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Irish Mathematics Teachers' Association

Founded to promote and assist the teaching of mathematics at all levels.

Membership is open to all those interested in mathematics and mathematics education.

Individual membership may be obtained through a Branch organisation.

Editorial

Welcome to the Autumn 2007 edition of the IMTA Newsletter.

This edition contains further riveting and entertaining mathematics both at the centre of our practice in schools and also at the periphery of exploration. Euklid is a powerful computer programme which allows geometry to 'come alive' on the screen-room for the pupil and the teacher. But is it better than Geometer's Sketchpad? David Kearney puts both programmes through their paces and gives his considered opinion.

His unrelated namesake, Fr. Brendan Kearney, looks back to the basics of measurement—and surprises us with some cool insights delivered with his traditional humour.

In his well-crafted exposition Des McHale shows us that there are hidden depths remaining in the humble quadratic equation. Food for aspiring pupils during Transition Year?

And we are reminded by Michael Brennan why the Chief Examiner commented in 1996 (and other years) that 'Questions on logs were unpopular'. But could new names for old ideas camouflage the difficulty of dealing with such concepts?

At the moment it is significant that the NCCA is pursuing a Project Maths programme. Use of ICT, problem-solving skills and relating maths to everyday life are some of the aims set out for teaching and learning maths in the future. There are three significant pieces of work in this edition which relate to this national development. From St. Patrick's College, Drumcondra, Dublin, and from University College, Cork, we have Third Level research and interest in Second Level mathematics. Donal Hurley introduces us to significant practical mathematics dealing with Finance, Economics and the Lotto. Sinead Breen, Joan Cleary and Ann O'Shea report on pupils' perspectives on mathematics. Why should a student do mathematics in the first place? Good career prospects? Sheer enjoyment of mathematics? The survey raises questions which we would do well to ponder before embarking on a path which still could miss the point of what really goes on when we do mathematics.

Thirdly, from the chalk-face, there is the heartening description of teachers' ownership of their own professional development, of developing their expertise in both the teaching and learning of mathematics. Brendan O'Sullivan shows us a picture of dedication and enjoyment combined. And while it is heartening it is not yet all light—the case for Continuous Professional Encouragement (CPE for CPD) is well made by Brendan in his call for yet more light in this vitally important area! And we totally agree.

A sincere thanks to all contributors and especially to Maurice O'Driscoll who has supplied the Solutions to Higher Maths Leaving Cert 2007.

The Team Maths Quiz Final 2007 was won by Castleknock College, Dublin. The venue was Trinity College Dublin. Next year the venue moves to Cork. The BT Young Scientist 2007 winner, Abdusalam Abubakar, has recently gone on to win at European Level with his maths project dealing with RSA encryption. Congratulations to the students and their teachers.

National Maths Week approaches (15th-19th Oct., 2007. See www.calmast.ie, and perhaps you are reading this as part of your enjoyment of the week.

In the next Newsletter: More from all corners of the globe!

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The views expressed in this Newsletter are those of the individual authors and do not necessarily reflect the position of the IMTA. While every care has been taken to ensure that the information in this publication is up-to-date and correct no responsibility will be taken by the IMTA for any errors that might occur.

Some Applications of Leaving Certificate Mathematics

Donal Hurley

1 Introduction

In recent years there has been much comment on the lack of inclusion of applications on the Leaving Certificate Mathematics Syllabus. In the Applied Mathematics Syllabus, the applications are related to Physics, in particular to Mechanics. But many people may wish to consider other types of applications. Obviously, there are applications to all areas of human endeavour, so there are many applications which do not require a knowledge of, or interest in, Mechanics.

In this talk, I wish to suggest a few such applications. The selection I've made are in the areas of Financial Mathematics, Economics and Probability. There are many other applications which I could have selected in, for example, the Life Sciences. I have selected applications where the Mathematics included in the Leaving Certificate Syllabus is sufficient to cover topics which, I think, students would find interesting and beneficial.

2 Mathematics of Finance

2.1 Annual Percentage Rate

The table below lists the interest rate and compounding period for deposits offered by three banks. How can we decide which of these deposit rates has the best return?

Advanta	6.95%	monthly
DeepGreen	6.96%	daily
CharterOne	6.97%	quarterly

To solve this we use the concept of **annual percentage rate**. This is determined as follows.

If a principal P is invested at the annual percentage rate r compounded m times a year, then the amount after 1 year is

[1]Text of Fr Ingram Memorial Lecture given at the Annual General Meeting of the Irish Mathematics Teachers Association held in Tralee on November 24th, 2006.

$$A = P\left(1 + \frac{r}{m}\right)^m.$$

The *simple interest* rate that will produce the same amount A in 1 year is called the **annual percentage rate** (APR), or the **annual percentage yield** (APY). The formula for APR is derived as follows.

$$\begin{aligned}
(\text{APR simple interest rate at end of one year}) &= (\text{Compound interest rate at end of one year}) \\
P(1 + APR) &= P\left(1 + \frac{r}{m}\right)^m \\
1 + APR &= \left(1 + \frac{r}{m}\right)^m \\
APR &= \left(1 + \frac{r}{m}\right)^m - 1
\end{aligned}$$

So using the APR for each of the banks in the table, we get the following.

Advanta :	$APR = \left(1 + \frac{0.0695}{12}\right)^{12} - 1 = 0.07176$	7.176%
DeepGreen :	$APR = \left(1 + \frac{0.0696}{365}\right)^{365} - 1 = 0.07207$	7.207%
CharterOne :	$APR = \left(1 + \frac{0.0697}{4}\right)^{4} - 1 = 0.07154$	7.154%

Comparing these APRs, we conclude that **DeepGreen** has the best return and **CharterOne** has the least.

The APR formula can also be used to solve the following type of question.

Example A savings bank wishes to offer a savings scheme with a monthly compounding rate that has an APR of 7.7%. What annual nominal rate, compounded monthly, should it use?

2.2 Annuities: Future Value

An **annuity** is any sequence of equal periodic payments. If the payments are made at the end of each time interval, then the annuity is called an **ordinary annuity**. We consider only ordinary annuities. The amount, or **future value**, of an annuity is the sum of all the payments plus all the interest earned.

Suppose you decide to deposit €100 every six months into an account that pays 6% compounded semiannually. If you make 6 deposits, one at the end of each interest payment period, over 3 years, how much money will be in the account after the last deposit has been made?

To solve this, using the compound interest formula, $A = P(1 + i)^n$, we can find the value of each deposit after it has earned compound interest up to the time of the sixth deposit. So we get

$$S = 100(1.03)^5 + 100(1.03)^4 + 100(1.03)^3 + 100(1.03)^2 + 100(1.03)^1 + 100.$$

We could now evaluate each of the future value terms with a calculator and add the results to get €646.84.

However, if we had a deposit scheme with a large number of payments, this would be tedious and a better approach is to apply the formula for the sum of a geometric series.

In general, if R is the periodic deposit, i the interest rate *per period*, and n the number of periods, then the future value is given by

$$S = R(1+i)^{n-1} + R(1+i)^{n-2} + R(1+i)^{n-3} + \cdots + R(1+i)^2 + R(1+i)^1 + R,$$

and now using the formula for the sum of a geometric series we get

$$S = R\left(\frac{(1+i)^n - 1}{i}\right).$$

This is called the **future value formula** for an annuity.

Example What is the value of an annuity at the end of 20 years if €100 is deposited each month into an account earning 8.5% compounded monthly?

$$\begin{aligned} A &= 200\left(\frac{\left(1+\frac{0.085}{12}\right)^{240}-1}{\frac{0.085}{12}}\right) \\ &= 125,400. \end{aligned}$$

The formula for the future value of an ordinary annuity has another important application. Suppose the parents of a newborn child decide that on each of the child's birthdays up to the 18th year, they will deposit, €R, in an account that pays 6% compounded annually. The money is to be used for college expenses. What should the annual deposit €R be in order that the amount in the account be €80,000 after the 18th deposit?

We are given A, i, and n in the formula and we need to find R. Thus

$$\begin{aligned} 80,000 &= R\left(\frac{(1.06)^{18}-1}{0.06}\right) \\ R &= 80,000\left(\frac{0.06}{(1.06)^{18}-1}\right) \\ &= 2,588.52. \end{aligned}$$

This is one of the many examples of a type that are referred to as *sinking fund problems*. In general, any account that is established for accumulating funds to meet future obligations or debts is called a **sinking fund**. If the payments are to be made in the form of an ordinary annuity, then the size of payments is determined by using the future value formula,,

$$R = A\left(\frac{i}{(1+i)^n - 1}\right).$$

Example A company estimates that it will have to replace a piece of equipment at a cost of €800,000 in 5 years. To have this money available in 5 years, a sinking fund is established by making equal monthly payments into an account paying 6.6% compounded monthly.

(a) How much should each payment be?

(b) How much interest is earned during the last year?

Solution:
(a) €11,290.42 per month.
(b) To find the interest earned during the fifth year, we first use the future value formula to find that the amount in the account after 4 years is €618,277.04. During the fifth year, the amount in the account grew from €618,277.04 to €800,000. A portion of this growth was due to the 12 monthly payments of €11,290.42. The remainder of the growth was interest. Thus the interest during the fifth year is

$$800,000 - 618,277.04 - (12)(11,290.42) = 46,237.92.$$

2.3 Annuities: Present Value

Now consider the following question. How much should one deposit in an account paying 6% compounded semiannually in order to be able to withdraw €1,000 every six months for the next 3 years? (After the last payment is made, no money is to be left in the account.)

We are interested in finding the **present value** of each €1,000 that is paid out during the 3 years. We can do this by solving for P in the compound interest formula, $A = P(1+i)^n$, for each of the €1,000 payments. Since $P = A(1+i)^{-n}$ we get that the present value of all the 6 payments is given by

$$1,000(1.03)^{-1} + 1,000(1.03)^{-2} + 1,000(1.03)^{-3} + \cdots + 1,000(1.03)^{-6}.$$

Evaluating each term and adding we get €5,417.19.

However, we can again use the techniques of summing geometric series. In general, if R is the periodic payment, i the interest rate per period, and n the number of periods, then the present value is given by

$$\begin{aligned} V_0 &= R(1+i)^{-1} + R(1+i)^{-2} + \cdots + R(1+i)^{-n} \\ &= R\left(\frac{1-(1+i)^{-n}}{i}\right). \end{aligned}$$

This formula,

$$V_0 = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

is called the **present value formula**.

Example What is the present value of an annuity that pays €200 per month for 5 years if money is worth 6% compounded monthly?
Solution: €10,345.11.

Retirement planning type problems involve both future and present value formulas. Consider the following example.

Example Recently, Benefit Life offered an ordinary annuity that earned 6.5% compounded annually. A person plans to make equal annual deposits into this account for 25 years in order to then make 20 equal annual withdrawals of €25,000, reducing the balance in the account to zero.

(a) How much must be deposited annually to accumulate sufficient funds to provide for these payments?

(b) How much total interest is earned during this entire 45-year process?

Solution:
(a) We begin by finding the present value necessary to provide for the 20 annual withdrawals. We get

$$\begin{aligned} V_0 &= 25,000\left(\frac{1-(1.065)^{-20}}{0.065}\right) \\ &= 275,462.68. \end{aligned}$$

Now we find the deposits that will produce a future value of €275,462.68 in 25 years. We get

$$\begin{aligned} R &= 275,462.68\left(\frac{0.065}{(1.065)^{25}-1}\right) \\ &= 4,677.76. \end{aligned}$$

The present value formula for an ordinary annuity has another important use. Suppose you borrow €25,000 from a bank to buy a car and agree to repay the loan in 36 equal monthly payments, including all interest due. If the bank charges 1% per month on the unpaid balance (12% per year compounded monthly), how much should each payment be to retire the total debt, including interest in 36 months?

Actually, the bank has bought an annuity from you. The question is: If the bank pays you €25,000, (present value) for an annuity paying them €R, per month for 36 months at 12% nominal interest, compounded monthly, what are the monthly payments? We use the present value formula to get

$$\begin{aligned} 25,000 &= R\left(\frac{1-(1.01)^{-36}}{0.01}\right) \\ R &= 830.358. \end{aligned}$$

At €830.358 per month, the car will be yours after 36 months. That is you have *amortized* the debt in 36 equal payments. (*Mort* means "death": you have "killed" the loan in 36 months.) In general, **amortizing a debt** means that the debt is retired in a given length of time by equal periodic payments that include compound interest. Solving the present value formula for R, we get the **amortization formula**

$$R = V_0\left(\frac{i}{1-(1+i)^{-n}}\right).$$

Example A couple borrow €300,000 to purchase a house and repay with a 35 year mortage at 4.5% compounded monthly.

(a) What are their monthly payments.

(a) How much interest do they pay over the 35-year term?

(c) If the interest rate increased to 5.5%, what will their monthly payments be?

Solution:
(a)

$$R = 300000\left(\frac{\frac{0.045}{12}}{1-(1+\frac{0.045}{12})^{-420}}\right) = 1419.77.$$

(b)

$$420\,(1419.77) - 300,000 = 296,303.$$

(c) €1,611.05.

3 Economics

3.1 Supply-Demand Functions

The quantity of a product that people are willing to buy during some period of time depends on its price. Generally, the higher the price, the less the demand; the lower the price, the greater the demand. Similarly, the quantity of a product that a supplier is willing to sell during some period of time also depends on the price. Generally, a supplier will be willing to supply more of a product at higher prices and less of a product at lower prices.

The simplest supply and demand model is a linear model where the graphs of a demand and a supply equation are straight lines.

Example Suppose that the supply and demand equations for printed T-shirts at a resort town for a particular week are

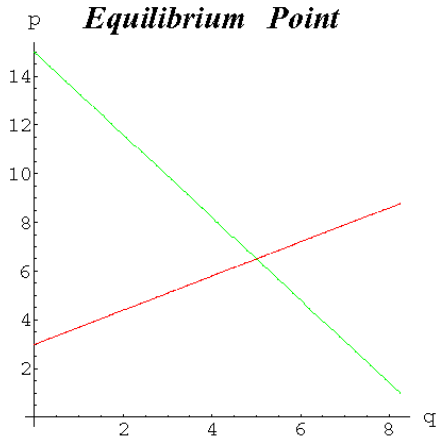
$$\begin{aligned} p &= 0.7\,q + 3 && \text{supply} \\ p &= -1.7\,q + 15 && \text{demand} \end{aligned}$$

where p is the price in Euro and q is the quantity in hundreds.

(a) Find the supply and the demand if T-shirts are priced at €9 each.

(b) Graph the two equations in the same coordinate system and identify the point where the supply equals demand.

Solution (a) Supply 857 T-shirts; demand:353 T-shirts.
(b)



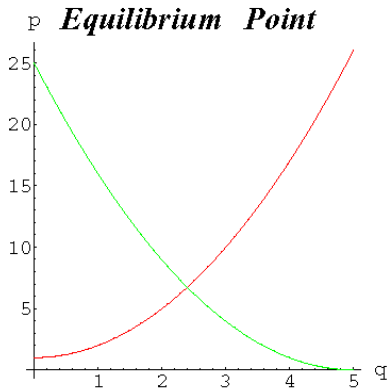
The price and quantity at which the supply equals demand are called the **equilibrium price** and **equilibrium quantity**, respectively. The point where the graphs of the two functions intersect is called the **equilibrium point**.

Example Given the following *supply* and *demand* functions

$$p = q^2 + 1$$
$$p = (q - 5)^2$$

where p denotes the price in Euro and q denotes the number of units of the product supplied or demanded in Millions, find the equilibrium point. Graph the supply and demand functions.

Solution €6.76 and 2.4 million units.



3.2 Marginals

In business and economics, the rates at which certain quantities are changing often provide useful insight into various economic systems. A manufacturer, for example, is interested not only in the total cost $C(q)$ at certain production levels, but is also interested in the rate of change of costs at various production levels.

In economics, the word **marginal** refers to a rate of change, that is to a derivative. Thus if $C(q)$ is the total cost of producing q items, then $C'(q)$ is the marginal cost at production level q.

Example Suppose that the total cost $C(q)$ (in thousands of Euro) for manufacturing sail boats per year is given by the function

$$C(q) = 575 + 25\,q - 0.25\,q^2 \qquad 0 \;\leq\; q \;\leq\; 50.$$

(a) Find the marginal cost at a production level of q boats per year.

(b) Find the marginal cost at a production of 40 boats per year, and interpret the result.

Solution

(a) The marginal cost at a production level of q boats is

$$C'(q) = 25 - 0.5\,q.$$

(b) The marginal cost at a production level of 40 boats is $C'(40) = €5{,}000$ per boat. The total cost is increasing at the rate of €5,000 per boat.

The interpretation of marginal cost given in the last Example is the usual way we use the derivative to describe the manner in which a quantity is changing. However, there is a special way to interpret the rate of change of a total cost function that is used extensively in economics. In the Example above, if the total cost is increasing at the rate of €5,000 per boat when 40 boats are produced per year, and if production is increased to 41 boats per year, the total cost will increase by approximately €5,000. But the increase in total cost is just the cost of producing the next boat. Thus the cost of producing the 41st boat must be approximately €5,000, the marginal cost at the production level of 40 boats per year. The *actual* cost of producing the 41st boat is $C(41) - C(40) = €4{,}750$.

The interpretation of the marginal cost function as the approximate cost of producing the next item is a special case that applies to total cost functions. This interpretation also applies to total revenue and total profit functions.

Example The market research department of a company recommends that the company manufacture and market a new CD player. After suitable test marketing, the research department presents the following **price-demand equation**:

$$q = 10{,}000 \;-\; 1{,}000\,p, \qquad q \text{ is demand at price } p.$$

Solving this for p,

$$p = 10 \;-\; 0.001\,q$$

where q is the number of CD players retailers are likely to buy at $€p$ per CD player.

The financial department provides the following **cost function**:

$$C(q) = 7,000 + 2\,q$$

where €7,000 is the estimate of fixed costs (tooling and overheads) and €2 is the estimate of variable costs per CD player (materials, labour, marketing, transportation, storage etc.).

(a) Find the marginal cost function $C'(q)$ and interpret.

(b) Find the revenue function $R(q)$.

(c) Find the marginal revenue at $q = 2000,\ 5000,$ and 7000. Interpret the results.

(d) Graph the cost function and the revenue function on the same coordinate system, find the intersection points of these two graphs and interpret the results.

(e) Find the profit function and sketch its graph.

(f) Find the marginal profit at $q = 1000,\ 4000,$ and 6000. Interpret the results.

Solution (a) The marginal cost is $C'(q) = 2$. Since this is a constant, it costs an additional €2 to produce one more CD player at any production level.

(b) The **revenue** is the amount of money, $R(q)$ received by the company for manufacturing and selling q CD players at €p per CD player and it is given by

$$R = \text{(number of CD players)(price per CD player)} = q\,p.$$

Thus the **revenue function** is

$$R(q) = q\,p = q\,(10 - 0.001\,q) = 10\,q - 0.001\,q^2.$$

(c) The **marginal revenue** is

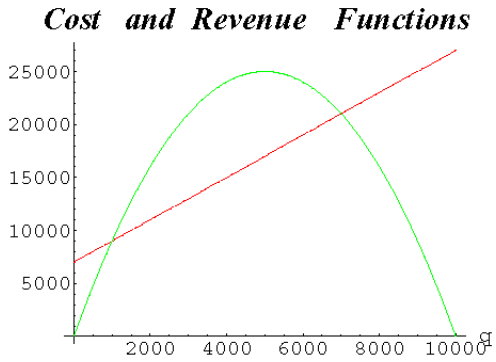
$$R'(q) = 10 - 0.002\,q.$$

For production levels of $q = 2000,\ 5000,$ and 7000, we have

$$R'(2000) = 6 \qquad R'(5000) = 0 \qquad R'(7000) = -4.$$

This means that at production levels of 2000, 5000, and 7000, the respective approximate changes in revenue per unit change in production are €6, €0 and - € 4. That is, at the 2000 output level, revenue increases as production increases; at the 5000 output level, revenue does not change with a "small" change in production; and at the 7000 output level, revenue decreases with an increase in production.

(d) When we graph $R(q)$ and $C(q)$ on the same coordinate system, we obtain the following graph.



