') + 3c = 0$
$\Rightarrow \quad ax' + 2ay' + 2bx' + by' + 3c = 0$
$\therefore f(L)$ is $(a + 2b)x' + (2a + b)y' + 3c = 0$

This is the equation of a line as it is in the form $ax + by + c = 0$.
We will now substitute x and y for x' and y' to show that every point in $f(L)$ has a corresponding point in L.

$x' = -x + 2y$ and $y' = 2x - y$

Substituting : $(a + 2b)(-x + 2y) + (2a + b)(2x - y) + 3c = 0$
$\Rightarrow \quad -ax + 2ay - 2bx + 4by + 4ax - 2ay + 2bx - by + 3c = 0$
$\Rightarrow 3ax + 3by + 3c = 0$
$\Rightarrow ax + by + c = 0$

Thus every point in $f(L)$ has a corresponding point in L
$\therefore f(L)$ is a line

(ii) Let M be $y = mx$

$f(M)$: $\dfrac{2x' + y'}{3} = m\left(\dfrac{x' + 2y'}{3}\right)$

$\Rightarrow 2x\bullet + y\bullet = mx\bullet + 2my\bullet$
$\Rightarrow 2x\bullet - mx\bullet + y\bullet - 2my\bullet = 0$
$\Rightarrow f(M) = x\bullet(2 - m) + y\bullet(1 - 2m) = 0$

If $f(M)$ is the same line as M, then the slopes are equal.

$\Rightarrow m = \dfrac{-(1 - 2m)}{2 - m}$
$\Rightarrow 2m - m^2 = -1 + 2m$
$\Rightarrow \quad -m^2 = -1$
$\Rightarrow \quad m^2 = 1 \Rightarrow m = \pm 1$
$\therefore$ the values of m are ± 1

4. (a) Show that $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$.

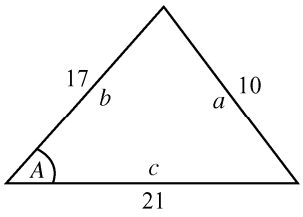
$(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2$
$= \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta$
$= 2\sin^2\theta + 2\cos^2\theta$
$= 2(\sin^2\theta + \cos^2\theta)$
$= 2(1)$
$= 2$

(b) The lengths of the sides of a triangle are 21, 17 and 10.
The smallest angle in the triangle is A.

(i) Show that $\cos A = \frac{15}{17}$.

(ii) Without evaluating A, find $\tan \frac{A}{2}$.

(i) Using the Cosine Rule :



$a^2 = b^2 + c^2 - 2bc \cos A$

$\Rightarrow 100 = 17^2 + 21^2 - 2(17)(21) \cos A$

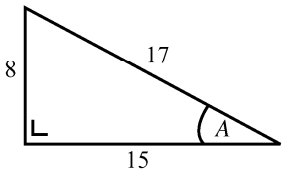
$\Rightarrow 100 = 289 + 441 - 714 \cos A$

$\Rightarrow 714 \cos A = 630$

$\Rightarrow \cos A = \frac{630}{714} = \frac{90}{102} = \frac{15}{17}$

$\therefore \cos A = \frac{15}{17}$

(ii) $\cos A = \frac{15}{17}$



$\Rightarrow \tan A = \frac{8}{15}$

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$\Rightarrow \tan A = \frac{2\tan \frac{A}{2}}{1-\tan^2 \frac{A}{2}}$

$\Rightarrow \frac{2\tan \frac{A}{2}}{1-\tan^2 \frac{A}{2}} = \frac{8}{15}$

Let $\tan \frac{A}{2} = t$

$\Rightarrow \frac{2t}{1-t^2} = \frac{8}{15}$

$\Rightarrow 8 - 8t^2 = 30t$

$\Rightarrow 8t^2 + 30t - 8 = 0$

$\Rightarrow 4t^2 + 15t - 4 = 0$

$\Rightarrow (4t-1)(t+4) = 0$

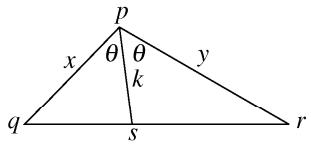
$\Rightarrow t = \frac{1}{4}$ or $t = -4$

$\Rightarrow t = \frac{1}{4}, \quad t \neq -4$ as t is acute

$\Rightarrow \tan \frac{A}{2} = \frac{1}{4}$

(c) The bisector of $\angle qpr$ meets $[qr]$ at s.

$|\angle qpr| = 2\bullet, |pq| = x,$
$|pr| = y$ and $|ps| = k.$



(i) Find the area of the triangle pqs in terms of x, k and θ.

(ii) Show that $k = \dfrac{2xy\cos\bullet}{x+y}$.

(i) Area $\Delta pqs = \dfrac{1}{2}(x)(k)\sin\bullet$

$= \underline{\underline{\dfrac{xk\sin\bullet}{2}}}$

(ii) Area of $\Delta psr = \dfrac{1}{2}(k)(y)\sin\bullet$

$= \dfrac{ky\sin\bullet}{2}$

Area of $\Delta pqr = \dfrac{1}{2}(x)(y)\sin 2\bullet$

$= \dfrac{1}{2}(x)(y)2\sin\bullet\cos\bullet$

$= xy\sin\bullet\cos\bullet$

Now $\Delta pqr = \Delta pqs + \Delta psr$

$\Rightarrow xy\sin\bullet\cos\bullet = \dfrac{xk\sin\bullet}{2} + \dfrac{ky\sin\bullet}{2}$

$\Rightarrow 2xy\sin\bullet\cos\bullet = xk\sin\bullet + ky\sin\bullet$

$\Rightarrow 2xy\cos\bullet = xk + ky$

$\Rightarrow 2xy\cos\bullet = k(x+y)$

$\Rightarrow \underline{\underline{k = \dfrac{2xy\cos\bullet}{x+y}}}$

5. (a) Find all the solutions of the equation $\cos^2 x - \cos x = 0$, where $0^\circ \le x \le 180^\circ$.

$\cos^2 x - \cos x = 0$

$\Rightarrow \cos x(\cos x - 1) = 0$

$\Rightarrow \cos x = 0 \quad \text{or} \quad \cos x - 1 = 0$

$\Rightarrow \cos x = 0 \quad \text{or} \quad \cos x = 1$

$\Rightarrow x = 90^\circ \quad \text{or} \quad x = 0^\circ$

$\therefore \underline{\underline{x = 0^\circ \quad \text{or} \quad 90^\circ}}$

(b) The function $f : x \to \sin^{-1} x$ is defined for $-1 \le x \le 1$.

(i) Copy and complete the table of values of f below.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$f(x)$			$-\frac{\pi}{6}$				

(ii) Draw the graph of $y = f(x)$ on graph paper, noting that $\frac{\sqrt{3}}{2} \approx 0.87$.

Scale the y-axis in terms of π.

(iii) State, with reason, whether each of the following statements is true.

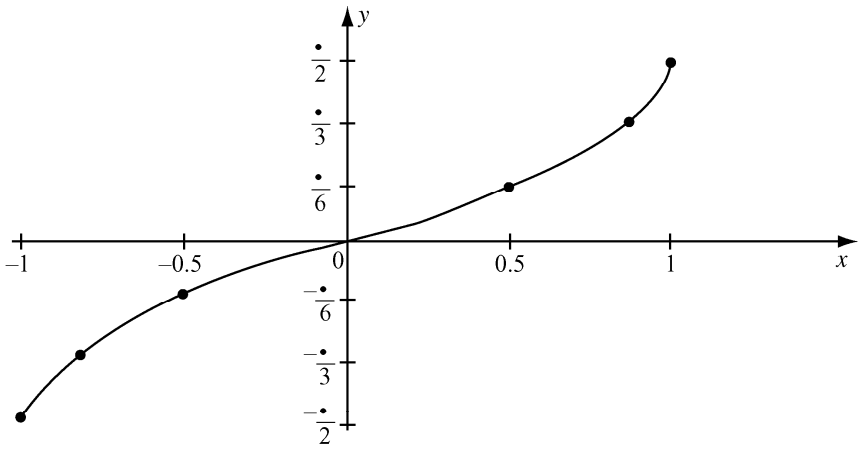
A : "If $\sin x_1 = \sin x_2$, then $x_1 = x_2$".

B : "If $\sin^{-1} x_1 = \sin^{-1} x_2$, then $x_1 = x_2$".

(i)

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$f(x)$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

(ii)



(iii) A : If $x_1 = 60°$ and $x_2 = 120°$, then

$\sin 60° = \frac{\sqrt{3}}{2}$ and $\sin 120° = \frac{\sqrt{3}}{2}$

Both are equal, but $x_1 \neq x_2$ ..... $(60° \neq 120°)$

$\Rightarrow$ the statement is false

B : True : $\sin^{-1} x_1$ and $\sin^{-1} x_2$ are both angles

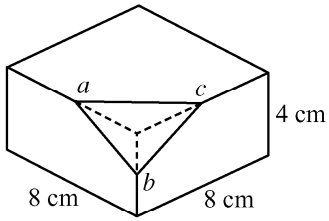
For these two angles to be equal, x_1 must be equal to x_2.