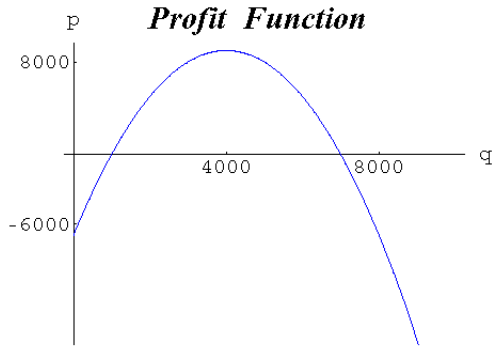
The intersection points are called the **break-even points** because revenue equals cost at these production levels. The break-even points are got by solving

$$\begin{aligned} C(q) &= R(q) \\ 7,000 + 2q &= 10q - 0.001q^2 \end{aligned}$$

The solutions are $q = 1000$ and 7000. At these points, $R(1000) = C(1000) = 9000$, and $R(7000) = C(7000) = 21000$.

(e) The **profit function** is

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= (10q - 0.001q^2) - (7,000 + 2q) \\ &= -0.001q^2 + 8q - 7,000. \end{aligned}$$



(f) The **marginal profit** is

$$P'(q) = -0.002q + 8.$$

For production levels of $1000, 4000$, and 6000, we have

$$P'(1000) = 6 \qquad P'(4000) = 0 \qquad P'(6000) = -4.$$

This means, for example, that at the 6000 output level, profits will decrease if production is increased. Since $P'(4000) = 0$ the maximum profit is at the 4000 output level.

Remark Since $P(q) = R(q) - C(q)$ then

$$P'(q) = R'(q) - C'(q),$$

so at the output level which maximizes profit, that is, when $P' = 0$, we have

$$R'(q) = C'(q).$$

Thus profit is maximized at the output level when the marginal revenue equals the marginal cost.

4 Probability

4.1 Odds

When the probability of an event E is known, it is often customary (particularly in gaming situations) to speak of *odds* for (or against) the event E, rather that the probability of the event E. The relationship between these two designations is as follows.

If $P(E)$ is the probability of the event E, we define:

(A) **Odds for** $E = \frac{P(E)}{1-P(E)} = \frac{P(E)}{P(E')}, \quad P(E) \neq 1$

(B) **Odds against** $E = \frac{P(E')}{P(E)}, \quad P(E) \neq 0$

Example If you roll a fair die once, the probability of rolling a **4** is 1/6, whereas the odds in favour of rolling a **4** are

$$\frac{P(E)}{P(E')} = \frac{1/6}{5/6} = \frac{1}{5},$$

and the odds against rolling a **4** are 5/1.

In terms of a fair game, if you bet €1 on a **4** turning up, you would lose €1 to the house if any number other that **4** turns up and you would be paid €5 by the house (and in addition your bet of €1 returned) if a **4** turns up. (An experienced gambler would say that the house pays 5 to 1 on a **4** turning up of a single roll of a die.)

In general, if the odds for an event are $a : b$, then **a game is fair** if your bet of €a is lost if event E does not happen, but you win €b (and in addition your bet of €a is returned) if event E does happen. (Gamblers would say that the house pays b to a on event E happening.)

Example

(a) What are the odds of rolling a sum of 8 in a single roll of two fair dice? [Ans. 5/31.]

(b) If you bet €5 that a sum of 8 will turn up, what should the house pay (plus returning your €5 bet) if a sum of 8 does turn up for the game to be fair? [Ans 31.]

Now we go in the other direction. If we are given the odds for an event, what is the probability of the event? The result is

If the odds for an event E are a/b, then the probability of E is

$$P(E) = \frac{a}{a+b}.$$

To see this, solve for $P(E)$ in the equation

$$\frac{a}{b} = \frac{P(E)}{P(E')} = \frac{P(E)}{1-P(E)}.$$

4.2 LOTTO

Recently, the LOTTO increased the selection numbers from 42 to 45. In the early days the numbers were up to 36 and this was increased to 42 as a result of some drama on a Holloween weekend several years ago.

When the numbers were up to 36, the number of possible selections of 6 numbers was

$$C(36,\,6) = \left(\begin{array}{c} 36 \\ 6 \end{array}\right) = \frac{36!}{30!\,6!} = 1,947,792.$$

Later, when selections were made from 42 numbers, the number of selections was

$$C(42,\,6) = \left(\begin{array}{c} 42 \\ 6 \end{array}\right) = \frac{42!}{36!\,6!} = 5,245,786.$$

Now with 45 numbers from which to choose, there are

$$C(45,\,6) = \left(\begin{array}{c} 45 \\ 6 \end{array}\right) = \frac{45!}{39!\,6!} = 8,145,060.$$

Something which should be observed is that, roughly, one of the two winning selections each week is a selection where at least one pair of successive numbers is included. We will now explain why this is the case and also notice that the probability of it occurring has decreased as the set of numbers, from which the selection is made, has increased. However, it is still just greater than 0.5.

First, we compute the number of ways one can select a set of 6 numbers from the set $S_{36} = \{1,\,2,\,3,\,\ldots,\,36\}$ so that there are **no** successive numbers. One uses binary sequences, that is sequences of **0** and **1**. Form such a sequence with 30 zeros and 6 ones. Any winning combination is matched with such a binary sequence where the positions of the ones coincide with the positions given by the winning numbers. For example the binary sequence

010011000100101000000000000000000000

corresponds to the winning selection $\{2, 5, 6, 10, 13, 15\}$ It is easy to see that each winning set of 6 numbers coincides with such a binary sequence, and visa versa. If we require that no pair of numbers are a successive pair, then there can't be a set of adjacent ones in the corresponding binary sequence. So we need to count the binary sequences with no pair of adjacent ones. One forms such a binary sequence by first placing the 30 zeros and then placing the 6 ones in 6 of the 31 possible slots,

- between the 30 zeros (29 such),
- or at the very beginning or at the very end (2 such).

We have $\binom{31}{6}$ ways to make this choice. If we wish to have at least one pair of successive numbers, then there are $\binom{36}{6} - \binom{31}{6}$ such winning selections. The probability is then

$$\frac{\binom{36}{6} - \binom{31}{6}}{\binom{36}{6}} = 1 - \frac{\binom{31}{6}}{\binom{36}{6}} = 0.622.$$

When the selections were made from 42 numbers, the probability of getting a successive pair was

$$1 - \frac{\binom{37}{6}}{\binom{42}{6}} = 0.557.$$

Now the selection is from 45 numbers so the probability of at least one pair of successive numbers is

$$1 - \frac{\binom{40}{6}}{\binom{45}{6}} = 0.529.$$

Exercise Show that if we select 6 numbers from the first 48 natural numbers, then the probability of at least one pair of successive numbers being selected is 0.503.

Mathematics Department,
University College Cork.

Comórtas Sóisearach Matamaitice Éireann 2008
(Irish Junior Mathematics Competition 2008)

This competition is organised by the Irish Mathematics Teachers Association (I.M.T.A.)

Eligibility First Year Students 2007/08

Format One set of question papers and answer key will be posted to each participating school some days before the competition date.

Each school will be responsible for photocopying the question paper and administering the First Round.

First Round Wednesday, March 12th, 2008 Time : 40 minutes

Final May 2008

The top students from the First Round may be invited to compete in the Final at venues to be arranged, ***provided a certain standard is reached.***

Entrance fee €25 **per school** (cheques payable to I.M.T.A)

If you wish your school to participate please return the completed Registration Form with the fee **no later than December 12th 2007**

Applications received after this date may not be accepted.

Applications should be sent to:

Michael D. Moynihan (Mícheál D. Ó Muimhneacháin)
Coláiste an Spioraid Naoimh,
Bishopstown,
Cork.

Phone : 087-2860666 / 021- 4870362 (Evenings)
email mmoynihan@eircom.net

Fax : 021 - 4543625

Registration Form 2007

Name of Teacher:_____

School:_____

School address_____

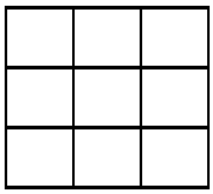
Phone number_____(Home)_____(School)

School Fax number_____ email address_____

Approximate number of students participating_____

:: LENGTH :: AREA :: VOLUME :: NUMBER

In about 1944 a diagram was chalked on the floor of the classroom to illustrate how area is measured:-



…this was to illustrate that there are 9 square feet in a square yard.

Today's Reader Today should be tolerant with this. As the Second World War was still in progress, the European Union had not got off the ground; there were 240 pennies in a pound and 30¼ square yards in a square perch – provided it was an English square perch and not an Irish square perch! (49 square yards in an Irish square perch).

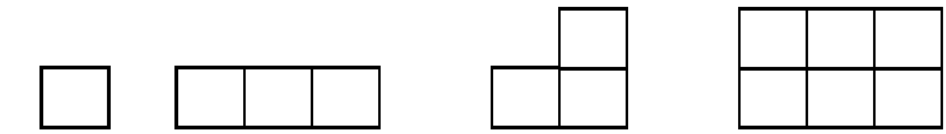
At least one member of that class could count 9 'panes of glass' making one 'window', but was puzzled by the number 9 measuring this physical configuration or shape. The concept of 'AREA' had not come across at all as yet. There were clearly nine squares with sides equal to one foot, and one square of sides of one yard each. But as a 'square foot' was thought to be four equal lengths in the shape of a square, there must be something underhand because most of the sides were used by two of the squares …

Area is a **number**, not a configuration of line segments or a conglomerate of points.
Length is a **number**, not a collection of points each having no magnitude.
Volume is a **number.** It is not a lot of plane figures squeezed together like plywood; it is not a shape, not a mass or a weight … actually a number!

Possibly the 30¼ square yards in a square perch has some advantages… It might help in the psychological transition to area as **number** from area as shape or as organized line segments.

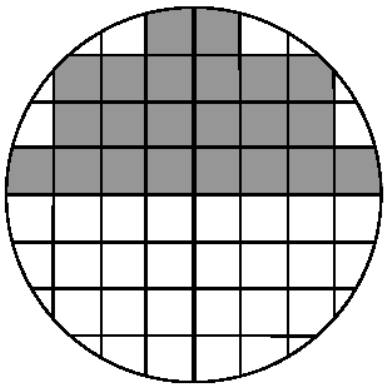
Time heals some wounds – if not all. Probably by the time one came to the geometry of 'parallelograms on the same base and between the same parallels', which begins with congruent triangles, and the usual proof of Pythagoras Theorem, where one is adding or subtracting parts of the figure – perhaps then, number as 'amount' of some of the diagram begins to assert itself.

Here are some figures of 1 'square unit', 3 'square units' and 6 'square units' …



'Square unit' seems to be an infelicitous expression. 'Unit of Area' is better. You might choose a triangle or a rectangle (with base and height unequal), or a hexagon as your basic unit of area. However, the main point just now is to show that an 'area' speaking mathematically, is a number, not a shape, not an arrangement of matchsticks or bits of line. Also, the area (number) is a multiple (or fraction) of the arbitrary unit of area.

This is a circle (or disc) with radius = 4. Counting the units of area on the upper half we find 8 + 6 + 6 + 2 = 22 units totally or 'almost totally' inside the circumference – these are shaded. Then we have 4 pairs of 'bits' which are still inside the upper half, and each pair seems to make up slightly less than one unit. So we might estimate the half area as 22 + 2 = 24, and the total area as about 48 ...
(This is 3 × 16 ... which is 'near enough' to 16π.)



Counting and adding in such ways as the above should bring **number** into prominence. The same 'amount' of area can apply to different shapes. One supposes that one was thinking numerically when the class learned from drawing rectangles what $(a + b)^2$ was equal to and, with cutting and pasting, that $a^2 - b^2 = (a + b)(a - b)$.

Yet, speaking for oneself, it was precisely the difficulty of thinking of area as a shape and not as a number ... that slowed up one's final satisfaction with the integral calculus! Even when one was reasonably comfortable with limiting processes, limiting values of fractions when numerator and denominator tend to zero, derivative as slope of tangent … the last fence to clamber, the last pothole on the road was: **Area is a number not a geometrical diagram.**

Before taking calculus as such, did you ever find occasion to ask your class:-
So you think area is got by multiplying length by breadth, do you?
So you think volume is got: length × breadth × height, do you?
Some of our clients believe that the unit of area pictured on the previous page, is a unit of area because the base is 1, the height is also 1 and 1 × 1 = 1.

This is just not so… the arithmetic is correct; the 'because' is wrong.
In the case of the square of unit side, the area = 1 is not a deduction but a definition!
(We can take it that if this unit square is moved about, or rotated or cut up into pieces, put together in some other way, NOT put together, but let lie there torn limb from limb, the area will remain one unit.
The last sentence can be taken as an axiom, one supposes. We do not think it is worth mentioning in a classroom… we are teaching honest-to-god Mathematics, not General Relativity, dammit!)

From the figure above of the six units of area we see how *Length x Breadth = Area* comes about. If we *define* (and it is a definition, not a deduction), that a unit of area is formed from a square of unit side, then two rows of three units of area must give us six units of area. This is how *Length x Breadth = Area* is deduced.

(Naturally if the unit of length is a perch, or a chain, or a furlong, the unit of area will have to be a square perch/square chain/square furlong. For the mathematician it does not matter whether the unit is an inch or an ell or seven leagues… the choice will matter for the physicist or geo-surveyor, of course.)

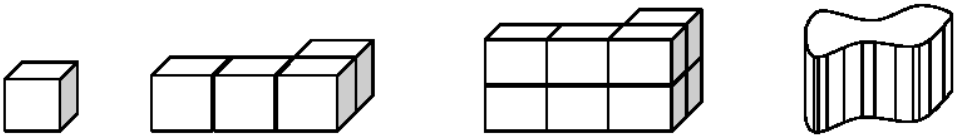
Some Psychologist/Educationalist – possibly Piaget or Binet – noticed from experiment that the concept of **volume** is not innate but must be learned. A child might see a mugful of water or fine sand

poured into a taller and narrower receptacle, and declare that the tall vessel contained 'more' water or sand than the cup did.

(How many students would be surprised to hear that one square inch/(square centimetre) could be the area of a rectangle with length equal to one mile/(kilometre))?

The unit of volume is *defined* – not deduced – to be the 'amount' of 'space' in a cube whose edge is one unit!

If we have 3 or 4 units of area in the base and the height is 2 units then the volume of our solid will be the number of units in the base multiplied by the number of units in the height.



Volume = 1
Base area = 1
Height = 1

Volume = 4
Base area = 4
Height = 1

Volume = 8
Base area = 4
Height = 2

Volume = 8
Base area = 4
Height = 2

In the 3rd figure the volume is twice that of the second. If the height in the 3rd case were 10 times the height of the second, the volume would be 40.
The last figure is a 'prism' or 'cylinder'. The sides rise vertically upwards. The area of the base is supposed to be 4 – however it is calculated – and the volume would be equal to the volume of the 3rd figure: Area of Base multiplied by the height.

Integral Calculus

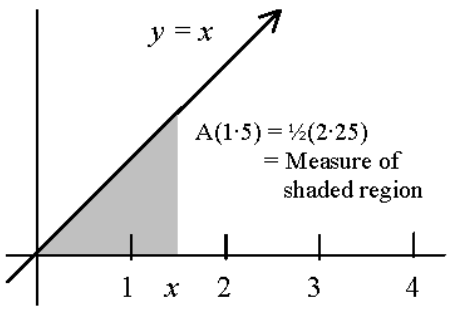
To find the area of the base in the last figure above will require some sort of limiting process. Sometimes it need not be necessary to use the Newton/Leibniz calculus for this as Archimedes did it without our algebraic notation. (See, for instance, '*100 Great Problems of Elementary Mathematics*', Dörrie, which was published by Dover Books). For the areas under the graphs of $y = x$, $y = x^2$, $y = x^3$, ... $y = x^n$, and so for any polynomial in x, it is possible to get an expression for rectangles 'under' and 'over' the curve and to see what is the limit of their summed areas as the base of each rectangle $\rightarrow 0$ and the number of rectangles increases without limit. To do a little of that would not be a waste of time, but it will hardly fit into this present submission to 'Newsletter'.

If we start with the graph of any straight line the numerical value of the area between it and the x-axis in any interval $a \leq x \leq b$ can be written down using 'ordinary' plane geometry.

The simplest case is the area under $y = x$ starting at the origin.

This function of the area to any value of x can be called $A(x)$... and, by elementary geometry, not calculus, $A(x) = \frac{1}{2}x^2$ … the measure of area to any x.

For any value of x to the right of the origin this gives a **number** (not a 'shape') for the value of the area up to x.



$A(x) = \frac{1}{2}(x^2)$

At $x = 3$, $A(x) = \frac{9}{2}$ which is the well-known:- half the base by the height.

At $x = 3.01$, $A(x) = \frac{9.0601}{2}$ … $\Delta A = ½ (0.0601) = 0.03 + 0.00025$.

$\frac{\Delta y}{\Delta x} = 3 + 0.025$ at $x = 3$ and $\Delta x = 0.01$.

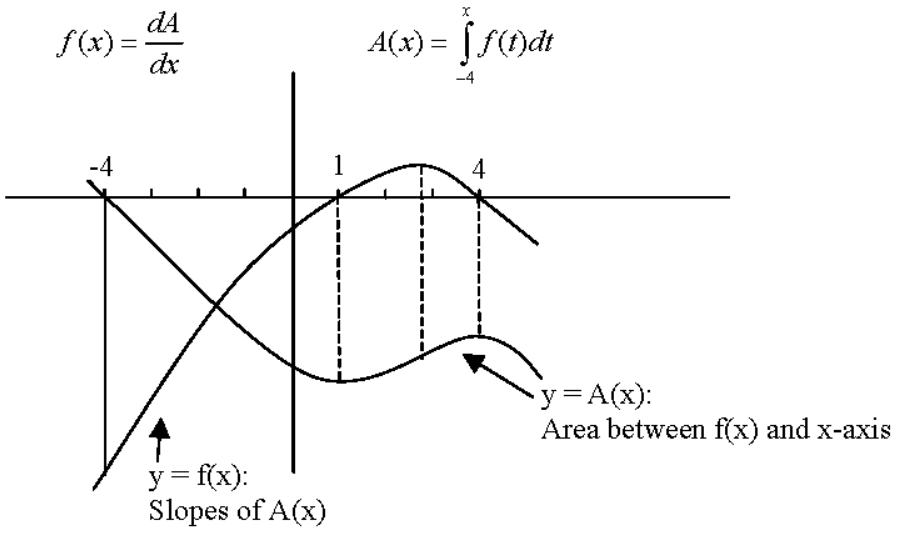
We can take smaller values of x and get closer to the value 3…
At $x = 5$ we come closer to 5 … etc.

The derivative at any x comes after some numerical values. It is, in this case, x itself – the value at the last ordinate of the curve in every case…

The Graph of an Area

Our clients quite readily become familiar with $\frac{d}{dx}(5x^3 + 2x^2 - 7x + 1) = 15x^2 + 4x - 7$ which is called the 'derivative' or 'derived function'. However, it seems important to bear in mind that the derivative is defined first at a fixed value. Only afterwards is it expressed in a formula like the above for all values of x.

Classroom discussion, (now called '*discernment*'??!) on the superimposed graphs of an area function, A(x), and its derivative, f(x), is worth while…
If the teacher wishes to do such an exercise, a few times in the year, it is advisable to prepare it beforehand. The clientele will get confused, and the teacher should try not to!
The *discernment* should be interesting, enlightening… even fun!



FROM ONE POINT OF VIEW: -- A(x) is the area between f(x) and the axis measured from x = -4 onwards. Since it is a function of x it can be graphed. A(-4) = 0.

As x progresses from –4 to +1 the area due to f(x) is negative and decreasing. Graph of A(x) is below the axis and getting lower.

Before x = 1, negative area is being accumulated by f(x)… Positive area comes after x = 1. A(x) decreases before x = 1, increases after… A(x) is stationary at x = 1.

From 1 to 4 we get some positive measure of area 'under' f(x) … A(x) increases.
Area is increasing just before x = 4, then decreases → Stationary for A(x) at x = 4.

FROM ANOTHER POINT OF VIEW: - f(x) is a graph of the various values of the slope of A(x).
When A(x) is decreasing, that is $-4 < x < 1$, and $x > 4$, f(x) is negative.
When A(x) is stationary, at 1 and 4, f(x) = 0.
When A(x) is increasing, in $1 < x < 4$, f(x) is positive…

All such discussion and discernment would reinforce that A(x) is a series of numbers →→ at each stage it measures the accumulated area as far as x. A(x) is a function of x and is graphed here on the Cartesian plane.