OR A horizontal line cannot cut the graph of $y = \sin^{-1} x$ more than once.

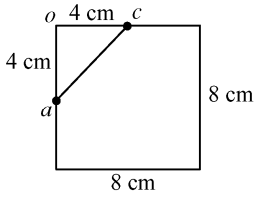
(c) A rectangular block of cheese measures 8 cm × 8 cm × 4 cm.

One corner is cut away from the block, in such a way that three of the edges are cut through their midpoints a, b and c.

Find the area of the triangular face abc created by the cut.

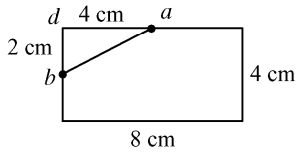


a and c are the midpoints of a square of side 8 cm

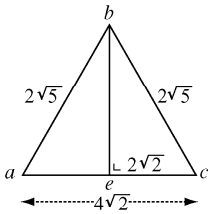


$$|ac|^2 = |oa|^2 + |oc|^2$$
$$= 4^2 + 4^2$$
$$= 32$$
$$\Rightarrow |ac| = \sqrt{32} = \sqrt{16}.\sqrt{2}$$
$$= 4\sqrt{2} \text{ cm}$$

a and b are the midpoints of a rectangle of sides 8 cm and 4 cm



$$|ab|^2 = |bd|^2 + |da|^2$$
$$= 4 + 16$$
$$\Rightarrow |ab| = \sqrt{20} = 2\sqrt{5}$$
$$|bc| = |ab| = 2\sqrt{5}$$



Δabc is isosceles, as shown

$$\text{Area of } \Delta abc = \frac{1}{2}|ac|.|be|$$
$$= \frac{1}{2}(4\sqrt{2})(\sqrt{12})$$
$$= \frac{1}{2}(4\sqrt{2})(2\sqrt{3})$$
$$= \frac{8\sqrt{6}}{2} = 4\sqrt{6}$$

$$|be|^2 = (2\sqrt{5})^2 - (2\sqrt{2})^2$$
$$= 20 - 8$$
$$= 12$$
$$\Rightarrow |be| = \sqrt{12}$$

$\therefore$ area of $\Delta abc = \underline{\underline{4\sqrt{6} \text{ cm}^2}}$

6. (a) A student taking a literature course has to read three novels from a list of ten novels.

(i) How many different selections of three novels are possible?

(ii) Two of the ten novels are by the same author. How many selections are possible if the student wishes to choose three novels by different authors?

(i) No. of selections $= \binom{10}{3}$

$$= \frac{10\times 9\times 8}{3\times 2\times 1} = \underline{\underline{120}}$$

(ii) Let A and B be the 2 books by the same author.
A, B xxxxxxxx
The selections will consist of
① A and 2 books from 8 … (B cannot be included)
or ② B and 2 books from 8 … (A cannot be included)
or ③ 3 books from the 8 (excluding A and B)

No. of selections : ① $1 \times \binom{8}{2} = 28$

② $1 \times \binom{8}{2} = 28$

③ $\binom{8}{3} = 56$

Number of selections = $28 + 28 + 56 = \underline{\underline{112}}$

(b) (i) In how many different ways can eight people be seated in a row?
(ii) Three girls and five boys sit in a row, arranged at random.
Find the probability that the three girls are seated together.
(iii) Three girls and n boys sit in a row, arranged at random.
If the probability that the three girls are seated together is $\frac{1}{35}$,
find the value of n.

(i) No. of ways = 8!
= 8· 7· 6· 5· 4· 3· 2· 1
= $\underline{\underline{40320}}$

(ii) G G G ☐ ☐ ☐ ☐ ☐

Regard the 3 girls as 1 unit
No. of arrangements = 6!
For each arrangement the girls can be
arranged among themselves in 3! ways
Total number of arrangements = 6! × 3!
= 720 × 6
= 4320

Probability that the 3 girls are together is

$$\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{4320}{40320} = \frac{432}{4032} = \frac{48}{448} = \frac{6}{56} = \frac{3}{28}$$

$\therefore$ probability $= \underline{\underline{\frac{3}{28}}}$

(iii) No. of arrangements of 3 girls and n boys if the 3 girls are seated together is $(n + 1)! \times 3!$

Probability is $\frac{(n+1)!\times 3!}{(n+3)!}$

$$= \frac{(n+1)!\times 6}{(n+3)(n+2)(n+1)!} = \frac{6}{(n+3)(n+2)}$$

Probability $= \frac{1}{35} \Rightarrow \frac{6}{(n+3)(n+2)} = \frac{1}{35}$

$\Rightarrow (n + 3)(n + 2) = 210$

$\Rightarrow n^2 + 5n + 6 = 210$

$\Rightarrow n^2 + 5n - 204 = 0$

$\Rightarrow (n - 12)(n + 17) = 0$

$\Rightarrow n = 12$ as $n \neq -17$

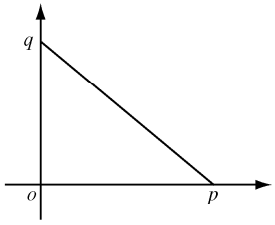
$\therefore \underline{\underline{n = 12}}$

(c) x and y are randomly selected integers with $1 \le x \le 10$ and $1 \le y \le 10$.
p is the point with coordinates $(x, 0)$ and q is the point with coordinates $(0, y)$.
Find the probability that

(i) the slope of pq is equal to -1

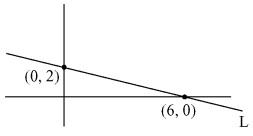
(ii) the slope of pq is greater than -1

(iii) the length of $[pq]$ is less than or equal to 5.



(i) Slope $= -1 \Rightarrow$ lines go through these points.
(1, 0), (0, 1) or (2, 0), (0, 2) or (3, 0), (0, 3), or ..... (10, 0), (0, 10)
Favourable lines = 10
Total number of lines = $10 \times 10 = 100$
Probability slope $= -1$ is $\frac{10}{100} = \underline{\underline{\frac{1}{10}}}$

(ii) Slope of given line L is
$\frac{0-2}{6-0} = -\frac{1}{3}$ which is > -1
For the slope to be greater than -1, the intercept on the x-axis must be greater than the intercept on the y-axis.
Thus the possible lines are
$x = 2 \quad y = 1$
$x = 3 \quad y = 1, 2$
$x = 4 \quad y = 1, 2, 3$



$x = 5$ $y = 1, 2, 3, 4$
$x = 6$ $y = 1, 2, 3, 4, 5$
$x = 7$ $y = 1, 2, 3, 4, 5, 6$
$x = 8$ $y = 1, 2, 3, 4, 5, 6, 7$
$x = 9$ $y = 1, 2, 3, 4, 5, 6, 7, 8$
$x = 10$ $y = 1, 2, 3, 4, 5, 6, 7, 8, 9$

Total number of lines $= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$
$= 45$

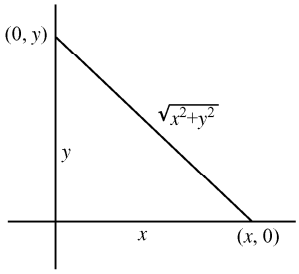
Total number of possible lines $= 100$

$$\text{Probability of slope} > -1 = \frac{\text{number of favourable lines}}{\text{number of possible lines}}$$

$$= \frac{45}{100} = \underline{\underline{\frac{9}{20}}}$$

(iii) If $\sqrt{x^2 + y^2} \leq 5$

and $x \geq 1$ and $y \geq 1$
then the possible values are
(1, 1), (1, 2), (1, 3), (1, 4)
(2, 1), (2, 2), (2, 3), (2, 4)
(3, 1), (3, 2), (3, 3), (3, 4)
(4, 1), (4, 2), (4, 3)



$$\text{Probability } \sqrt{x^2 + y^2} \leq 5 \text{ is } \frac{\text{total favourable lines}}{\text{total of possible lines}}$$

$$= \frac{15}{100} = \underline{\underline{\frac{3}{20}}}$$

7.

(a) The prices of four food items in a shopping basket are €3, €5, €1 and €6. Find the weighted mean price of these items using the weights 2, 3, 4 and 1 respectively.

Variable	€3	€5	€1	€6
Weight	2	3	4	1

$$\text{Weighted mean} = \frac{(2\times3)+(3\times5)+(4\times1)+(1\times6)}{2+3+4+1}$$

$$= \frac{6+15+4+6}{10} = \frac{31}{10} = 3.1$$

$\therefore$ $\underline{\underline{\text{weighted mean} = €3.10}}$

(b) (i) Solve the difference equation $u_{n+2} - 6u_{n+1} + 5u_n = 0$, where $n \geq 1$, given that $u_1 = 0$ and $u_2 = 20$.

(ii) Find an expression in n for the sum of the terms $u_1 + u_2 + u_3 + ... + u_n$.