The discernment could proceed further, in 0 to 1 f(x) is generating negative area more slowly than it was at –4; this is why the area function A(x) is going down more slowly just before x = 1 than it was at x = -4. The different starting points for measurement of the area 'under' f(x) will give the same curve A(x) translated up or down.

Measure of triangle, pyramid, prism

It can be shown, using integration, but without using Cartesian geometry, that the area of any triangle is the celebrated ½×*base*×*height*. The mathematical tables used to say, rather blithely, that the volume of any cone or pyramid was one-third the volume of the prism on the same base and having the same height. --- thus the cone is one-third the corresponding cylinder… And it need not be a right circular cone. It could be on an elliptical base and it could be skewed off to one side, neither right nor circular. One may be permitted to discourse on these matters in a future submission to Newsletter.

Units again…

Gasoline pumps in the United States give volume in gallons, not litres. Normally, accounts of excavations or avalanches talk of 'cubic yards', not even cubic feet.

What became of square and cubic perches, or former friends? If one treats of the volume of the pyramid (base being any polygonal shape) in a future Newsletter, it should be in cubic perches for Old Time's Sake.

Brendan Kearney S.J., California, U.S.A. Quack!

FACTORING INTEGER QUADRATICS

An integer quadratic polynomial is an expression of the type $ax^2 + bx + c$, where $a \neq 0$, b and c are integers. Examples are $x^2 + 6x + 7$, $2x^2 - 7x + 6$, $-5x^2$, $25x^2 - 9$ and $3x^2 - 9x$. Young people in school spend a fair proportion of their time attempting to factor such quadratics i.e. express $ax^2 + bx + c$ as $(mx + n)(sx + t)$ for integers m, n, s and t.

It is not always clear to students precisely why such factorisation is desirable. The school textbooks I have consulted on the matter offer a bewildering variety of methods and rules for factoring integer quadratics into the product of two linear polynomials; however, all of these methods are somewhat experimental and decidedly hit and miss. What we need is an efficient algorithm – a constructive method in a few steps that will always solve the problem. Before we proceed, let us examine a few examples taken at random from textbooks.

Example 1 Factorise $x^2 - x - 56$

Now, $x^2 - x - 56 = x^2 - 8x + 7x - 56 = x(x-8) + 7(x-8) = (x+7)(x-8)$.
But teacher, why did you write $-x$ as $-8x + 7x$?
Because it works! And could we have a bit more respect and less cheeky questions from the back of the class please.

Students are justifiably mystified as to what decomposition of the middle term to choose to apply this method when faced with an unseen example.

Another line of reasoning often used is the following:
Clearly $x^2 - x - 56 = (x-p)(x-q) = x^2 - (p+q)x + pq$ where $pq = -56$ and $p + q = 1$.
Eliminating $q = 1 - p$ we get $p(1-p) = -56$ which simplifies to $p^2 - p - 56 = 0$.
But to solve this equation we need to be able to factor a polynomial that looks suspiciously like the one with which we started!
In fact, this 'method' is essentially inspection i.e. guess the answer.

Example 2 Factorise $6x^2 - 11x - 10$

Students are advised to try, by long division or maybe even the Remainder Theorem, all of the following possibilities for factors:
$x \pm 1$, $x \pm 2$, $x \pm 5$, $x \pm 10$
$2x \pm 1$, $2x \pm 2$, $2x \pm 5$, $2x \pm 10$
$3x \pm 1$, $3x \pm 2$, $3x \pm 5$, $3x \pm 10$
$6x \pm 1$, $6x \pm 2$, $6x \pm 5$, $6x \pm 10$

i.e. up to 32 potential factors of the type $px + q$ where $p|\ 6$ and $q|\ \pm 10$ until they hit the jackpot which in this case is $(3x + 2)(2x – 5)$.

This is intolerable and more suited to the mentality of a computer than a mathematician. With no disrespect to Gauss, we submit that it is not very efficient, although it does always work if we make minor adjustments for quadratics with zero constant term.

Another method used is the so-called 'guide number' method which works as follows. Clearly, $ax^2 + bx + c$ has integer factors if and only if $a(ax^2 + bx + c) = a^2x^2 + abx + ac$ has integer factors. Putting $y = ax$, this expression becomes $y^2 + by + ac$, and to factorise this expression, we need only consider factorisations of the type $(y + p)(y + q)$, where p and q are integer divisors of ac. So, for example, to factorise $6x^2 + 19x + 10$, we consider $y^2 + 19y + 60$ which factors as $(y + 4)(y + 15) = (6x + 4)(6x + 15) = 6(3x + 2)(2x + 5)$ giving the factorisation of $6x^2 + 19x + 10$ as $(3x + 2)(2x + 5)$. While this method has some advantages, it is still experimental, requiring a trawl through all the positive and negative integer factors of ac.

The purpose of this note is to give a quick and efficient method of deciding when a given integer quadratic has integer factors and, when it has, to find them in a few steps. It involves a little healthy calculation and nowadays that can be another advantage.

It is based on the fact that $(x + s)^2 - r^2 = (x + s + r)(x + s - r)$ for all rational numbers s and r, a fact that involves no mysterious operations and is easily verifiable by multiplying out both sides.

First, we demonstrate the method with a specific example.

Example 3 Factorise $12x^2 – 7x – 12$

Now, $12x^2 – 7x – 12 = 12(x^2 - \frac{7}{12}x - 1)$.

Let us see if we can write $x^2 - \frac{7}{12}x - 1$ in the form $(x + \alpha)^2 - \beta^2$, for some rational numbers α and β.

We get $x^2 - \frac{7}{12}x - 1 = x^2 + 2\alpha x + \alpha^2 - \beta^2$ for all x so that $2\alpha = -\frac{7}{12}$, forcing $\alpha = -\frac{7}{24}$.

Next, $\alpha^2 - \beta^2 = -1$ so $\beta^2 = \alpha^2 + 1 = (-\frac{7}{24})^2 + 1 = \frac{49 + 24^2}{24^2} = \frac{625}{24^2} = \frac{25^2}{24^2}$ (*)

Thus $x^2 - \frac{7}{12}x - 1 = (x - \frac{7}{24})^2 - (\frac{25^2}{24^2}) = (x - \frac{7}{24} - \frac{25}{24})(x - \frac{7}{24} + \frac{25}{24}) = (x - \frac{4}{3})(x + \frac{3}{4})$.

So $12x^2 - 7x - 12 = 12(x^2 - \frac{7}{12}x - 1) = 12(x - \frac{4}{3})(x - \frac{3}{4}) = (3x - 4)(4x + 3).$

We remark that if at stage (*), β^2 does not turn out to be the square of a rational number, we may stop the process because the given integer quadratic does not have integer linear factors, as in the following:

Example 4 Factorise $x^2 - 3x + 1$

If $x^2 - 3x + 1 = (x + \alpha)^2 - \beta^2 = x^2 + 2\alpha x + \alpha^2 - \beta^2$,

then $2\alpha = -3$ so $\alpha = -\frac{3}{2}$; $\beta^2 = \alpha^2 - 1 = \frac{9}{4} - 1 = \frac{5}{4}$.

Now, $\frac{5}{4}$ is not the square of a rational number so no factorisation of the desired type exists. In fact, since the denominator of β^2 is an integer square, we can draw this conclusion from the fact that 5, the numerator of β^2, is not an integer square.

Here is a more demanding example numerically – however, our method is far quicker and more reliable than any of the experimental methods already outlined:

Example 5 Factorise $27x^2 - 15x - 112$

Now, $27x^2 - 15x - 112 = 27(x^2 - \frac{5}{9}x - \frac{112}{27})$

If $x^2 - \frac{5}{9}x - \frac{112}{27} = (x + \alpha)^2 - \beta^2$, then

$2\alpha = -\frac{5}{9}$ so $\alpha = -\frac{5}{18}$ and $\beta^2 = \alpha^2 + \frac{112}{27} = \frac{25 + 1344}{18^2} = \frac{1369}{18^2} = \frac{37^2}{18^2}$

Thus the given expression has integer linear factors and these are given by

$27(x - \frac{5}{18} + \frac{37}{18})(x - \frac{5}{18} - \frac{37}{18})$

which simplifies in a few steps to the desired answer $(9x + 16)(3x - 7)$.

Of course we freely admit that lurking in the undergrowth is the famous quadratic formula for the solution of a quadratic equation, but students are expected to factor integer quadratics long before they encounter the quadratic formula. Indeed if we derive the general result for the method proposed in this note, we get a very natural motivation

for the quadratic formula and a novel connection between it and the problem we are discussing.

So let ax^2+bx+c be an integer quadratic polynomial with a ≠ 0.

Writing $ax^2+bx+c = a(x^2+\frac{b}{a}x+\frac{c}{a}) = a[(x+\alpha)^2-\beta^2]$, we get

$\alpha = \frac{b}{2a}$ and $\beta^2 = \alpha^2 - \frac{c}{a} = (\frac{b}{2a})^2 - \frac{c}{a} = \frac{b^2-4ac}{(2a)^2}$

If b^2-4ac is not the square of an integer, we stop, because this implies that ax^2+bx+c has no integer linear factors. If $b^2-4ac = h^2$ for some integer *h,* then $ax^2+bx+c =$ $a(x+\alpha+\beta)(x+\alpha-\beta) = a(x+\frac{b+h}{2a})(x+\frac{b-h}{2a})$ which when simplified will still give the desired factorisation.

If we follow up the mysterious fact that although we appear to have only one term 'a' to cancel with two '2a' terms in the denominators, the final factors will have all integer coefficients, we will soon arrive at the wonderful lemma of Gauss, already referred to.

To summarise: All factorisations of integer quadratics into the product of two integer linear factors can be derived from the factorisation $(x+s)^2-r^2 = (x+s+r)(x+s-r)$ for some rational number *s* and *r*. Moreover, an integer quadratic ax^2+bx+c has integer linear factors if and only if $b^2-4ac = h^2$ for some integer $h \geq 0$.
Under this condition the integer linear factors are given by
$ax^2+bx+c = a(x+\frac{b+h}{2a})(x+\frac{b-h}{2a})$.

So, without going through the derivation of this result each time, here are some examples of the lazy person's method of factorisation:

Example 6 Factorise $2x^2+7x-8$

Here, $b^2-4ac = 49+64 = 113$, which is not the square of an integer, so no factorisation of the desired type exists.

Example 7 Factorise $2x^2-5x-12$

Here, $b^2-4ac = 121 = 11^2 = h^2$, so we proceed.
The factorisation is given by

$$2(x+\frac{-5+11}{4})(x+\frac{-5-11}{4}) = 2(x+\frac{3}{2})(x-\frac{16}{4}) = (2x+3)(x-4)$$

We remark that in writing ax^2+bx+c as $a((x+\frac{b}{2a})^2-\frac{b^2-4ac}{(2a)^2})$, where $a \neq 0$, b and c are real numbers, quickly leads to the following results without the use of calculus:

(i) If $a>0$, the minimum value of ax^2+bx+c is $\frac{4ac-b^2}{4a}$ which occurs for $x=-\frac{b}{2a}$

(ii) If $a<0$, the maximum value of ax^2+bx+c is $\frac{4ac-b^2}{4a}$ which occurs for $x=-\frac{b}{2a}$

(iii) If $b^2-4ac>0$, the maximum or minimum value of ax^2+bx+c lies on the perpendicular bisector of the line segment joining the zeros of $f(x)=ax^2+bx+c$.

Clearly, the methods of this note can easily be adapted to cover factorisations of quadratics of the type $ax^2+bxy+cy^2$ for integers $a \neq 0$, b and c.

Des MacHale
Department of Mathematics
University College Cork
Ireland

A Demonstration of Central Point Symmetry

A playing card (but not all of them) gives a very useful practical demonstration of central point symmetry. Stick a pin in the centre of the card and spin it around to see the images interchange:

Neil Hallinan

'Let there be light!'
The Professional Development of the Mathematics Teacher

Every so often we see the letters 'CPD' appear in one context or another, generally muted as the hallmarks of a profession. The professional status of the secondary school teacher was questioned a few years ago and many members of the public suggested that teaching was a vocation. While this was firmly rejected by teachers and cogent arguments against this view were put forward, it is a sad fact that we are still lacking many of the requisite features that should be associated with our profession.

Continuing Professional Development (CPD), as the title suggests, is a continuous process of personal and professional growth. The aims are to improve the capability and fully realise the true potential of professionals in the work place. The means of achieving this is by gaining and developing a wide range of skills, knowledge and experience, which are not normally obtained during pre-service training or the day-to-day work routine. Only by attaining this plethora of skills can one maintain enough competence to practice.

The world might rotate slowly but it is forever changing and that pace can often be relentless. New technology, new methods of working, legislative changes such as the Education Act and the effects of a changing society all place added strains and pressures on teachers in the classroom. We have been given a wide range of responsibilities that have greatly extended the amount of time invested in our work. Given this unrelenting maelstrom, there is a real need for constant updating of knowledge and skills in order to maintain professional competence. We must also be conscious that Irish education policy espouses the promotion of lifelong learning within society and it is a remit that persistently falls onto the plate of the teacher.

At the moment, In-service training is the most common answer that is offered to the needs of the teacher for CPD. This effectively is in response to curriculum change. We must keep in mind that it is of a short-term duration usually lasting a school day and it concentrates on knowledge content and teaching methodologies that are confined to areas of the curriculum. Our most recent example is the Junior Mathematics Support Service that provides on-going professional support for teachers of Junior Certificate Mathematics. We have recently seen the Department of Education and Science extend their services into the current school year. However, CPD goes far beyond this and really looks towards teacher learning. Within the CPD framework, the teacher is viewed as a lifelong learner and it promotes critical reflection while encouraging teachers to improve their knowledge base in the areas that they have identified as lacking.

On a more personal level, a number of mathematics teachers from the Cork area have just completed a two-year interdisciplinary postgraduate programme in Mathematics and the Teaching of Mathematics, entitled the MA in Curriculum Studies (Mathematics). Only offered once before, it was delivered by the Education Department at UCC and the UCC school of Mathematics, Applied Mathematics and Statistics. It consisted of six units of mathematics and two units concerning the teaching of mathematics each year. A unit consisted of twenty-four hours of lecture time. Originally thirty teachers expressed interest in the course and twenty-one took up the challenge. By the end of the first year, this number had been whittled down to twelve. It demonstrates the long demanding journey that the course represented. Each Wednesday from three until nine and every Saturday morning from ten until one, teachers made their way from as far away as Waterford and Kilkenny to attend lectures at the university. It also shows the commitment shown by the lecturers who were willing to put together such an extensive programme and work such anti social hours!

Projects and assignments were an integral part of the course and the summers were spent studying and preparing for examinations in August of the following year. So essentially it was a full two years of study. Everyone started off attending lectures in September and completed the year by sitting examinations in August! It could be said that we were given a chance to experience what we expect of our students each year! Despite this gruelling timetable, the teachers embraced the course with vigour and they enjoyed the new kinds of mathematics that were presented to them as well as engaging in deep consideration of how mathematics should be taught and how best the problems that are inherent in the current system could be approached. They received their MA degrees in December of last year and I was proud to be part of this company of dedicated professionals. However, it must be remembered that the course was funded by the teachers themselves and like all postgraduate degrees, it made a significant dent into the savings of individuals.

The only real encouragement at the moment from the Department of Education and Science for these courses is the increased allowance given for honours master's degrees. Given that most people entering the profession now have an honours degree and very often an honours master's degree in one of their subject areas, this is not much of an inducement. As far back as 1991, there have been calls for a proper system of CPD to be set up. The *OECD Review of Irish Education*, in the same year, criticised the lack of coherent and structured planning for teacher professional development at departmental level. While there was commitment shown in the 1995 White Paper 'Charting Our Education Future', we are still awaiting the emergence of a strategic policy in response to the ever-growing need for continuous professional development for teachers. While the recent budget does assign funds towards CPD, it is such a small amount that it suggests that we will be waiting longer than eleven years to see a proper policy for CPD introduced. We have hope with the foundation of the Teaching Council and its intention, as its director Áine Lawlor put it, to establish criteria and procedures relating to the education and

UCC graduations: The class of the MA in Curriculum Studies (Mathematics)

Front Row: Pat Piggott, Alicia Cussen, Brendan O'Sullivan, Eric Graham, Seamus Knox

Back Row: Joe Shanahan, Brian Murray, Tim Hanley, Bernice O'Leary, Tom O'Connor, Tom Carroll (Lecturer in Mathematics), John Paul Sheehan

Absent from Picture: Joe Kennedy, Donal Hurley (Course Organiser)

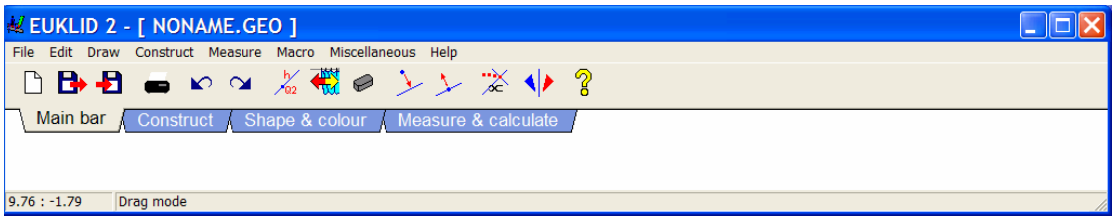
training of teachers, including their continuing professional development.

Mathematics teaching today is not the same as it was twenty years ago, the curriculum has changed and it has been tweaked this way and that. The role of the teacher in the classroom has undergone major extension and new pressures and responsibilities have been placed on our shoulders. Despite this, teachers have risen to the occasion and answered the call made on them. More change can be expected and it is slowly creeping up on us. Back in November 2005 the National Council for Curriculum and Assessment launched 'Everybody Counts'. There is now wide consultation on mathematics in post-primary education taking place and we can expect new recommendations and proposals that will directly impact on our working lives. Change is not a bad thing in itself and very often it is for the best. However, it is only fair to expect that teachers should be provided with adequate support and resources to assist them in the adjustment to this change. Hence CPD should not be forgotten or provided in a halfhearted sense; instead there should be a proper system set up as soon as possible. As we are aware, mathematics teachers are among the most dedicated in the teaching profession and deserve every opportunity to achieve as much professional growth as is possible. Any gardener knows that certain conditions are required to achieve growth. Irish teachers have spent enough time shrouded in darkness, let there be light!

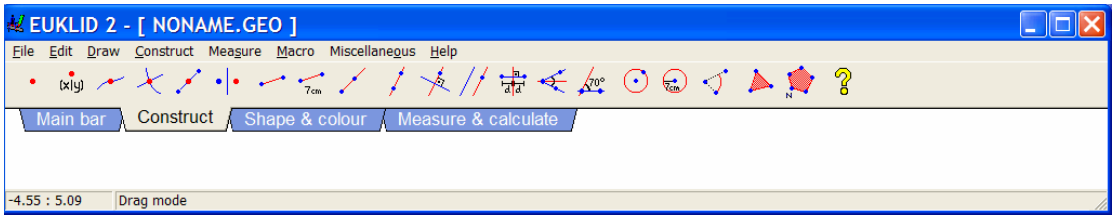
Brendan O'Sullivan, Cork

Dynamic Geometry: a comparison of Euklid and Geometer's Sketchpad

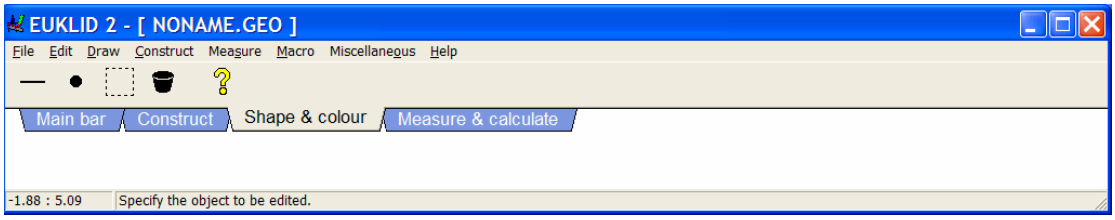
Euklid starts with an intuitive interface. There are four main Tabbed windows:



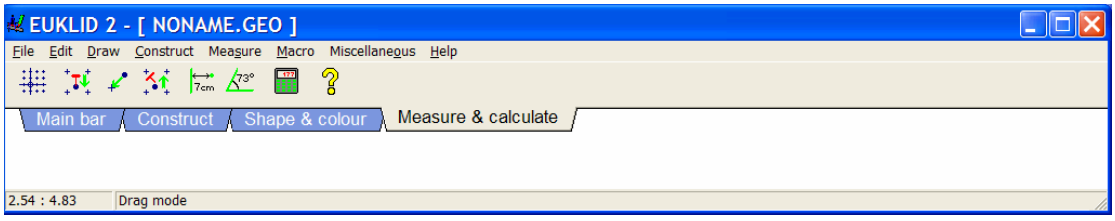
Main (shown above) controls the general management features



Construct (shown above) presents a menu of possible construction structures.