(i) $u_{n+2} - 6u_{n+1} + 5u_n = 0$

Characteristic equation is $x^2 - 6x + 5 = 0$

$x^2 - 6x + 5 = 0 \Rightarrow (x - 1)(x - 5) = 0$

$\Rightarrow \quad x = 1 \quad \text{or} \quad x = 5$

If α and β are the roots, then $\alpha = 1$ and $\beta = 5$

$$u_n = l(\alpha)^n + m(\beta)^n$$
$$= l(1)^n + m(5)^n$$

$u_1 = 0 \Rightarrow \quad l + 5m = 0 \cdots\cdots (1)$

$u_2 = 20 \Rightarrow l(1)^2 + m(5)^2 = 20$

$\Rightarrow \quad l + 25m = 20 \cdots (2)$

$$\underline{l + 5m = 0}$$
$$20m = 20$$

$\Rightarrow \quad m = 1$

From (1) : $l + 5(1) = 0 \Rightarrow l = -5$

$\therefore \; u_n = -5(1)^n + 1(5)^n = -5 + 5^n$

$$= \underline{\underline{5^n - 5}}$$

(ii) $u_1 + u_2 + u_3 + \cdots + u_n$

$u_1 = 5^1 - 5$

$u_2 = 5^2 - 5$

$u_3 = 5^3 - 5$

$\cdots$

$\underline{u_n = 5^n - 5}$

$S_n = 5^1 + 5^2 + 5^3 + \cdots + 5^n - [5 + 5 \cdots n \text{ times}]$

$= S_n$ of GP $- [5n]$

$$= \frac{a(r^n - 1)}{r - 1} - 5n$$

$$= \frac{5(5^n - 1)}{5 - 1} - 5n$$

$$= \frac{5}{4}(5^n - 1) - 5n$$

$$\Rightarrow u_1 + u_2 + u_3 + \ldots + u_n = \underline{\underline{\frac{5}{4}(5^n - 1) - 5n}}$$

(c) The two numbers a and b have mean $\bar{x}$ and standard devation σ_1.
The three numbers c, d and e have mean $\bar{x}$ and standard deviation σ_2.
Find the standard deviation of the five numbers a, b, c, d and e in terms of σ_1 and σ_2.

$$\bar{x} = \frac{a+b}{2} \text{ and } \sigma_1 = \sqrt{\frac{(\bar{x}-a)^2 + (\bar{x}-b)^2}{2}}$$

$$\Rightarrow a+b = 2\bar{x} \quad \Rightarrow (\sigma_1)^2 = \frac{(\bar{x}-a)^2 + (\bar{x}-b)^2}{2}$$

$$\Rightarrow 2\sigma_1^2 = (\bar{x}-a)^2 + (\bar{x}-b)^2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Also } \bar{x} = \frac{c+d+e}{3} \text{ and } \sigma_2 = \sqrt{\frac{(\bar{x}-c)^2 + (\bar{x}-d)^2 + (\bar{x}-e)^2}{3}}$$

$$\Rightarrow c+d+e = 3\bar{x} \quad \Rightarrow \quad (\sigma_2)^2 = \frac{(\bar{x}-c)^2 + (\bar{x}-d)^2 + (\bar{x}-e)^2}{3}$$

$$\Rightarrow 3(\sigma_2)^2 = (\bar{x}-c)^2 + (\bar{x}-d)^2 + (\bar{x}-e)^2 \ldots\ldots\ldots\ldots\ldots (2)$$

For the five numbers *a, b, c, d, e*

$$\text{mean} = \frac{a+b+c+d+e}{5}$$

$$= \frac{2\bar{x}+3\bar{x}}{5} = \frac{5\bar{x}}{5} = \bar{x}$$

$$\text{Standard deviation, } \sigma_3 = \sqrt{\frac{(\bar{x}-a)^2 + (\bar{x}-b)^2 + (\bar{x}-c)^2 + (\bar{x}-d)^2 + (\bar{x}-e)^2}{5}}$$

$$\Rightarrow \quad \sigma_3{}^2 = \frac{\left[(\bar{x}-a)^2 + (\bar{x}-b)^2\right] + \left\{(\bar{x}-c)^2 + (\bar{x}-d)^2 + (\bar{x}-e)^2\right\}}{5}$$

$$= \frac{2(\sigma_1)^2 + 3(\sigma_2)^2}{5}$$

$$\Rightarrow \quad \sigma_3 = \sqrt{\frac{2(\sigma_1)^2 + 3(\sigma_2)^2}{5}}$$

∴ the standard deviation of the 5 numbers is

$$\underline{\underline{\sigma_3 = \sqrt{\frac{2(\sigma_1)^2 + 3(\sigma_2)^2}{5}}}}$$

8. (a) Use integration by parts to find $\int xe^{4x}dx$.

Let $u = x$ and $dv = e^{4x}dx$

$\Rightarrow du = dx$ and $v = \int e^{4x}dx$

$$\Rightarrow v = \frac{e^{4x}}{4}$$

$$\int udv = uv - \int vdu$$

$$\Rightarrow \int xe^{4x}dx = x \cdot \frac{e^{4x}}{4} - \int \frac{e^{4x}}{4}dx$$

$$= \underline{\underline{\frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + c}}$$

(b) (i) Derive the first four terms of the Maclaurin series for $f(x) = \sqrt{1+x}$.

(ii) Given that this series converges for $-1 < x < 1$, use these four terms to find an approximation for $\sqrt{17}$, as a fraction.

(i) $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 + \ldots$$

$$\Rightarrow \underline{\underline{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3}}$$

(ii) $\sqrt{17} = \sqrt{16+1} = \sqrt{16\left(1+\frac{1}{16}\right)} = 4\sqrt{1+\frac{1}{16}}$

$= \sqrt{1+x} = \sqrt{17} = 4\sqrt{1+\frac{1}{16}}$

We now let $x = \frac{1}{16}$ in the series found in (i) above

$$\sqrt{1+x} = 1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3$$

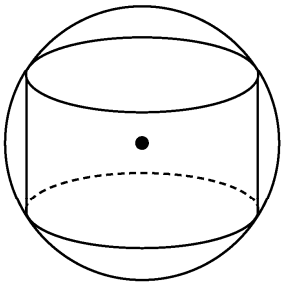
$$= 1 + \frac{1}{32} - \frac{1}{2048} + \frac{1}{65536}$$

$$= \frac{65536 + 2048 - 32 + 1}{65536} = \frac{67553}{65536}$$

$$\Rightarrow \sqrt{17} = 4\left(\frac{67553}{65536}\right) = \frac{67553}{16384}$$

$$\Rightarrow \underline{\underline{\sqrt{17} = \frac{67553}{16384}}}$$

(c) The diagram shows a cylinder inscribed in a sphere.
The cylinder has height $2x$ and radius r.
The sphere has fixed radius a.



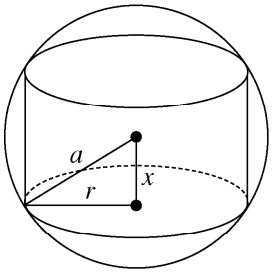
(i) Express r in terms of a and x.

(ii) Find, in terms of a, the maximum possible volume of the cylinder.

(i) $a^2 = r^2 + x^2$

$\Rightarrow r^2 = a^2 - x^2$

$\Rightarrow \underline{\underline{r = \sqrt{a^2 - x^2}}}$



(ii) Volume $= \pi r^2 h$

$= \pi(a^2 - x^2)2x$

$= 2\pi a^2 x - 2\pi x^3$

$$\frac{dv}{dx} = 2 \bullet a^2 - 6 \bullet x^2$$

For maximum or minimum volume $\frac{dv}{dx} = 0$

$\Rightarrow 2\pi a^2 - 6\pi x^2 = 0$

$\Rightarrow a^2 - 3x^2 = 0$

$$\Rightarrow x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Volume $= \bullet\ (a^2 - x^2)2x$

$$= \bullet \left[a^2 - \frac{a^2}{3}\right]\frac{2a}{\sqrt{3}}$$

$$= \bullet \left[\frac{2a^2}{3}\right]\frac{2a}{\sqrt{3}} = \frac{4 \bullet a^3}{3\sqrt{3}} = \frac{4\sqrt{3} \bullet a^3}{9}$$

For maximum or minimum volume, we find $\frac{d^2v}{dx^2}$.

$$\frac{dv}{dx} = 2 \bullet a^2 - 6 \bullet x^2$$

$$\Rightarrow \frac{d^2v}{dx^2} = -12 \bullet x$$

$$= -12 \bullet \left(\frac{a}{\sqrt{3}} \right) = \text{negative} \Rightarrow \text{maximum value}$$

$$\therefore \text{maximum volume} = \underline{\underline{\frac{4\sqrt{3} \bullet a^3}{9}}}$$

9. (a) A and B are independent events such that $P(A) = 0.25$ and $P(A \cup B) = 0.55$. Find $P(B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow 0.55 = 0.25 + P(B) - P(A \cap B)$$