Shape and Colour (shown above) presents the options for colouring etc.

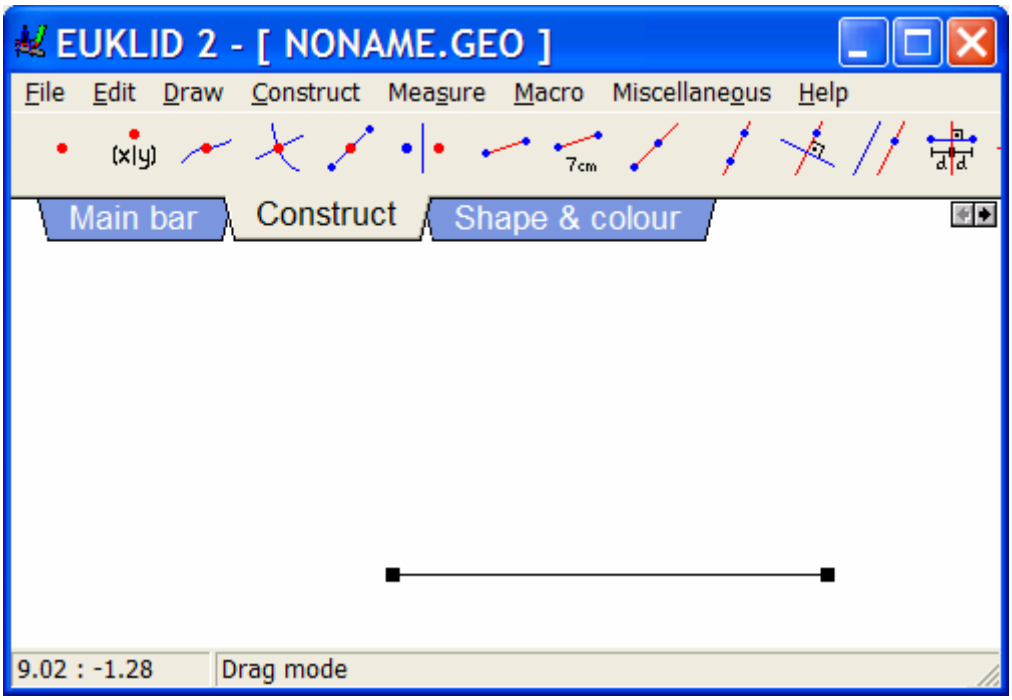


Measure and Calculator presents the options for measuring.

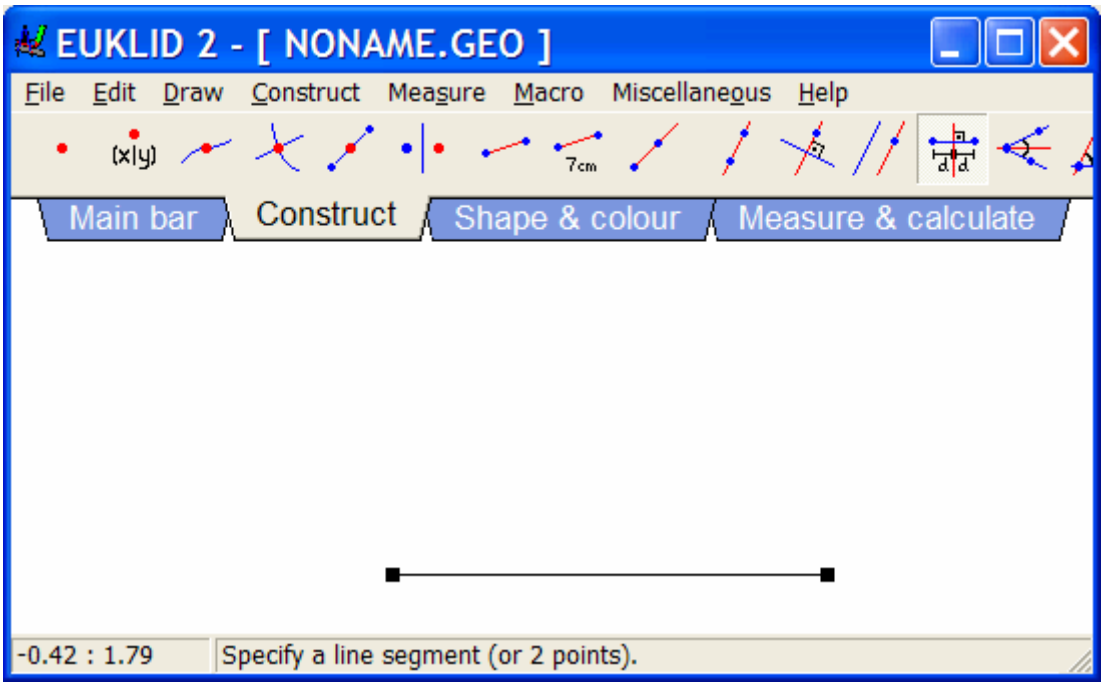
In order to evaluate Euklid, I will undertake two standard (for Junior and Leaving Cert. Courses) constructions.

Construction 1: Bisect a Line Segment

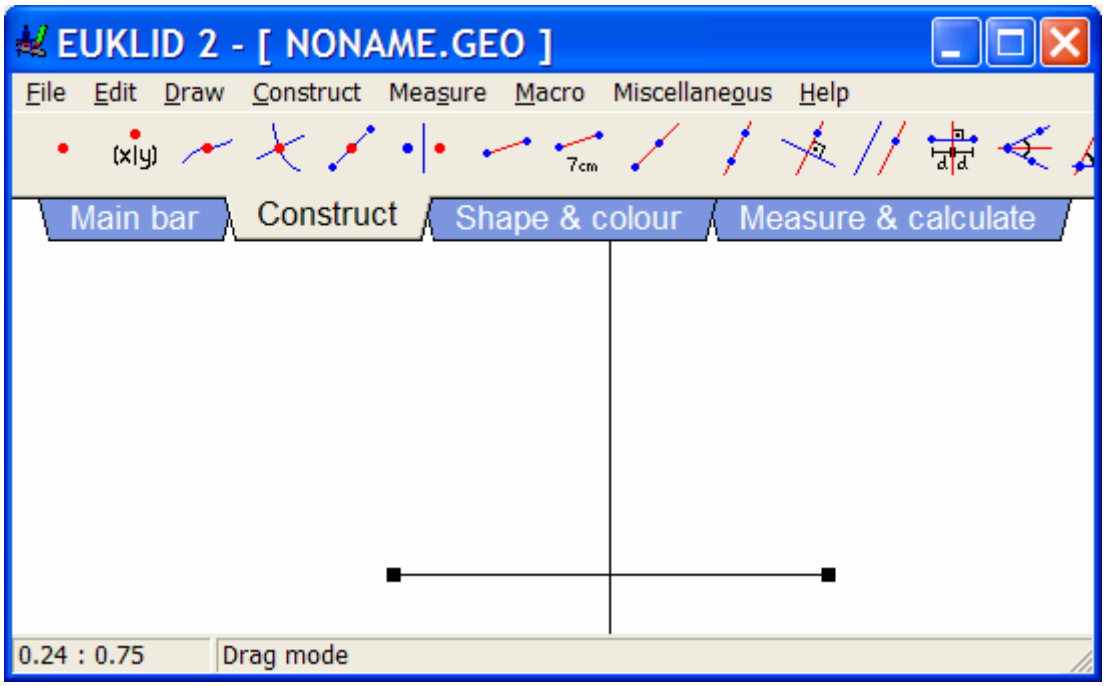
Euklid: From the Construct Tab, choose the button for 'Line Segment'; click at one position and at a second – the line segment is automatically drawn.



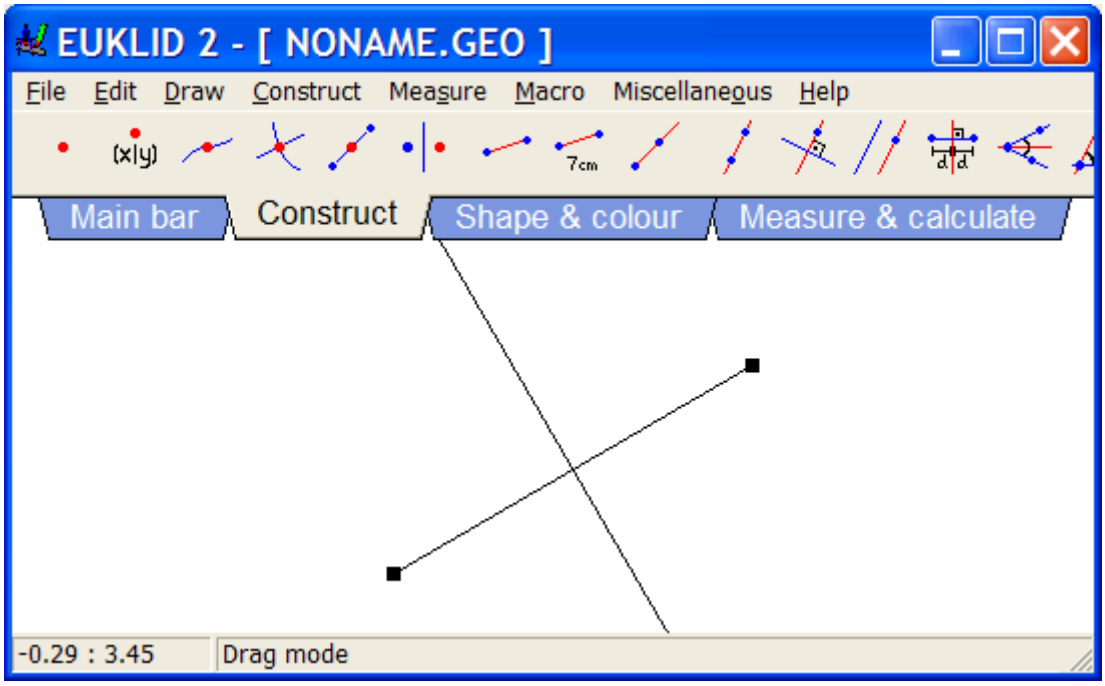
Choose the button for Perpendicular Bisector (shown clearly by the screen tip) …the Status bar suggests that a line segment (or two points) must be chosen:



Clicking anywhere along the segment results in the bisector being drawn:

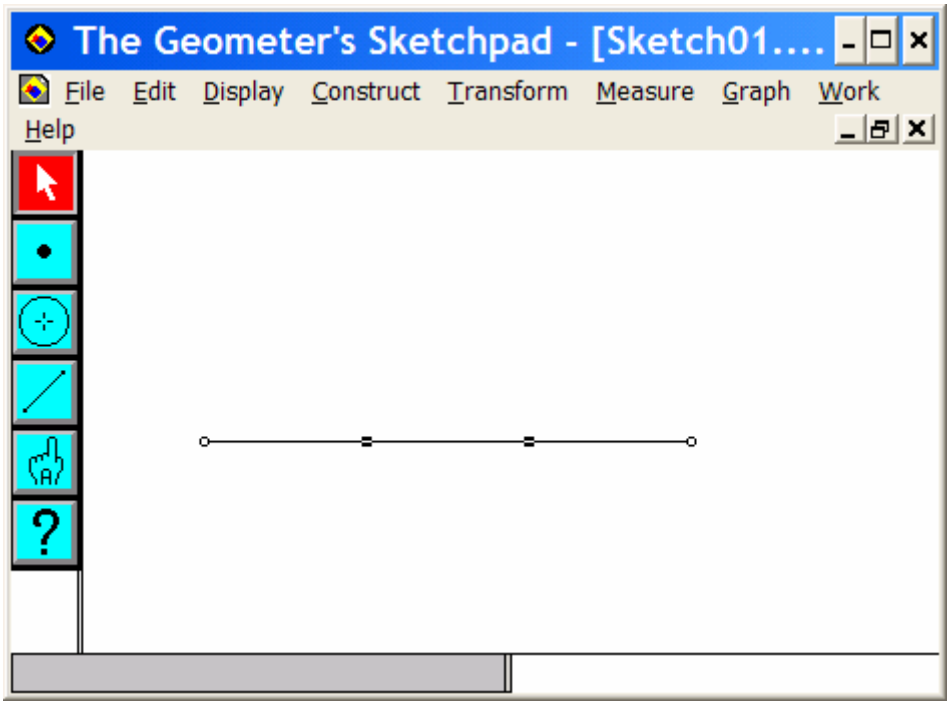


To see the interactive nature of the product, position the mouse pointer over an endpoint of the line segment – the cursor will change to a Pliers/Gripper shape – click and drag to a new position:

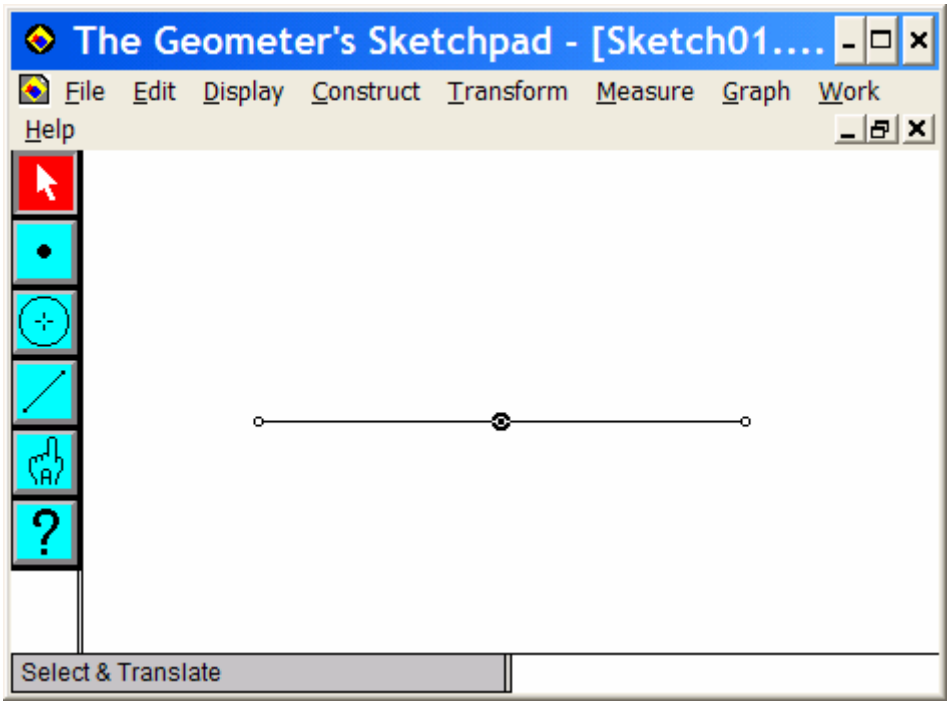


If a student is unhappy with the result the message from the Edit menu is “undo last construction”.

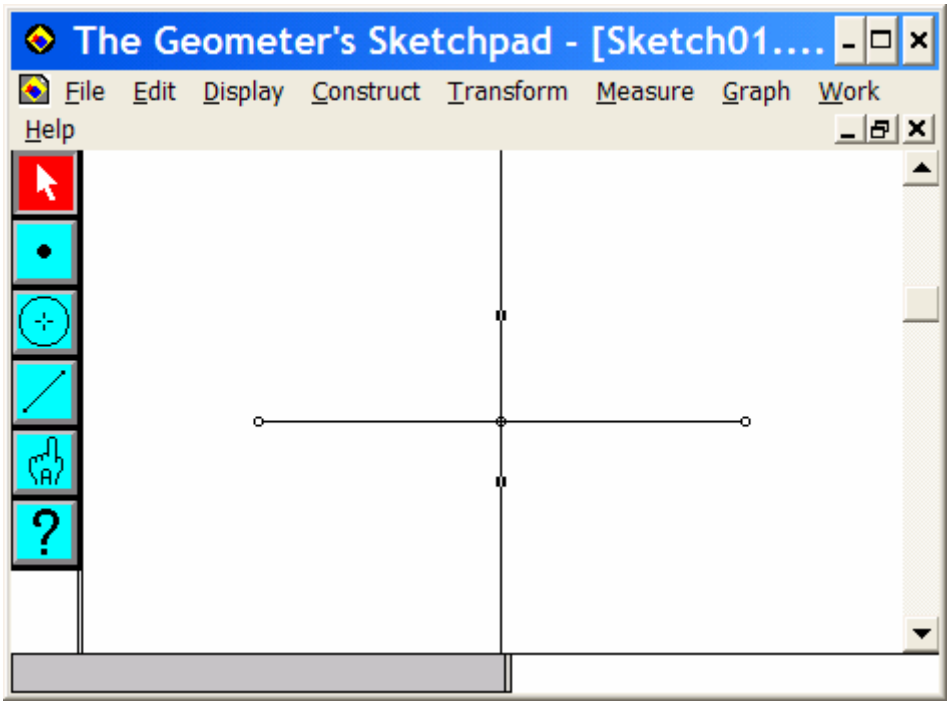
The finished work can be saved in native GEO format, or BMP, WMF or EMF formats
In Geometer’s Sketchpad: Choose the Line Segment button …click at one position …hold ...and click at the second position – the line segment is automatically drawn



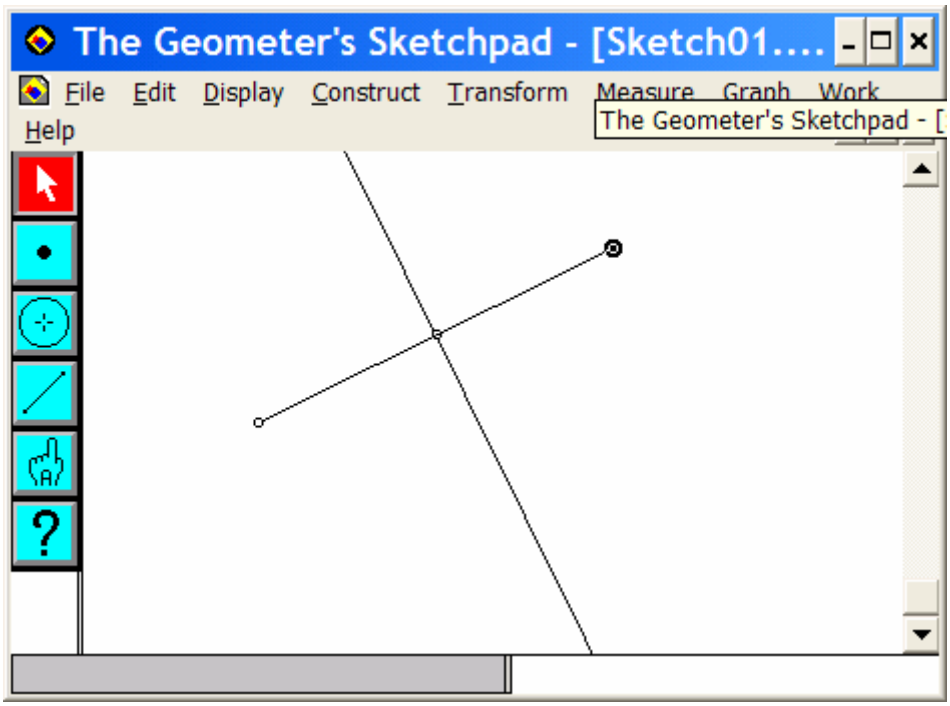
From the Construct menu, choose Point at Midpoint:



…and then select (by clicking on) the line segment, and also select (by holding Shift, and clicking on) the midpoint. With both items selected, you can then choose Perpendicular Line from the Construct menu.