Since A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$

$$\Rightarrow 0.55 = 0.25 + P(B) - P(A) \cdot P(B)$$

$$= 0.25 + P(B) - 0.25P(B)$$

$$\Rightarrow 0.55 = 0.25 + 0.75P(B)$$

$$\Rightarrow 0.75P(B) = 0.55 - 0.25$$

$$= 0.3$$

$$\Rightarrow P(B) = \frac{0.3}{0.75} = \frac{30}{75} = \underline{\underline{\frac{2}{5}}} = \underline{\underline{0.4}}$$

(b) A person plays a game that involves throwing five hoops at a peg. The following table gives the probability distribution for the number of hoops that land on the peg.

x	0	1	2	3	4	5
$P(x)$	0.01	0.08	0.23	0.34	0.26	0.08

Find the mean and the standard deviation of the distribution.

$$\text{Mean} = \frac{\sum xP(x)}{\sum P(x)}$$

$$= \frac{0(0.01) + 1(0.08) + 2(0.23) + 3(0.34) + 4(0.26) + 5(0.08)}{0.01 + 0.08 + 0.23 + 0.34 + 0.26 + 0.08}$$

$$= \frac{3}{1} = \underline{\underline{3}}$$

$$\text{Standard deviation} = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}$$

$$= \sqrt{\frac{0.01(0-3)^2 + 0.08(1-3)^2 + 0.23(2-3)^2 + 0.34(3-3)^2 + 0.26(4-3)^2 + 0.08(5-3)^2}{1}}$$

$$= \sqrt{0.01(9) + 0.08(4) + 0.23(1) + 0.34(0) + 0.26(1) + 0.08(4)}$$

$$= \sqrt{1.22} = 1.1$$

$$\therefore \underline{\underline{\text{Standard deviation} = 1.1}}$$

(c) A coin is slightly bent and is thought to favour heads. Accordingly, it is tossed 100 times to test the null hypothesis that it is fair against the alternative hypothesis that it favours heads. In this experiment, 55 heads are observed.

(i) Show that this result is not significant at the 5% level.

(ii) How many times would the coin have to be tossed in an experiment in order that an observation of 55% heads *would* be regarded as significant at the 5% level?

(i) This is a binomial distribution where

$$p = \frac{1}{2},\ q = \frac{1}{2} \text{ and } n = 100$$

$$\bar{x} = np = 100\left(\frac{1}{2}\right) = 50$$

$$\sigma = \sqrt{npq} = \sqrt{100\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \sqrt{25} = 5$$

$$z = \frac{x-\bar{x}}{\sigma} = \frac{55-50}{5} = \frac{5}{5} = 1$$

Since $z < 1.645$, we reject the alternative hypothesis and conclude that the coin is not biased in favour of heads.

(ii) Let n be the number of times the coin is tossed.

$$\bar{x} = \frac{n}{2} \quad \text{and} \quad \sigma = \sqrt{npq}$$

$$= \sqrt{n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \sqrt{\frac{n}{4}} = \frac{\sqrt{n}}{\sqrt{4}} = \frac{1}{2}\sqrt{n} = \frac{\sqrt{n}}{2}$$

$$\text{Heads} = \frac{55n}{100} = x$$

For $z > 1.96 \Rightarrow \dfrac{x-\bar{x}}{\sigma} > 1.645$

$$\Rightarrow \frac{\frac{55n}{100} - \frac{n}{2}}{\frac{\sqrt{n}}{2}} > 1.645$$

$$\Rightarrow \frac{\frac{55n}{100} - \frac{50n}{100}}{\frac{\sqrt{n}}{2}} > 1.645$$

$$= \frac{5n}{100} \times \frac{2}{\sqrt{n}} > 1.645$$

$$= \frac{n}{10\sqrt{n}} > 1.645$$

$$\Rightarrow \sqrt{n} > 16.45$$

$$\Rightarrow n > 270.6$$

$$\Rightarrow n = 271$$

Therefore, the coin would have to be tossed 271 times.

10. (a) If a is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, find $a \circ a$.

$$a \circ a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

(b) The set {1, 2, 4, 5, 7, 8} is a group under multiplication modulo 9.

(i) Draw up a Cayley table for the group.

(ii) Find a generator of the group.

(iii) Hence, or otherwise, find a subgroup of order 2 and a subgroup of order 3.

(i) mod 9

×	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

(ii) $2^1 = 2$; $2^2 = 4$; $2^3 = 8$; $2^4 = 7$; $2^5 = 5$; $2^6 = 1$

$\therefore$ 2 is a generator of the group

5 is also a generator since

$5^1 = 5$, $5^2 = 7$, $5^3 = 8$, $5^4 = 4$, $5^5 = 2$, $5^6 = 1$

(iii) Subgroup of order 2 = {1, 8}

Subgroup of order 3 = {1, 4, 7}

(c) $(G, \circ)$ and $(H, *)$ are two groups with identities e_G and e_H respectively.
If $\bullet : G \to H$ is an isomorphism, prove that

(i) $\bullet(e_G) = e_H$.

(ii) $\bullet(x^{-1}) = [\bullet(x)]^{-1}$, for all $x \in G$.

(i) Let $x \in G$.

$\bullet(x \circ e_G) = \bullet(x) * \bullet(e_G)$, because of isomorphism.

$\bullet(x) = \bullet(x) * \bullet(e_G)$, as $x \circ e_G = x$.

$\Rightarrow \bullet(e_G)$ is the identity in $(H, *)$.

$\therefore \bullet(e_G) = e_H$.

(ii) $\bullet(x \circ x^{-1}) = \bullet(x) * \bullet(x^{-1})$

$\therefore \bullet(e_G) = \bullet(x) * \bullet(x^{-1}) \Rightarrow \bullet(x)$ and $\bullet(x^{-1})$ are inverses.

$\therefore \bullet(x^{-1}) = [\bullet(x)]^{-1}$.

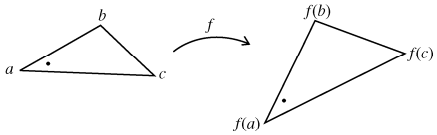
11. (a) Find the equation of the ellipse with foci $(\sqrt{7},0)$ and $(-\sqrt{7},0)$ and with eccentricity $\frac{\sqrt{7}}{4}$.

$e=\frac{\sqrt{7}}{4}$ and $ae=\sqrt{7}\Rightarrow a=4.$

$b^2=a^2(1-e^2)=16\left(1-\frac{7}{16}\right)=9.$

$\therefore$ ellipse : $\frac{x^2}{16}+\frac{y^2}{9}=1.$

(b) A transformation f is a similarity transformation if there exists a fixed number k such that $|f(a)f(b)|=k|ab|$, for all a and b.
Show that angle measure is invariant under a similarity transformation.



The triangle abc is mapped onto the triangle $f(a)f(b)f(c)$ under a similarity transformation f. $\angle\bullet$ is mapped onto $\angle\bullet$.

$$\cos\angle\bullet=\frac{|f(a)f(b)|^2+|f(a)f(c)|^2-|f(b)f(c)|^2}{2|f(a)f(b)|\cdot|f(a)f(c)|}=\frac{k^2|ab|^2+k^2|ac|^2-k^2|bc|^2}{2k^2|ab|\cdot|ac|}.$$

$$\therefore\cos\angle\bullet=\frac{|ab|^2+|ac|^2-|bc|^2}{2|ab|\cdot|ac|}=\cos\angle bac=\cos\angle\bullet.$$

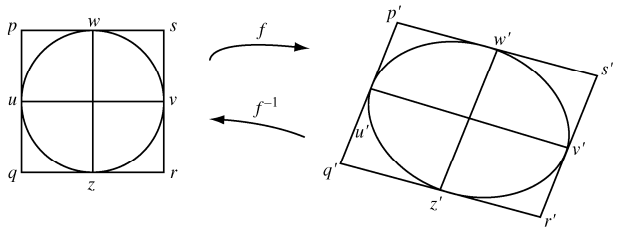
$\therefore|\angle\bullet|=|\angle\bullet|$, since both are in the range 0° to 180°

Hence angle measure is invariant under a similarity transformation.

(c) (i) Define the term *conjugate diameters* of an ellipse.
(ii) Prove that all parallelograms circumscribed to a given ellipse at the end points of conjugate diameters have the same area.

(i) If $|pq|$ is a diameter of an ellipse E, then there is a second diameter $|rs|$, such that $|pq|$ bisects all chords of E on lines parallel to $|rs|$ and vice versa. $|pq|$ and $|rs|$ are called *conjugate diameters* of the ellipse.

(ii)



$[u'v']$ and $[w'z']$ are conjugate diameters of ellipse E.
Tangents at their end points form the parallelogram $p'q'r's'$.
Under an affine transformation f^{-1}, the ellipse maps to the circle $x^2+y^2=1$ and $p'q'r's'$ is mapped to $pqrs$.
$[uv]$ and $[wz]$ are conjugate diameters of the circle and $uv\perp wz$.
The square $pqrs$ has fixed area 4 sq units.
$\therefore$ Area $p'q'r's'=|\det.(f)|\cdot$ area $pqrs=4\det(f)$.
But $\det(f)$ is constant $\Rightarrow$ area $p'q'r's'$ is constant.
$\therefore$ Areas of all parallelograms at end points of conjugate diameters of an ellipse are equal.