To see the interactive nature of the product, position the mouse pointer over an endpoint of the line segment – the cursor will change to a Left Arrow shape – click and drag to a new position:



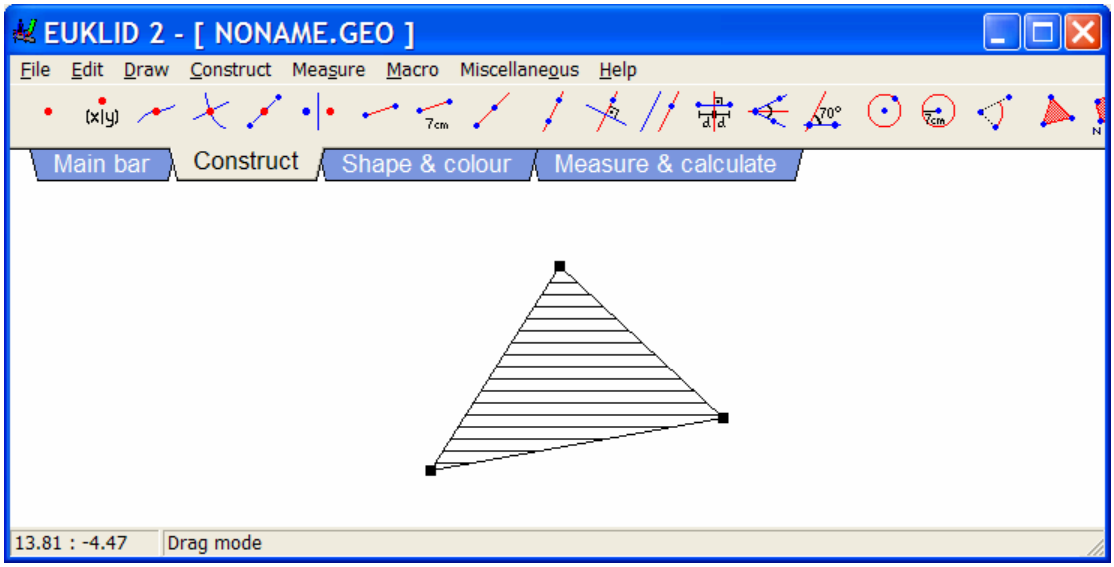
If a student is unhappy with the result the message from the Edit menu is “undo translate point”.

The finished work can be saved in native GSP format, or WMF formats.

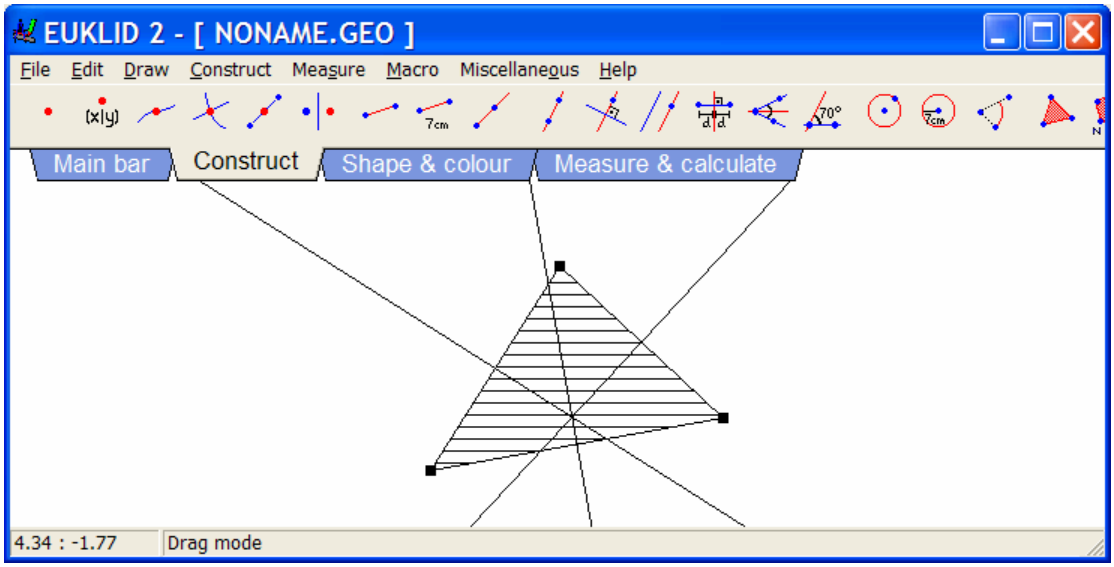
Construction 2: Construct the Circumcircle of a Triangle

Euklid: Choose File …New – you will then receive a request to Save the existing drawing.

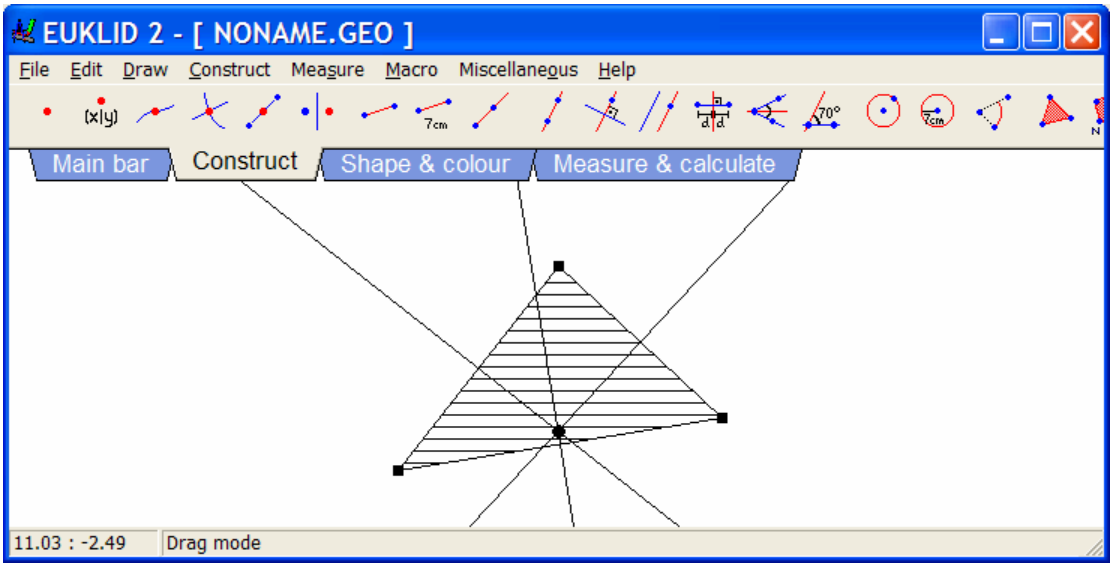
From the Construct Tab, click the Triangle button, position and click at three separate points – the triangle is automatically drawn (and shaded).



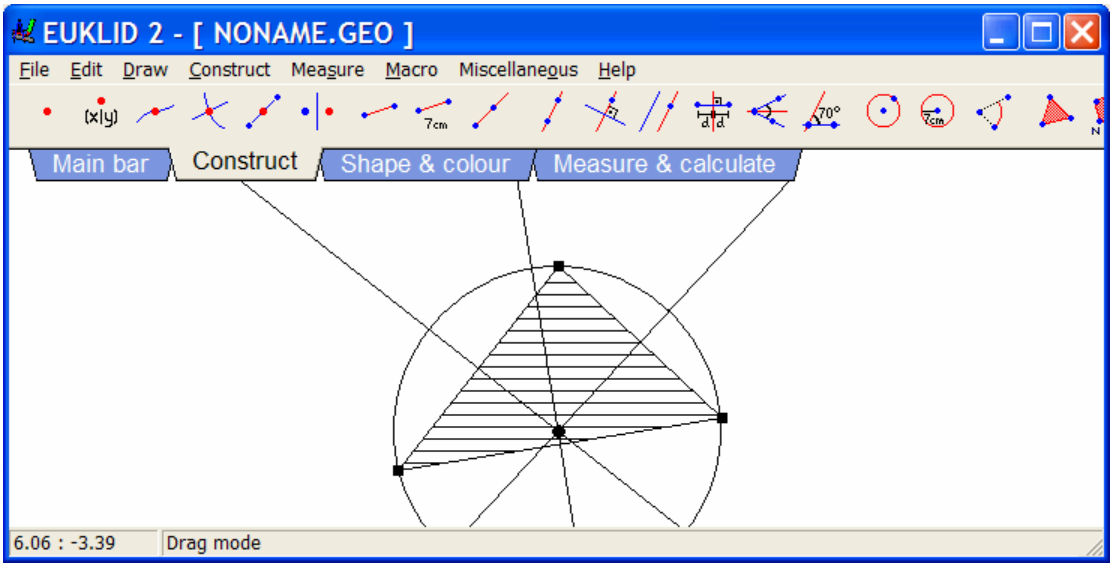
Click the Perpendicular Button …click on one side – the first bisector will be draw. Repeat the process for the other two sides:



From the Construct Menu, choose 'Intersection of two lines': you will be prompted (in the status bar) to specify two lines – click anywhere along two of the bisectors, and the intersection point will then be highlighted. Repeat for another pair of lines – the same point will be highlighted – this is the Circumcentre.



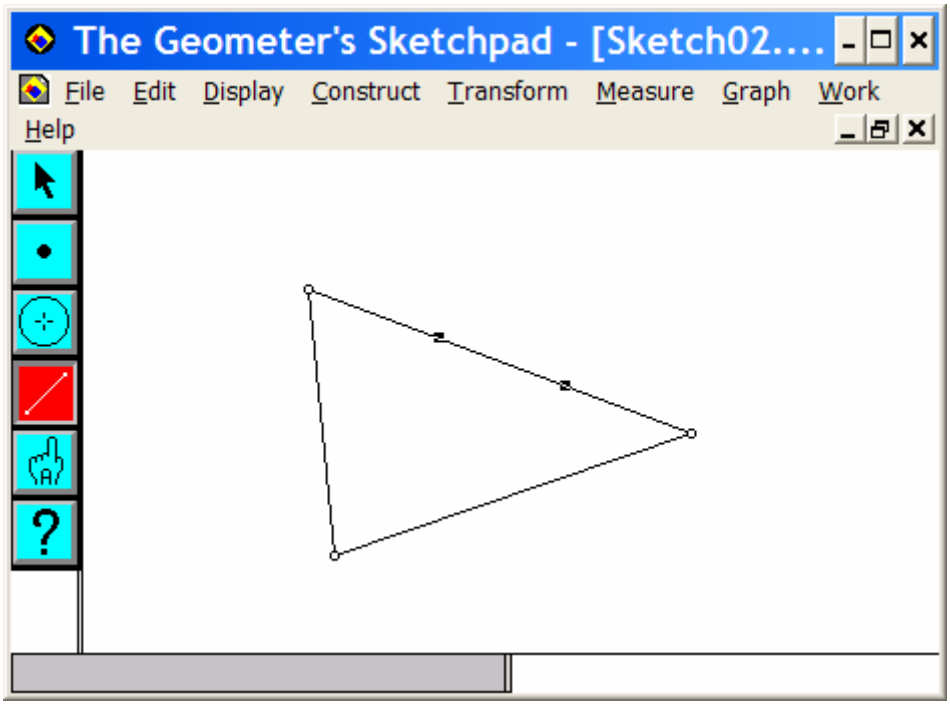
Click on the Circle Button: you will be asked (in the Status Bar) to specify a Centre and a Point. Click on the Circumcentre and any of the vertices: the Circumcircle will be drawn:



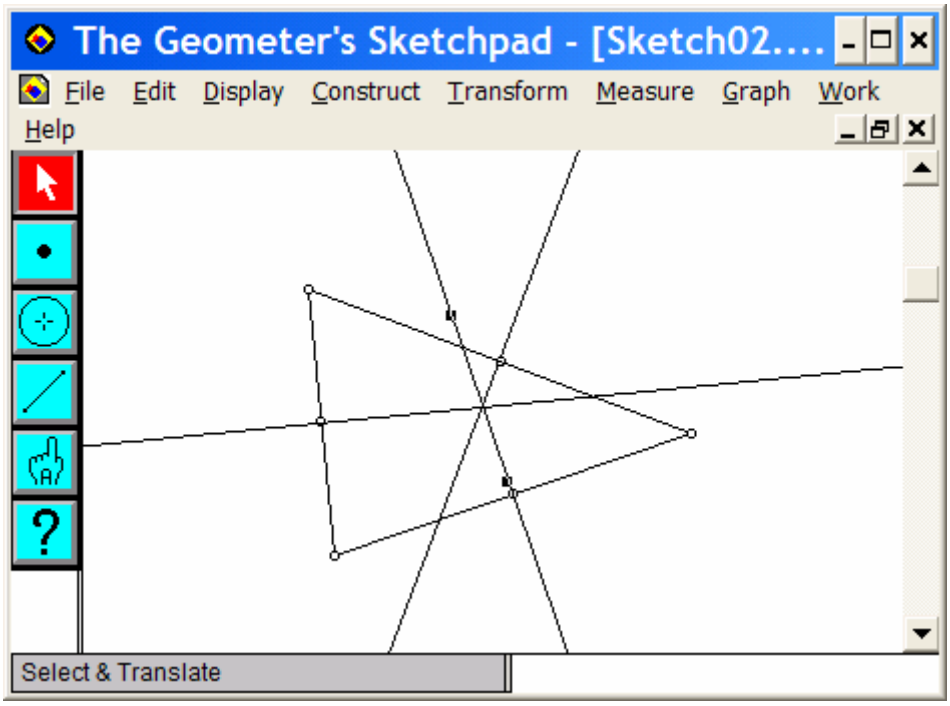
Sketchpad: Choose File …New – you will not receive a request to Save the existing drawing – your original work can later be viewed by switching Windows using the Work Menu.

To begin the construction, all items in the Construct Menu are all greyed out – leaving the novice user wondering how to begin.

The more experienced user will choose the Line Segment button: click, hold and move to complete one line segment, then, without moving the mouse from the last drawn point, click, hold and drag to draw the second, and, again without moving the mouse from the last drawn point, click, hold and drag towards the first-drawn point to complete the triangle:

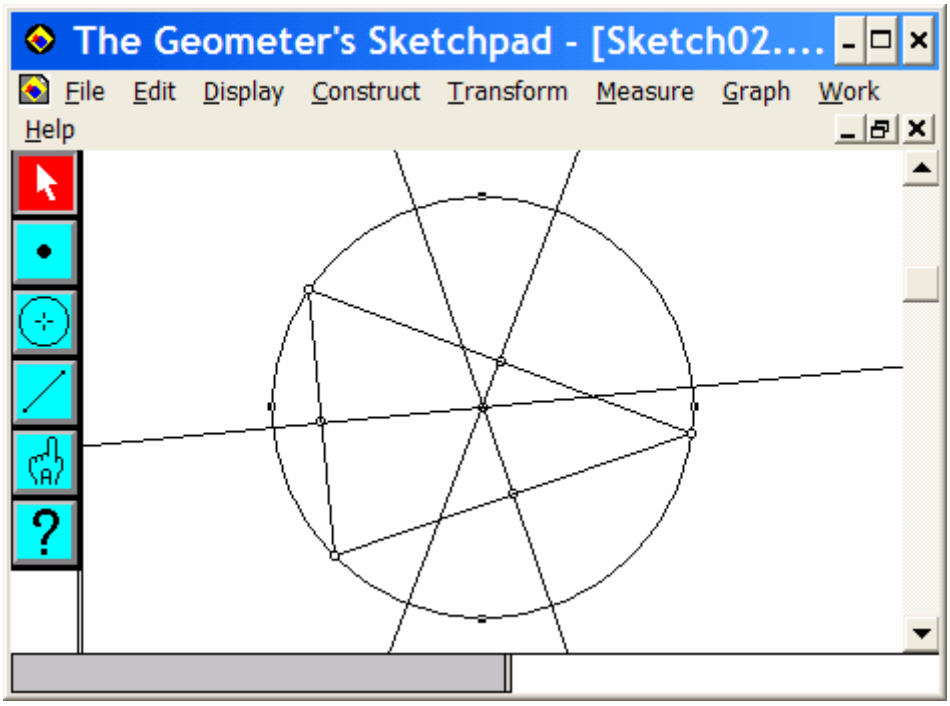


With a side highlighted (as shown by the two dots along the line segment) go to the Construct menu and choose (from the short list of greyed/available) 'Point at Midpoint' – the midpoint will then be created and highlighted …go back to the Construct menu and choose …no options will be available: you will have to return to the drawing and using the Shift method, highlight (provided that you have also switched back to Select mode as indicated by the Left Panel) both the midpoint and the line segment, then return to the Construct menu and select Perpendicular Line. Repeat to create the other two bisectors.



Select one bisector, and while holding Shift, select another. From the Construct Menu, choose 'Point at Intersection' and the intersection point will then be highlighted. Repeat for another pair of lines – the same point will be highlighted –this is the Circumcentre.

Click on this Circumcentre and, while holding Shift, click on one of the vertices. From the Construct Menu, choose 'Circle by Centre and Point': the Circimcircle will then be drawn.



Comparison:

Both Euklid and Sketchpad fully achieve their aim of bringing Interactive Geometry to the desktop. Sketchpad is probably the more powerful of the two – very advanced users can use scripting to animate a previous construction, but Euklid also has its Macro command

Euklid succeeds better for novice/younger users. Its construction options are presented in clearly labelled (with screen tips) and thumbnailed (visually represented) buttons.

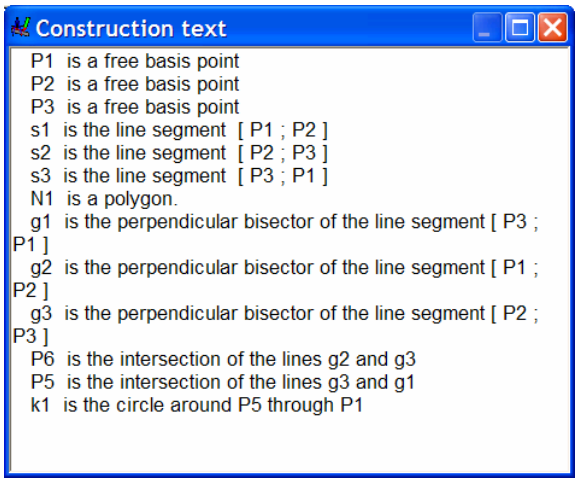
Euklid starts from a Task Orientation (e.g. a student is required to draw a Circle) – the user is then prompted to provide the subsidiary information (Centre and Radius).

Sketchpad puts the focus on the more traditional method: the student must envisage what subsidiary information is a pre-requisite: only when this is in place can the Circle be completed. This method enforces the rigour of the geometric method.

Again for younger users – and for visual learners who could be attracted to Geometry – the simple option to 'colour in' the final construction is easily accomplished in Euklid: switch to the Colouring Tab, choose your colour (from a 'traditional' Fill Can' button) and select the object to which it is to be applied. Meanwhile, in Sketchpad, you must select the three vertices (using the Shift method), then from the Construct Menu, choose 'Polygon Interior' – the triangle will then be filled in the current Fill Colour (this latter is chosen from the Display Menu …Color submenu).

The language – while still being technically correct – is more accessible in Euklid.

Finally, the work can be revised in Euklid by choosing the Miscellaneous Menu …Show Construction Text (while in Sketchpad, this can be achieved by recording a Script from the beginning of the construction process).



Euklid is a very useful dynamic geometry alternative. Its licensing is very attractive, both in terms of cost and in terms of structure (covering not only the school but also the homes of teachers and of students). It can be downloaded from the "English pages" of http://www.mechling.de (now at http://www.dynageo.com/)

David Kearney, Dublin

Mathematics Education Experiences

In February 2006, a questionnaire was administered to first year students in three institutions across the third-level sector. Some of the information collected from these students in relation to their achievements in, experiences of and attitudes to Mathematics and Mathematics Education is reported here.

Profile of Sample

146 students participated altogether: 36 of these were Civil Engineering students at the Institute of Technology, Tralee (ITT) studying Maths as a compulsory subject, 76 were B.A. students at National University of Ireland, Maynooth (NUIM) who have chosen Mathematical Studies as one of their subjects and 34 were B.Ed. or B.A. students at St Patrick's College, Drumcondra (SPD) who have chosen Mathematics as one of their academic subjects. Of the students who sat the Leaving Certificate Mathematics papers, half (71) sat Higher Level (HL) Mathematics and 63% of the overall sample was female (92 students). However, only 35% of the males surveyed took HL, compared with 58% of females.

Administration of Questionnaire

The questions to which students were asked to respond could be grouped into five categories: namely personal information, second-level experience of Maths, third-level experience of Maths, general attitudes to Maths and study habits. Responses were anonymous so that students would be more likely to answer frankly. It should be noted that the questionnaire was administered at the beginning of the second semester of first year when the students had only a limited experience of third-level.

Higher Level versus Ordinary Level

There has been debate recently about the numbers of students taking HL Maths at Leaving Certificate. We were interested in the reasons why students chose to study Maths at a particular level. We asked students to tell us at what level they studied Maths at Junior Certificate and at Leaving Certificate. We asked them for the reasons for studying at this level and if they had changed from HL to Ordinary Level (OL), we asked them when and why they changed.

As stated above exactly half the students surveyed studied Mathematics at HL for Leaving Certificate. When these students were asked why they had chosen to study HL Maths, 66% said it was because they liked Mathematics and 14% said it was because they were able to. Only 4% of

these students said they studied Mathematics with a view to a future career.

82% of the students surveyed took HL Mathematics for Junior Certificate. However, only 50% of the whole group sat the HL Leaving Certificate Mathematics papers. If points are awarded to Junior Certificate grades in a similar way to that by which the Leaving Certificate points are awarded (for example Junior Certificate grade A at HL was awarded 100 points), then the average drop in points from Junior to Leaving Certificate is 17.89 (or a whole letter grade). Half of the students who changed from HL to OL did so during 5th year. 41 students responded to a question asking why they had changed from HL to OL. The most popular answer to this question (32%) was that Maths was too time-consuming while 29% said it was because HL was too hard. One often hears students report that HL Maths is too time-consuming. It would be interesting to quantify how much time is needed to study HL Maths versus HL English for example.

Four NUIM students stated that HL was not available at their school. All four students scored OL A1 in Leaving Certificate. Three of these students were school leavers and one was male. This indicates that there are still some schools in the country where HL Maths is not taught. The Department of Education & Science was unable to provide statistics as to the number of schools in this position.

Grinds

We were interested to see if many students had had Maths grinds while at school and if these grinds had improved their grades. The percentage of students who had Mathematics grinds while at school was 27%, with most of these grinds (54%) undertaken in 6th year. Leaving Certificate grade and having grinds are not independent, with weaker students being more likely to have had grinds. When the Junior Certificate grade achieved by students is held fixed, their Leaving Certificate grade and whether they had grinds or not are independent, which suggests that grinds do not increase grades significantly.

Transition Year

We asked students if they had taken Transition Year (TY). In the group as a whole, 37% took TY and 63% did not. (However, we neglected to ask students whether or not TY was available at their school). We found that in our sample 46% of HL students took TY while only 29% of OL did. A question one could ask is whether taking TY encourages students to take HL at Leaving Certificate and tests gave some evidence that taking TY and taking HL maths are related. However, tests undertaken while holding the Junior Certificate grades achieved by the students fixed, give no evidence to suggest that taking TY improves the Leaving Certificate grade. There is also no significant difference between the proportions of male and female students taking TY.

The students were asked whether any Leaving Certificate material was covered in TY. 82% of

students who took TY said such material was covered. Topics covered included algebra, trigonometry, calculus, and probability. Some students mentioned that for their Maths classes in TY they were put into a 5th year class. Others mentioned that without the extra year of working on Leaving Certificate material, they felt that they would not have taken HL. Perhaps this is one reason why taking TY and studying at HL seem to be related.

Attitudes to Mathematics

When asked whether they found Mathematics difficult at school, 37% of students agreed while 49% disagreed. (The remainder were neutral.) There was no significant difference in the responses of males and females, or in those of HL and OL students. An overwhelming number of students reported Maths to be interesting (89%) and enjoyable (78%) at school. However, the responses of males and females were significantly different, as were those of HL and OL students. For instance, 96% of females and 78% of males found Maths to be interesting at school, together with 96% of HL students and 83% of OL students. Moreover, the differences observed were more pronounced when students were asked if they enjoyed Maths at second level: 90% of females and only 56% of males agreed, while 93% of HL students enjoyed Maths at school compared with 43% of OL students.

Confidence

Students were asked to rate, on a scale of 1 to 5, whether they felt confident in approaching Mathematics. Not surprisingly, HL students felt more confident than OL students with 61% of HL students but only 27% of OL students saying they were confident or very confident. In general, males were found to be more confident than females with tests showing that the odds of a female being confident are no more than 53% of the odds for a male with the same Leaving Certificate grade. For example, at OL A1 grade, 86% of males rated themselves as confident but only 25% of females did. We found no significant differences between the confidence levels of the students who had taken TY and those that had not.

Natural Ability

Students were asked whether it is possible to improve natural mathematical ability. 81% answered yes, 12% said no and 7% were unsure. When asked to comment, 63% said that practice and hard work can help improve ability, many of these believing such work entailed completing lots of examples and learning formulae. This attitude seems to stem from the view of Mathematics as a series of procedures rather than a series of concepts. A further 7% said ‘you either have it or you don’t’. However, many of the responses were very encouraging, with students commenting that it is always possible to improve any skill, or that an open mind or a willingness to challenge oneself can lead to improvements.

Transition to Third-Level

Students were asked to comment on how they found the transition from school Mathematics to Mathematics at third level. 56% of HL students reported that they had no problems in making the transition from second to third-level Maths, whereas only 25% of OL students could say the same. There is also a significant difference between the experiences of mature students and those of students having recently finished school, with mature students having more difficulties in general.

When asked about the differences between second- and third-level Mathematics, the majority of students (55% of those who answered) said the main difference was that more depth is required at third-level. When asked what they liked about third-level Mathematics, the most popular answer was the depth, understanding or content involved (28%), followed closely by its enjoyable, challenging or interesting nature (23%). Not surprisingly, HL students rated their third-level experience of Maths significantly higher than OL students, finding it more interesting and enjoyable and less difficult than was the case for OL students. One reason for this may be that many of the OL students are taking Maths at third-level as a compulsory subject.

Final Comments

The responses to this questionnaire have generated some interesting results and, indeed, many interesting questions. A PISA-style test was administered to the students in conjunction with this survey and the results of the test will be reported elsewhere. Furthermore we hope to conduct the study again in February 2007 with a new group of first year students. Any comments on the findings reported here or suggestions for future investigations of a similar type would be most welcome.

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The Role of Language in the Demise of Logs

1. Introduction

"You cannot teach logarithms to illiterates"
- "Stand and Deliver", Warner Brothers, 1988

Since June 2002 logarithm tables have not been used in public examinations in the Republic of Ireland, bringing to an end more than a century of association between log tables and school mathematics.

Thirty years ago *every* student who took the Leaving Certificate Examination was examined not only in the use of log tables but also in his/her knowledge of the properties of logs. The following is part of a "Pass" Level specimen question from 1970:

"$\log_{10} 2 = x$ $\log_{10} 3 = y$.
Express in terms of x and y, (i) $\log_{10} 6$, (ii) $\log_{10} 24$ (iii) $\log_2 3$ (iv) $\log_{10} \sqrt{8}$ "

In the years in between, the number of students staying on to take the Leaving Certificate exam has grown from 30% to 83% of the population cohort, and the range of ability has widened. This has necessitated the introduction of syllabi at three ability-levels, and only at the highest level of these do questions such as the one quoted above appear. The first year (second-level) class of September 2000 was the first to be allowed to use calculators, and the "theory" of logs was from that time removed from all three Junior Certificate syllabi.

Questions on logs were unpopular. The Chief Examiner's Reports for the Junior Certificate exam of 1996 and 1998, and the Leaving Certificate exam of 2000 show that logarithms were the least or second least popular question on the exam papers.

Some teachers themselves find logs difficult. A team of experienced U.S. textbook authors, Hornsby and Lial, conceded in 1999 in background material to their College Algebra book:

"Without a doubt, the concept of the logarithm is one of the most difficult for algebra students to grasp. The authors of your text admit that even they did not fully understand the concept until taking follow-up courses and teaching logarithms in their classes! So if you find this topic difficult, don't feel as if you're alone".

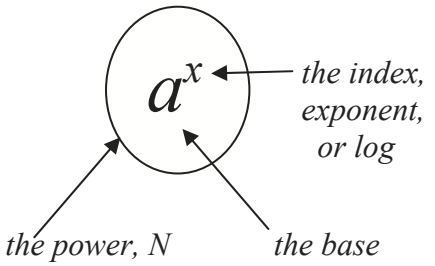
This article examines possible reasons for the unpopularity and difficulty of logarithms.

2. The Tradition

Consider how logs are introduced in textbooks.

Textbooks introduce logs in one of two ways – either algebraically or analytically. In the latter, the *base a exponential function* $y = a^x$ is firstly defined, and then $y = \log_a x$ ($a>0$, $a\neq1$) is defined to be the inverse of $y = a^x$ (Finney *et al.*, 2001).

The alternative, algebraic, treatment is more intuitive and is more likely to have been the mode experienced by school students up to the present day. In the tradition of Irish and British school-mathematics, Hall and Knight's *Elementary Algebra for Schools*, first published in 1885, is a seminal book. The eighth edition published in 1907 was, along with its companion text *A School Arithmetic* by Hall and Stevens, reprinted more than 30 times and remained with schools, in Ireland at least, all the way to the New Maths of the late 1960s.



Apropos what follows, note that the "2" in the symbol x^2 is known as the "index" in Irish and British schools. In other countries it is called the "exponent".

The Eighth Edition of Hall and Knight defines *log* as follows:

"The **logarithm** of any number to a given base is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, x is called the logarithm of N to the base a".

This is a correct definition and is illustrated here.

Hall & Stevens (1908) had a similar definition and they followed on with**:** "Since every logarithm is an *index*, it follows that the rules which govern the use of logarithms are deducible from the laws of indices".

But within a few years we had:

"The logarithm of a number to a given base is the power to which the base must be raised to equal the number" (Jones, 1913). Note the omission of "index".

And from the New Maths era onwards came:

"Logarithms are another form of indices…The logarithm of a number to a base is the power to which the base must be raised to give the number" (Holland and Madden, 1976)

"The logarithm of a positive number N to the base *a* is defined as the power of *a* which is

equal to N" (Bunday and Mulholland, 1983).

"The *logarithm* function is the inverse of the index function ″ (Solomon, 1997)

"**Logarithm** is another word for **power**, **exponent** or **index"** (Sherran and Crawshaw, 1998)

"Logarithms are mirror images of exponentials…the logarithms are the exponents" (Strang, 1992)

" $\log_b x$ is defined to be that exponent to which b must be raised to produce x" (Anton, 1999).

The intended meanings of the above definitions are the same. The language however is inconsistent. A logarithm is variously defined as an index, exponent, or power. While the first two of these are synonymous, they are not synonymous with the third.

Also, while a logarithm is indeed an exponent the adjectives "logarithmic" and "exponential" are given opposing (inverse) meanings.

3. Power

The greatest confusion surrounds the word "power". *Power* and the phraseology associated with it are so embedded in the language of algebra since the sixteenth century that they are difficult to deconstruct.

8 is a power. It is, among other things, the 3^{rd} power of 2.

The use of the ordinal, 3^{rd}, appears safe until we start "raising" things. When we read $8 = 2^3$ as "8 is 2 raised to the 3^{rd} power", "*raised to* " points to the superscript 3, as if *it* were the power as well as being the "exponent".

Repeated use of the cardinal, "8 is 2 to the power (of) 3" or "raised to the power 3" leads to the situation, common at present, in which *power* is used as another term for the exponent, and the original meaning of power as a product of copies of the base is ignored. Yet this meaning is used by the teaching profession on occasions such as "$x + x^2$+… higher powers of x".

Similarly, while $\log_2 M = x$ is correctly called the "log form" of the relationship between M and x, its inverse, $M = 2^x$, is often referred to as the "index or exponential form" (which should mean log form!) - when its correct name is "power form".

Because of the identification, in textbooks and teaching, of the concept of an exponent/index with the term "power" it is possible to deduce from terminology in current use that

logarithm = exponent = **power** = antilogarithm

and that

index form = inverse of log form = inverse of index form

The fact that students do not protest at these inconsistencies is not proof that they are comfortable with logs. The evidence points to the contrary. It is tempting to say that it is possible to teach logarithms *only* to illiterates.

A likely result of the inconsistency is that the students lose confidence in their ability to understand the logarithm concept, and settle for engaging with logs at the procedural level only.

4. Teaching Logarithms

In an effort to counter-act the inconsistency of the language the following is used by the author to teach logs to first year college students:

A basic revision of powers and indices is firstly given, along with substantial drilling in the "rules of indices". Students are then introduced to the terms used to describe the *power equation*

$8 = 2^3$, namely the power is 8, or 2^3, the base is 2 and index is 3.

"Logarithm" ("log" for short) is given as another, synonymous name for the index when the base is > 0 and not 1.

A natural-language description of $8 = 2^3$ is next transformed, progressively, to mathematical shorthand, like this:

$8 = 2^3$ **…**"power form"

The index of 8, when 8 is written as a power of the base 2, is 3.

The index of 8 when 8 is written to the base 2 = 3.

The log of 8 when 8 is written to the base 2 = 3.

log of 8 to the base 2 = 3.

$\log_2 8 = 3$ **…**"log form" or "index form"

Drill is then given in transforming directly between *power form* and *index form* equations. Subsequent work emphasises the identification of **log** with **index** by reinterpreting a subset of the rules of indices as rules of logs and giving drill-exercises in these. Finally logs are applied to solving equations, such as $A = P(1 + {}^{r}/_{100})^{n}$, that contain an unknown exponent.

In a recent attempt to measure the effectiveness of this approach a class was asked as part of a

Christmas assessment: Q1. What is a log? Q2. What are logs useful for?

Roughly half the class answered that a log is “an index or exponent” or, less often, “a power”. The answers to Question 2 showed that their knowledge of a *usage* for logs was not dependent on their being able to say what a log is. This might lead one to think that a student’s ability to parrot that “a log is an exponent/index” is of no great consequence. The opposite however is the case.

5. Can logs become popular?

The experience of the author in teaching both at second-level and third-level is that while logs will surely never move to the top of the popularity list in examination questions, they can certainly move off the bottom.

The language problem must first be solved. Napier's term *logarithmus* (logos+arithmos, a "reckoning number"), in the form *logarithm* or *log* does not offer an intuitive notion of the role that a log performs, in the way that, say, *index* (which 'points' to how often a base is multiplied by itself), *base*, and to an extent *power* do for their roles. A suggestion is made later for abandoning the term "logarithm" altogether. Until that happens it needs to be introduced carefully. The key to this lies in admitting to our students that "logarithm" is a superfluous word, hallowed by tradition, for a concept with which they are familiar - the 3 in x^3. They already know this object as the "index". (Of course in the opposite direction every index is not a log. It is only a log when the base is positive and not 1).

But the freedom to change the word "logarithm" into "index", can be a lifeline in solving problems so students should be shown the semantic transformation described above, from $8 = 2^3$ to $\log_2 8 = 3$. Next should come drill in changing between $A = x^n$ and $\log_x A = n$. The "laws of logs", such as logAB = logA + logB, can be presented as rewordings of the rules of indices – the base being restricted.

Students should come to feel that in using a redundant word like "log" they are only humouring their teacher/lecturer. They will not be intimidated by drill - exercises such as

"expand $\log_a \dfrac{P^3Q^4}{R}$ as a sum or difference of logs"

or its opposite

"write $2\log_4 D + 3\log_4 F - {}^1/_2\log_4 G$ as a single log".

At all times they know that there is an unbroken thread between the land of *logs* and the familiar ground of *indices/exponents*. It is a thread of two strands: the translation at any stage of

log to *index*, and the skill to switch fluently between a *log form* equation and its *power form*. By this means students can be led to questions such as the one quoted from 1970, as well as to equations requiring logs for their solution. They can even reach a stage where the thread is no more than an underlying confidence, a feeling that if required they could get back safely to indices/exponents! And this is enough. It should then be as easy to go forward to harder material as to go back, and they can advance without the insecurity that ill-defined terminology brings and without the fear that their leader isn't sure of the ground.

6. Changing the Terminology (i): *Index* for *log*

In an ideal world none of this remedial work would be necessary. Mathematics would not have redundant terminology whose work could be done by words already in the field. If the job of *logarithm* can be done by *index*/*exponent,* then fair play for our students, not to mention logic and a history of failure would suggest that the word *logarithm* should be dropped from Mathematics. And despite its restricted use internationally *index* would be a better candidate to take the place of *log* than would *exponent.*

$81 = 3^4$ could be semantically transformed in the style shown earlier to $index_3\, 81 = 4$, or for short, $ind_3\, 81 = 4$, or $Ind_3\, 81 = 4$. These forms of $index_a$ would require that the base a be > 0 and not 1. (Such a restriction is not outlandish to our students: the fraction $^a/_b$ of primary school becomes restricted by $b \neq 0$ in secondary school).

When the base is 10, i.e. *d*ecimal, the form *Ind* 10000 = 4 could be used.

When the base is the *n*atural number e, the form *In* 5 = 1.609… could be used.

In reverse, $Ind_3\, 81 = 4$, the *index* form, transforms to $81 = 3^4$, the *power* form.

Rules of Indices having been justified, $x^n.x^m = x^{n+m}$ would imply that the index of a product equals the sum of the indices of the factors, or

$$\text{Ind } AB = \text{Ind } A + \text{Ind } B$$

(Strang (1992) uses a direct approach in this sense to justify the properties of logs).

Functions

Of course if *logarithm* were dropped from algebra there would be consequences for the language of functions. In an ideal world one would attempt a new terminolgy:

x^4 is a *power* function, *variable-based*.

4^x is a *fixed-base power* function, as is e^x.

e^x could be distinguished as the *ebasepower* function, or *ebp* for short,

So, $$ebp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

It is hard to teach old dogs new tricks, but the next generation would adapt to a term such as "fixbasepower growth" instead of the misnomer "exponential growth".

We could even shorten the new term to "power growth" .Confusion with x^a-type behaviour can be avoided by describing the latter as "polynomial growth".

Finally, *logarithmic* functions would become *indicial* functions.

7. Changing the Terminology (ii): *exponent* for *log*

Exponent is used in many countries instead of *index*. The difficulty in choosing *exponent* to replace *logarithm* would lie in the fact that variations of *exponent* are already in use and have well-entrenched meanings. The details are omitted here.

8. Acknowledging the Difficulties

But would the mathematical community rise to the challenge of changing the terminology? Perhaps not, but something needs to be done if logs are not to remain as indigestible as they have been since Victorian times. With the abandonment of logs in arithmetic there is time and space to improve their accessibility in algebra. The terms in current use gives rise to the mystification of logs, and the latter's popularity is as low as ever.

As educators we should not stand idly by. Either we purge the notation of its redundancy at an early stage, which now has to be Higher Leaving Cert Maths or students will continue to flounder when they are expected to be familiar with logs in disciplines such as business and engineering.

Part of the remedial work must be to acknowledge the difficulties of the topic in the manner of Hornsby and Lial above. Such openness can both have a reassuring effect on the student, who may think "I'm not so stupid after all", and act as a challenge to him/her to rise to the task required.

In the case of logs, an acknowledgement of (and an apology for) a non-intuitive and careless

terminology should be part of every introduction to the material. If the mathematical community were prepared to shoulder their responsibility and start untangling the language it would be interesting to assess the effect of this on the understanding and application of logarithms in the years that follow.

A start could be made by reducing to one the number of names for *exponent/index/log* and using this name with the usual restriction to replace *logarithm*. Simultaneously we need to be more careful of how we use the term "power", not an easy task. A full and rational rolling-back of the linguistic mess may be too much to hope for, yet not to attempt some change is to be complacent about a terminology that is non-intuitive, archaic and inconsistent. We should not continue to accept, with our students, that logarithms belong to a nineteenth century glass case from which they are drawn out once in a while, manipulated in mysterious ways and put back again. June 2002 was historical. On that day we admitted that logarithms are not for consumption by the vast majority of our students. It is time for us to reflect on our part in this failure.

(The above is adapted from an article in the Proceedings of the 2nd International Conference on the Teaching of Mathematics, Crete,2002).

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Are You Ready For Maths Week?

Maths Week Ireland was established in 2006. The purpose of Maths Week Ireland is to promote the understanding, awareness and appreciation of mathematics.

Maths Week Ireland was established and co-ordinated by CALMAST – the Centre for the Advancement of Learning of Maths, Science & Technology at W.I.T.— http://www.calmast.ie/

Last year from October 16 to October 20 students and teachers heard presentations on the uses of various aspects of mathematics. They indulged in the fun and games which can be associated with mathematics as well as whetting their appetites for further study of the Queen of Sciences.

Eoin Gill and Dr. Sheila Donegan, CALMAST
Waterford Institute of Technology

This year Maths Week Ireland runs from Monday 15th to Friday 19th October. During this week highlight events include the annual Hamilton Walk in Dublin on the 16th October (info:fiacre.ocairbre@maths.nuim.ie) The RIA Hamilton Lecture (info: www.ria.ie), the PRISM contest from Galway, the Dr. Maths Magic Show and lectures organised by various Colleges and Associations around the country. Joining local maths presenters are several visitors from the UK. In addition to Steve Humble a.k.a Dr Maths will be Andrew Jefferey (magic maths), Kjartan Poskitt (murderous maths), Jenny Piggot (NRICH), Paul Shepherd (Stadium Design), Neil Challis and Chris Budd. Steve Humble will be running a special session with IMTA Dublin on 18th October.

Check out the CALMAST website (www.mathsweek.ie) for full information about centres near you and think perhaps could you join in by organising some 'mathematical explorations' within your own school or classroom.

The resource links on the CALMAST site are of benefit both inside and outside of Maths Week. For instance, here is a puzzle found through the Primary School pages link http://www.mathsisfun.com/puzzles/

On your travels you come to an old man on the side of the road holding three cards from a standard deck face down. Trying to make conversation you ask him what the three cards are. He tells you, "To the left of the queen, are one or two jacks. To the right of the jack, are one or two jacks. To the right of the club, are one or two diamonds. To the left of the diamond, are one or two diamonds." What are the three cards?

Neil Hallinan, Dublin; Eoin Gill, Waterford

A Practical Application of Geometry

Last summer there was such an abundance of apples on our old apple trees that the branches could not bear the weight and were so bent over that the apples were lodging on the ground. I decided to make some trestles to put underneath the sagging branches as support so that the apples would be raised off the ground. The trestles were to be made from old planks of wood. It seemed reasonable to me to slant the legs at an angle of 60° to the horizontal. To achieve this I had to reason out how to make two cuts on each leg, a horizontal cut so that the leg sat at an angle of 60° to the ground and a vertical cut so that the top of the leg would sit flush with the top plank of the trestle. A sketch and some geometry came to my aid.

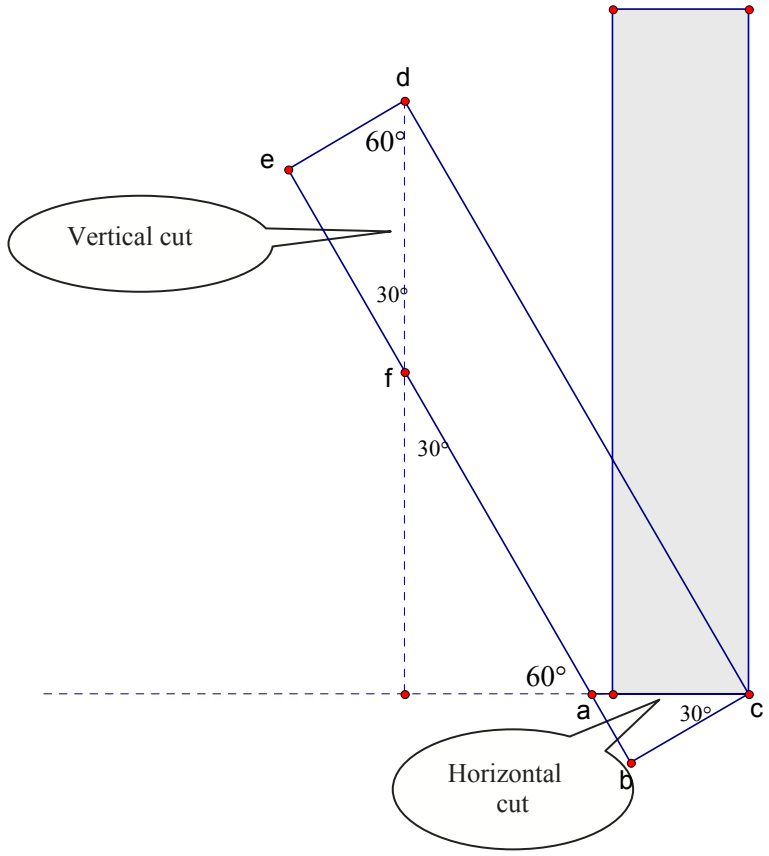


Figure 1

The shaded rectangle is the leg standing vertically and the rectangle debc is the leg in its slanted position before any cuts were made. It then becomes clear that I needed to cut away the triangle abc at the bottom and triangle def at the top. Having worked out the angles it becomes obvious that there are three similar triangles in the diagram, starting with triangle abc at the bottom, then the middle triangle and finally triangle def at the top.

Δ abc and Δdef, although similar and having |bc| = |de|, are not congruent as [bc] has 90° and 30° at its ends whereas [de] has 90° and 60°. This illustrates the danger in using AAS in proving congruence.

We can generalise that if we need the leg to be at an angle α to the horizontal then we need to cut away an angle $(90 - \alpha)°$ at the bottom and $\alpha°$ at the top.

Michael O'Loughlin, Navan

SOLUTIONS

Leaving Certificate Mathematics
Higher Level 2007—Paper 1

1. (a) Simplify $\dfrac{x^2 - xy}{x^2 - y^2}$.

$$\frac{x^2 - xy}{x^2 - y^2} = \frac{x(x-y)}{(x+y)(x-y)} = \underline{\underline{\frac{x}{x+y}}}$$

(b) Let $f(x) = x^2 + (k+1)x - k - 2$, where k is a constant.

(i) Find the value of k for which $f(x) = 0$ has equal roots.

(ii) Find, in terms of k, the roots of $f(x) = 0$.

(iii) Find the range of values of k for which both roots are positive.

(i) $f(x) = x^2 + (k+1)x - k - 2$

If the roots are equal, then $b^2 - 4ac = 0$

$\Rightarrow (k+1)^2 - 4(1)(-k-2) = 0$

$\Rightarrow k^2 + 2k + 1 + 4k + 8 = 0$

$\Rightarrow k^2 + 6k + 9 = 0$

$\Rightarrow (k+3)^2 = 0$

$\Rightarrow \underline{k = -3}$

(ii) $f(x) = 0 \Rightarrow x^2 + (k+1)x - k - 2 = 0$

Factorising: $(x-1)(x+k+2) = 0$

$\Rightarrow x - 1 = 0$ or $x + k + 2 = 0$

$\Rightarrow \underline{x = 1 \text{ or } x = -k - 2}$

OR

(ii) Using the formula to solve the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-(k+1) \pm \sqrt{(k+1)^2 - 4(1)(-k-2)}}{2}$$

$$= \frac{-(k+1) \pm \sqrt{k^2 + 6k + 9}}{2}$$

$$= \frac{-(k+1) \pm \sqrt{(k+3)^2}}{2}$$

$$= \frac{-(k+1) \pm (k+3)}{2}$$

$$\Rightarrow x = \frac{-k-1+k+3}{2} \quad \text{or} \quad x = \frac{-k-1-k-3}{2}$$

$\Rightarrow x = \frac{2}{2}$ or $x = \frac{-2k-4}{2}$

$\Rightarrow x = 1$ $\Rightarrow x = -k-2$

$\therefore$ $\underline{\underline{x = 1 \text{ or } x = -k-2}}$

(iii) Roots positive $\Rightarrow -k-2 > 0$

$\Rightarrow -k > 2$

$\Rightarrow \underline{\underline{k < -2}}$

(c) $x + p$ is a factor of both $ax^2 + b$ and $ax^2 + bx - ac$.

(i) Show that $p^2 = -\frac{b}{a}$ and that $p = \frac{-b-ac}{b}$.

(ii) Hence show that $p^2 + p^3 - c$.

$(x + p)$ is a factor of $ax^2 + b$

$\Rightarrow f(-p) = 0$

$\Rightarrow a(-p)^2 + b = 0$

$\Rightarrow ap^2 + b = 0$ $\Rightarrow \underline{\underline{p^2 = -\frac{b}{a}}}$

$(x + p)$ is a factor of $ax^2 + bx - ac$

$(x + p)$ is a factor $\Rightarrow f(-p) = 0$

$\Rightarrow a(-p)^2 + b(-p) - ac = 0$

$\Rightarrow \quad ap^2 - bp - ac = 0$

But $ap^2 = -b$ … from above

$\Rightarrow -b - bp - ac = 0$

$\Rightarrow \quad bp = -b - ac \Rightarrow \underline{\underline{p = \frac{-b-ac}{b}}}$

OR

$$\begin{array}{r|l} & ax - ap \\ \hline x + p & ax^2 + b \\ & \underline{ax^2 + apx} \\ & \quad -apx + b \\ & \quad \underline{-apx \quad -ap^2} \\ & \quad\quad b + ap^2 \end{array}$$

Since $(x + p)$ is a factor, the remainder is zero

$\Rightarrow b + ap^2 = 0 \Rightarrow \underline{\underline{p^2 = \frac{-b}{a}}}$

$$\begin{array}{r|l} & ax + (b - ap) \\ \hline (x+p) & ax^2 + bx - ac \\ & \underline{ax^2 + apx} \\ & \quad x(b-ap) - ac \\ & \quad \underline{x(b-ap) + p(b-ap)} \\ & \quad\quad -p(b-ap) - ac \end{array}$$

Since remainder = 0 $\Rightarrow -p(b-ap)-ac=0$

$\Rightarrow -pb+ap^2-ac=0$

But $ap^2=-b$ … (from above)

$\Rightarrow -pb-b-ac=0$

$\Rightarrow pb=-b-ac$

$\Rightarrow \underline{\underline{p=\frac{-b-ac}{b}}}$

(ii) $p^2+p^3=p^2(1+p)$

$$=\left(-\frac{b}{a}\right)\left(1+\frac{-b-ac}{b}\right)$$

$$=\left(-\frac{b}{a}\right)\left(\frac{b-b-ac}{b}\right)$$

$$=\left(-\frac{b}{a}\right)\left(-\frac{ac}{b}\right)=\frac{abc}{ab}=c$$

$\Rightarrow \underline{\underline{p^2+p^3=c}}$

2. (a) Solve the simultaneous equations

$x+y+z-2$ … ①

$2x+y+z-3$ … ②

$x-2y+2z-15$ … ③

②: $2x+y+z=3$

①: $x+y+z=2$

$\underline{x\qquad\quad=1}$

Put $x=1$ into ②: $2(1)+y+z=3$

$\Rightarrow y+z=1$ … ④

Put $x=1$ into ③: $1-2y+2z=15$

$\Rightarrow -2y+2z=14$

$\Rightarrow -y+z=7$ … ⑤

④: $y+z=1$

⑤: $\underline{-y+z=7}$

$2z=8$

$z=4$

④ $y+z=1$

$y+4=1$

$\Rightarrow y=-3$

$\therefore$ $\underline{x=1,\ y=-3 \text{ and } z=4}$ are the solutions

(b) α and β are the roots of the equation $x^2 - 4x + 6 - 0$.

(i) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

(ii) Find the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

(i) α and β are the roots of $x^2 - 4x + 6 = 0$

$$\Rightarrow \alpha + \beta = -\frac{b}{a} = \frac{4}{1} = 4 \text{ and } \alpha\beta = 6$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \quad \underline{\underline{\frac{1}{\alpha} + \frac{1}{\beta} = \frac{2}{3}}}$$

(ii) The equation is $x^2 - x$ (sum of roots) + product of roots = 0

$$x^2 - x\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \frac{1}{\alpha} \cdot \frac{1}{\beta} = 0$$

$$\Rightarrow x^2 - x\left(\frac{2}{3}\right) + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \frac{2x}{3} + \frac{1}{6} = 0$$

$$\Rightarrow \underline{\underline{6x^2 - 4x + 1 = 0}}$$

(c) (i) Prove that $x + \frac{9}{x+2} \geq 4$, where $x + 2 > 0$.

(ii) Prove that $x + \frac{9}{x+a} \geq 6 - a$, where $x + a > 0$.

(i) Prove $x + \frac{9}{x+2} \geq 4$

Multiply across by $(x + 2) > 0$.

$$x(x + 2) + 9 \geq 4(x + 2)$$
$$x^2 + 2x + 9 \geq 4x + 8$$
$$\Rightarrow \quad x^2 - 2x + 1 \geq 0$$
$$\Rightarrow \quad (x + 1)^2 \geq 0 \ldots \text{ since } (n)^2 > 0$$
$$\Rightarrow \quad \underline{\underline{x + \frac{9}{x+2} \geq 4}}$$

(ii) Prove that $x + \frac{9}{x+a} \geq 6 - a$, where $x + a > 0$

Multiply across by $(x + a) > 0$

$$x(x + a) + 9 \geq (6 - a)(x + a)$$
$$\Rightarrow \quad x^2 + ax + 9 \geq 6x + 6a - ax - a^2$$
$$\Rightarrow \quad x^2 + 2ax - 6x - 6a + a^2 + 9 \geq 0$$

$\Rightarrow\ x^2 + 2(a-3)\,x + a^2 - 6a + 9 \geq 0$

$\Rightarrow\ x^2 + 2(a-3)\,x + (a-3)^2 \geq 0$

Let $(a-3) = y$

$\Rightarrow\ x^2 + 2xy + y^2 \geq 0$

i.e. $(x+y)^2 \geq 0$

$\Rightarrow\ (x + a - 3)^2 \geq 0$ ..... since $(n)^2 \geq 0$

$\Rightarrow\ x + \dfrac{9}{x+a} \geq 6 - a$, where $x + a > 0$

3. (a) Let $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix}$. Find $A^2 - 2A$.

$$A^2 = A.A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix}$$

$$2A = 2\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix}$$

$$A^2 - 2A = \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}}$$

(b) Let $z = -1 + i$, where $i^2 = -1$.

(i) Use De Moivre's theorem to evaluate z^5 and z^9.

(ii) Show that $z^5 + z^9 = 12z$.

(i) $z = -1 + i$

$z = r(\cos\theta + i\sin\theta)$

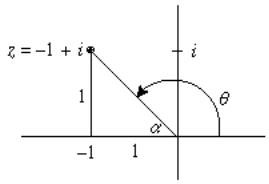
$= \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

$= 2^{\frac{1}{2}}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

$z^5 = \left[2^{\frac{1}{2}}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)\right]^5$

$= 2^{\frac{5}{2}}\left[\cos\dfrac{15\pi}{4} + i\sin\dfrac{15\pi}{4}\right]$

$= 2^{\frac{5}{2}}\left[\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right]$



$r = |z| = \sqrt{(-1)^2 + 1}$

$= \sqrt{2}$

$= 2^{\frac{1}{2}}$

$\tan\alpha = 1$

$\Rightarrow \alpha = \dfrac{\pi}{4}$

$\Rightarrow \theta = \dfrac{3\pi}{4}$

$$= 2^{\frac{5}{2}}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right]$$

$$= 4\sqrt{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right]$$

$$= 4\,(1 - i)$$

$$\Rightarrow \quad \underline{z^5 = 4 - 4i}$$

$$z^9 = \left[2^{\frac{1}{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^9$$

$$= 2^{\frac{9}{2}}\left[\cos\frac{27\pi}{4} + i\sin\frac{27\pi}{4}\right]$$

$$= 2^{\frac{9}{2}}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$= 2^{\frac{9}{2}}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$$

$$= 2^4 \cdot 2^{\frac{1}{2}}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$$

$$= 2^4[-1 + i]$$

$$= 16[-1 + i]$$

$$\Rightarrow \quad \underline{z^9 = -16 + 16i}$$

(ii) $z^5 + z^9 = 4 - 4i - 16 + 16i$

$$= -12 + 12i$$

$$= 12\,(-1 + i)$$

$$\Rightarrow \underline{z^5 + z^9 = 12z} \quad \ldots \quad z = -1 + i$$

(c) (i) Find the two complex numbers $a + bi$ for which $(a + bi)^2 - 15 + 8i$.
(ii) Solve the equation $iz^2 + (2 - 3i)z + (-5 + 5i) - 0$.

(i) $(a + bi)^2 = 15 + 8i$

$\Rightarrow a^2 + 2abi + b^2i^2 = 15 + 8i$

$\Rightarrow (a^2 - b^2) + (2ab)i = (15) + (8)i$

Equation real coefficients to real, etc, we have

$a^2 - b^2 = 15 \quad \ldots \quad ①$ and $2ab = 8$

$\Rightarrow ab = 4$

$\Rightarrow \quad b = \frac{4}{a}$

We now put $\frac{4}{a}$ for b into equation ①.

① $\quad a^2 - b^2 = 15$

$$\Rightarrow a^2 - \left(\frac{4}{a}\right)^2 = 15$$

$$\Rightarrow a^2 - \frac{16}{a^2} = 15$$
$$\Rightarrow a^4 - 15a^2 - 16 = 0$$
$$\Rightarrow (a^2 - 16)(a^2 + 1) = 0$$
$$\Rightarrow a^2 = 16 \quad \text{or} \quad a^2 = -1$$
$$\Rightarrow a = \pm 4 \quad \text{and} \quad a^2 \neq -1$$
$$a = 4 \Rightarrow b = \frac{4}{a} = \frac{4}{4} = 1 \text{ i.e. } b = 1$$
$$a = -4 \Rightarrow b = \frac{4}{-4} = -1 \text{ i.e. } b = -1$$

The two complex numbers are: $\underline{\underline{4 + i \text{ and } -4 - i}}$

(ii) $iz^2 + (2 - 3i)z + (-5 + 5i) = 0$

$a = i$
$b = (2 - 3i)$
$c = (-5 + 5i)$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\Rightarrow z = \frac{-(2-3i) \pm \sqrt{(2-3i)^2 - 4(i)(-5+5i)}}{2i}$$
$$= \frac{-2 + 3i \pm \sqrt{4 - 12i - 9 + 20i + 20}}{2i}$$
$$= \frac{-2 + 3i \pm \sqrt{15 + 8i}}{2i}$$
$$= \frac{(-2 + 3i) \pm (4 + i)}{2i} \quad \ldots\ldots \sqrt{15 + 8i} = \pm(4 + i) \text{ from (i) above}$$
$$= \frac{-2 + 3i + 4 + i}{2i} \text{ or } \frac{-2 + 3i - 4 - i}{2i}$$
$$= \frac{2 + 4i}{2i} \text{ or } \frac{-6 + 2i}{2i}$$
$$z_1 = \frac{2 + 4i}{2i} = \frac{i(2 + 4i)}{2i^2} = \frac{-4 + 2i}{-2} = \frac{4 - 2i}{2} = 2 - i$$
$$z_2 = \frac{-6 + 2i}{2i} = \frac{i(-6 + 2i)}{2i^2} = \frac{-2 - 6i}{-2} = \frac{2 + 6i}{2} = 1 + 3i$$

$\therefore$ The roots are: $\underline{\underline{z_1 = 2 - i}}$ and $\underline{\underline{z_2 = 1 + 3i}}$

4. (a) Show that $\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2}$ for all natural numbers $n \geq 2$.

$$\text{LHS}: \binom{n}{1} + \binom{n}{2} = \frac{n}{1} + \frac{n(n-1)}{1.2}$$
$$= \frac{n}{1} + \frac{n^2 - n}{2}$$
$$= \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2}$$

$$\text{RHS}: \binom{n+1}{2} = \frac{(n+1)(n)}{1.2}$$

$$= \frac{n^2+n}{2}$$

$$\therefore \quad \underline{\underline{\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2} = \frac{n^2+n}{2}}}$$

(b) $u_1 - 5$ and $u_{n+1} = \frac{n}{n+1}u_n$ for all $n \geq 1$, $n \in$ N.

(i) Write down the value of each of u_2, u_3 and u_4.

(ii) Hence, by inspection, write an expression for u_n in terms of n.

(iii) Use induction to justify your answer for part (ii).

(i) $u_1 = 5$ and $u_{n+1} = \frac{n}{n+1}u_n$

$$u_2 = \frac{1}{1+1}u_1 = \frac{1}{2}(5) = \underline{\underline{\frac{5}{2}}}$$

$$u_3 = \frac{2}{2+1}\left(\frac{5}{2}\right) = \frac{2}{3}\left(\frac{5}{2}\right) = \underline{\underline{\frac{5}{3}}}$$

$$u_4 = \frac{3}{3+1}\left(\frac{5}{3}\right) = \frac{3}{4}\left(\frac{5}{3}\right) = \underline{\underline{\frac{5}{4}}}$$

The terms are: $\frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}$

(ii) $\underline{\underline{u_n = \frac{5}{n}}}$

(iii) To prove by induction that $u_n = \frac{5}{n}$

$P(1): n = 1 \quad u_1 = \frac{5}{1} = 5 \Rightarrow$ true for $n = 1$

$P(k)$: assume true for $n = k$

$$\Rightarrow u_k = \frac{5}{k}$$

$P(k+1)$: Test for $n = (k+1)$

$$u_{k+1} = \frac{k}{k+1}(u_k) = \frac{k}{k+1}\left(\frac{5}{k}\right) = \frac{5}{k+1}$$

Therefore the proposition is true for $n = k + 1$

$\therefore$ it is true for all $n \geq 1$.

(c) The sum of the first n terms of a series is given by $S_n - n^2\log_e 3$.

(i) Find the n^{th} term and prove that the series is arithmetic.

(ii) How many of the terms of the series are less than $12\log_e 27$?

(i) $S_n = n^2\log_e 3 \Rightarrow S_{n-1} = (n-1)^2\log_e 3$

$u_n = S_n - S_{n-1}$

$= n^2\log_e 3 - (n-1)^2\log_e 3$

$= \log_e 3\left[n^2 - (n-1)^2\right]$

$= \log_e 3\left[n^2 - n^2 + 2n - 1\right]$

$\Rightarrow \quad \underline{\underline{u_n = \log_e 3[2n-1]}}$

$u_n = (2n-1)\log_e 3$ and $u_{n+1} = \left[2(n+1)-1\right]\log_e 3$

$= (2n+1)\log_e 3$

$u_{n+1} - u_n = (2n+1)\log_e 3 - (2n-1)\log_e 3$

$= \log_e 3\left[2n+1-(2n-1)\right]$

$= \log_e 3\left[2n+1-2n+1\right] = 2\log_e 3$

Since $u_{n+1} - u_n$ = a constant i.e. $2\log_e 3$, the series is arithmetic,

i.e. $d = 2\log_e 3$

(ii) $12\log_e 27 = 12\log_e(3^3) = 36\log_e 3$

Let $u_n = (2n-1)\log_e 3 \le 36\log_e 3$

$\Rightarrow 2n - 1 \le 36$

$2n \le 37$

$n \le 18\frac{1}{2}$

$\Rightarrow \quad u_{18} < 12\log_e 27$

Therefore there are 18 terms $< 12\log_e 27$.

5. (a) Plot, on the number line, the values of x that satisfy the inequality $|x+1| \le 2$, where $x \in \mathbf{Z}$.

$|x+1| \le 2$

$\Rightarrow \quad (x+1)^2 \le 4$ …… (square both sides)

$\Rightarrow \quad x^2 + 2x + 1 \le 4$

$\Rightarrow \quad x^2 + 2x - 3 \le 0$

$\Rightarrow \quad (x+3)(x-1) \le 0$

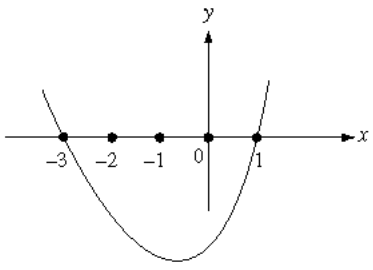
$\Rightarrow \quad x = -3$ or $x = 1$

From the graph

$(x+3)(x-1) \le 0$ for

$-3 \le x \le 1, \ x \in \mathbf{Z}$

i.e. $\underline{x = -3, -2, -1, 0, 1}$

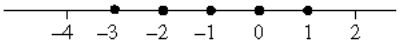


OR

$|x+1| \le 2, \ x \in \mathbf{Z}$

$\Rightarrow -2 \le x + 1 \le 2$

(a) $-2 \le x + 1$
$\Rightarrow -3 \le x$
$\Rightarrow -3 \le x \le 1, \ x \in Z$
$\Rightarrow \ x = -3, -2, -1, 0, 1$



(b) $x + 1 \le 2$
$\Rightarrow x \le 1$

(b) In the expansion of $\left(2x - \frac{1}{x^2}\right)^9$,

(i) find the general term

(ii) find the value of the term independent of x.

(i) $\left(2x - \frac{1}{x^2}\right)^9 = \left[2x + \left(-\frac{1}{x^2}\right)\right]^9$

$u_{r+1} = \binom{9}{r}(2x)^{9-r} \cdot \left(-\frac{1}{x^2}\right)^r$

$= \binom{9}{r}(2)^{9-r} \cdot x^{9-r} \cdot \left(-x^{-2}\right)^r$

$= k \,.\, x^{9-r} \,.\, x^{-2r}$

$\underline{\underline{u_{r+1} = k \,.\, x^{9-3r}}}$

(ii) Let u_{r+1} be the term with x^0.
$\Rightarrow \quad x^{9-3r} = x^0$
$\Rightarrow \quad 9 - 3r = 0$
$\Rightarrow \quad 3r = 9$
$\Rightarrow \quad r = 3$
$\Rightarrow u_4$ is the term independent of x

$u_4 = \binom{9}{3}(2x)^6\left(-\frac{1}{x^2}\right)^3$

$= \frac{9.8.7}{1.2.3}\left(64x^6\right)\left(-\frac{1}{x^6}\right)$

$\Rightarrow \underline{\underline{u_4 = -5376}}$

OR

$\left[2x + \left(-\frac{1}{x^2}\right)\right]^9 = (2x)^9 + \binom{9}{1}(2x)^8\left(-\frac{1}{x^2}\right) + \binom{9}{2}(2x)^7\left(-\frac{1}{x^2}\right)$

$+ \binom{9}{3}(2x)^6\left(-\frac{1}{x^2}\right)^3 + \cdots$

u_4 has x^0

$$u_4 = \binom{9}{3}(2x)^6\left(-\frac{1}{x^2}\right)^3$$

$$= \frac{9.8.7}{1.2.3}\cdot 2^6\left(x^6\right)(-1)^3\cdot\frac{1}{x^6}$$

$$= \underline{-5376}$$

(c) The n^{th} term of a series is given by nx^n, where $|x|<1$.

(i) Find an expression for S_n, the sum of the first n terms of the series.

(ii) Hence, find the sum to infinity of the series.

(i)
$$\begin{aligned} S_n &= x + 2x^2 + 3x^3 + \cdots\cdots\cdots + (n-1)x^{n-1} + nx^n \\ xS_n &= \quad x^2 + 2x^3 + \cdots\cdots\cdots + (n-2)x^{n-1} + (n-1)x^n + nx^{n+1} \end{aligned}$$

$$S_n - xS_n = x + x^2 + x^3 + \cdots\cdots\cdots + x^{n-1} \quad + x^n \quad - nx^{n+1}$$

$$= \left[x + x^2 + x^3 + \cdots\cdots\cdots + x^{n-1} + x^n\right] - nx^{n+1}$$

$$S_n - xS_n = [\text{G.P. to } n \text{ terms} : a = x : r = x] - nx^{n+1}$$

$$\Rightarrow S_n[1-x] = \frac{x(1-x^n)}{1-x} - nx^{n+1}$$

$$\Rightarrow \quad \underline{\underline{S_n = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{(1-x)}}}$$

(ii) $|x|<1,\ n\to\infty,\ x^n\to 0$

$$S_\infty = \lim_{n\to\infty}\left(S_n\right)$$

$$= \lim_{n\to\infty}\left[\frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}\right]$$

$$= \frac{x(1-0)}{(1-x)^2} - 0$$

$$\underline{\underline{S_\infty = \frac{x}{(1-x)^2}}}$$

6. (a) Differentiate $\dfrac{x^2-1}{x^2+1}$ with respect to x.

$$y = \frac{x^2-1}{x^2+1}$$

$$\frac{dy}{dx} = \frac{\left(x^2+1\right)2x - \left(x^2-1\right)2x}{\left(x^2+1\right)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{\left(x^2+1\right)^2}$$

$$\frac{dy}{dx} = \frac{4x}{\left(x^2+1\right)^2}$$

(b) (i) Differentiate $\frac{1}{x}$ with respect to x from first principles.

(ii) Find the equation of the tangent to $y = \frac{1}{x}$ at the point $\left(2, \frac{1}{2}\right)$.

(i) $f(x) = \frac{1}{x}$

$$f(x+h) = \frac{1}{x+h}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$$

$$= \frac{x-(x+h)}{x(x+h)}$$

$$= \frac{-h}{x(x+h)}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{-1}{x(x+h)} \quad \cdots \text{ (divide across by } h\text{)}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(\frac{-1}{x(x+h)}\right)$$

$$= \frac{-1}{x^2}$$

(ii) From (i) above, if $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$

$\therefore$ the slope of the tangent to the curve is $-\frac{1}{x^2}$

$$m = -\frac{1}{x^2} = -\frac{1}{4} \text{ at the point } \left(2, \frac{1}{2}\right)$$

Equation of tangent: point $= \left(2, \frac{1}{2}\right)$; slope $= -\frac{1}{4}$

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = -\frac{1}{4}(x-2)$

$4y - 2 = -x + 2$

$\Rightarrow x + 4y - 4 = 0$ is the equation of the tangent

(c) Let $f(x) = \tan^{-1}\frac{x}{2}$ and $g(x) = \tan^{-1}\frac{2}{x}$, for $x > 0$.

(i) Find $f'(x)$ and $g'(x)$.

(ii) Hence, show that $f(x) + g(x)$ is constant.

(iii) Find the value of $f(x) + g(x)$.

(i) Standard derivative : $\tan^{-1}\left(\frac{x}{a}\right) \Rightarrow f'(x) = \frac{a}{a^2 + x^2}$

$$f(x) = \tan^{-1}\left(\frac{x}{2}\right) \qquad g(x) = \tan^{-1}\left(\frac{2}{x}\right)$$

$$\underline{\underline{f'(x) = \frac{2}{4 + x^2}}} \qquad g'(x) = \frac{1}{1 + \left(\frac{2}{x}\right)^2} \cdot \left(2 \cdot \frac{-1}{x^2}\right)$$

$$= \frac{1}{1 + \frac{4}{x^2}}\left(\frac{-2}{x^2}\right)$$

$$= \frac{x^2}{x^2 + 4}\left(\frac{-2}{x^2}\right)$$

$$\underline{\underline{g'(x) = \frac{-2}{4 + x^2}}}$$

(ii) $f'(x) + g'(x) = \frac{2}{4 + x^2} + \frac{(-2)}{4 + x^2}$

$$= \frac{2}{4 + x^2} - \frac{2}{4 + x^2} = 0$$

Since the derivative of a constant is zero,

$\Rightarrow \underline{\underline{f(x) + g(x) = \text{a constant}}}$

(iii) Find the value of $f(x) + g(x)$.

Let $f(x) + g(x) = k$

$$\tan^{-1}\left(\frac{x}{2}\right) + \tan^{-1}\left(\frac{2}{x}\right) = k$$

Let $x = 2$: $\tan^{-1}(1) + \tan^{-1}(1) = k$

$$\frac{\pi}{4} + \frac{\pi}{4} = k$$

$$\Rightarrow \underline{\underline{k = \frac{\pi}{2}}}$$

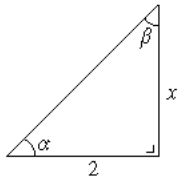
(**Note**: Any value for x may be used.)

OR

$$\alpha+\beta=\frac{\pi}{2}$$

$$\tan\alpha=\frac{x}{2}\Rightarrow\alpha=\tan^{-1}\left(\frac{x}{2}\right)$$

$$\tan\beta=\frac{2}{x}\Rightarrow\beta=\tan^{-1}\left(\frac{2}{x}\right)$$



$$\tan^{-1}\left(\frac{x}{2}\right)+\tan^{-1}\left(\frac{2}{x}\right)=\alpha+\beta$$

$$=\frac{\pi}{2}$$

7. (a) Taking 1 as the first approximation of a root of $x^3 + 2x - 4 - 0$, use the Newton-Raphson method to calculate the second approximation of this root.

$f(x) = x^3 + 2x - 4$ $\qquad$ $f'(x) = 3x^2 + 2$

$f(1) = (1)^3 + 2(1) - 4$ $\qquad$ $f'(1) = 3(1)^2 + 2$

$= -1$ $\qquad$ $= 5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{(-1)}{5} = 1 + \frac{1}{5} = 1\frac{1}{5}$$

$$\therefore \quad x_2 = 1\frac{1}{5}$$

(b) (i) Find the equation of the tangent to the curve $3x^2 + y^2 - 28$ at the point $(2, -4)$.

(ii) $x - e^t\cos t$ and $y - e^t\sin t$. Show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

(i) $3x^2 + y^2 = 28$

$$6x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -6x$$

$$\frac{dy}{dx} = \frac{-6x}{2y} = \frac{-3x}{y}$$

At $(2, -4)$: $m = \frac{dy}{dx} = \frac{-3(2)}{-4} = \frac{3}{2}$

Equation of tangent at $(2, -4)$ with slope $m = \frac{3}{2}$

$$y - y_1 = m(x - x_1)$$

$$y + 4 = \frac{3}{2}(x - 2)$$

$$2y + 8 = 3x - 6$$

$$\Rightarrow \underline{\underline{3x - 2y - 14 = 0 \text{ is the equation}}}$$

(ii) $x = e^t(\cos t)$ | $y = e^t(\sin t)$

$$\frac{dx}{dt} = e^t(-\sin t) + \cos t\left(e^t\right) \quad \left| \quad \frac{dy}{dt} = e^t(\cos t) + \sin t\left(e^t\right)\right.$$

$$\frac{dx}{dt} = e^t\cos t - e^t\sin t \quad \left| \quad \frac{dy}{dt} = e^t\cos t + e^t\sin t\right.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t\cos t + e^t\sin t}{e^t\cos t - e^t\sin t} = \underline{\underline{\frac{x+y}{x-y}}}$$

(c) $f(x) = \log_e 3x - 3x$, where $x > 0$.

(i) Show that $\left(\frac{1}{3}, -1\right)$ is a local maximum point of $f(x)$.

(ii) Deduce that the graph of $f(x)$ does not intersect the x-axis.

(i) $f(x) = \log_e 3x - 3x, \qquad (x > 0)$

$$f'(x) = \frac{1}{3x}(3) - 3$$

$$f'(x) = \frac{1}{x} - 3$$

$$f''(x) = -\frac{1}{x^2}$$

Local max/min point: $f'(x) = 0 \Rightarrow \frac{1}{x} - 3 = 0$

$$\frac{1}{x} = 3$$

$$\Rightarrow \quad x = \frac{1}{3}$$

$$\text{At } x = \frac{1}{3} : f''(x) = -\frac{1}{x^2} = -\frac{1}{\left(\frac{1}{3}\right)^2} = -9 < 0$$

$\therefore$ local maximum point at $x = \frac{1}{3}$

$$\text{At } x = \frac{1}{3}, \ f(x) = \log_e(3x) - 3x$$

$$f\left(\frac{1}{3}\right) = \log_e(1) - 1$$
$$= 0 - 1 = -1$$

$\therefore$ Local maximum point at $\left(\frac{1}{3}, -1\right)$

Note: Since $f'(x) = 0$ has only one root, there is only one turning point.

OR

$f(x) = \log_e(3x) - 3x$

$f'(x) = \frac{1}{3x}(3) - 3 \Rightarrow f'(x) = \frac{1}{x} - 3$

At $x = \frac{1}{3}$: $f'(x) = \frac{1}{\left(\frac{1}{3}\right)} - 3 = 3 - 3 = 0$

$\Rightarrow$ turning point at $x = \frac{1}{3}$

$f''(x) = -\frac{1}{x^2} < 0$ for all x

$\Rightarrow$ local maximum point at $x = \frac{1}{3}$.

$x = \frac{1}{3}$: $y = \log_e(3x) - 3x$

$= \log_e(1) - 1$
$= 0 - 1 = -1$

$\Rightarrow$ local maximum at $\left(\frac{1}{3}, -1\right)$

(ii) $f'(x)$ has only one root.

This implies only one turning point

$f(x)$ is continuous, so the maximum point above is an absolute maximum point.

Since the maximum point $\left(\frac{1}{3}, -1\right)$ is below the x-axis, the whole graph must lie below the x-axis. Hence, the graph will not cut the x-axis.

Note: (i) $f(x) = 0$ has no roots since the graph does not cut the x-axis

(ii) Since $f''(x) = \frac{-1}{x^2} \neq 0$, the graph has no point of inflection.

OR

There is only one turning point – a maximum point at $\left(\frac{1}{3}, -1\right)$.

There are no minimum points.

The graph lies completely below the x-axis.
$\Rightarrow$ the graph does not cut the x-axis

8. (a) Find (i) $\int x^3 dx$ (ii) $\int \frac{1}{x^3} dx$.

(i) $\int x^3 dx = \underline{\underline{\frac{x^4}{4} + c}}$

(ii) $\int \frac{dx}{x^3} = \int x^{-3} dx = \frac{x^{-2}}{-2} = \underline{\underline{\frac{-1}{2x^2} + c}}$

(b) (i) Evaluate $\int_0^4 x\sqrt{x^2+9}\,dx$.

(ii) f is a function such that $f'(x) - 6 - \sin x$ and $f\left(\frac{\pi}{3}\right) = 2\pi$.

Find $f(x)$.

(i) $\int_0^4 x\sqrt{x^2+9}dx$ Let $u = x^2 + 9$

$du = 2xdx$

$\Rightarrow xdx = \frac{1}{2}du$

$= \int \left(x^2+9\right)^{\frac{1}{2}} \cdot xdx$

$= \frac{1}{2}\int u^{\frac{1}{2}} du$

$= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) = \frac{1}{3}\left(u^{\frac{3}{2}}\right)$

$= \frac{1}{3}\left[\left(x^2+9\right)^{\frac{3}{2}}\right]_0^4$

$= \frac{1}{3}\left[25^{\frac{3}{2}} - 9^{\frac{3}{2}}\right]$

$= \frac{1}{3}[125 - 27]$

$\Rightarrow \underline{\underline{\int_0^4 x\sqrt{x^2+9}\,dx = \frac{98}{3}}}$

$25^{\frac{3}{2}} = 5^{2\cdot\frac{3}{2}}$

$= 5^3$

$= 125$

(ii) $f'(x) = 6 - \sin x$

$\Rightarrow f(x) = 6x + \cos x + c$

$$f\left(\frac{\pi}{3}\right) = 6\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) + c = 2\pi$$

$$\Rightarrow 2\pi + \frac{1}{2} + c = 2\pi$$

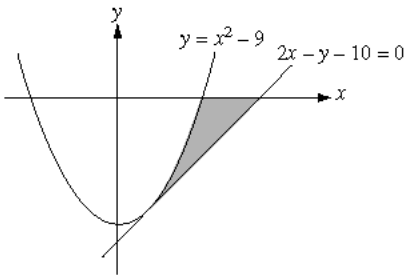
$$\Rightarrow \quad c = -\frac{1}{2}$$

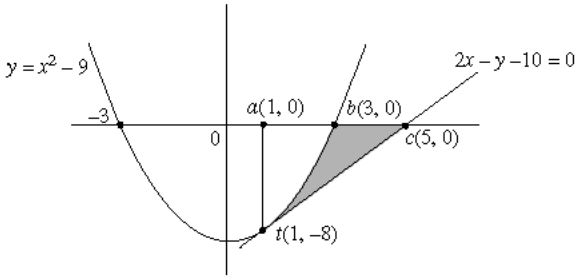
$$\Rightarrow f(x) = 6x + \cos x - \frac{1}{2}$$

(c) The line $2x - y - 10 - 0$ is a tangent to the curve $y - x^2 - 9$, as shown.

The shaded region is bounded by the line, the curve and the x-axis.

Calculate the area of this region.





The line intersects the curve at $2x - y - 10 = 0 \cap y = x^2 - 9$

From the line $y = 2x - 10$

$y = x^2 - 9 \Rightarrow \quad 2x - 10 = x^2 - 9$

$\Rightarrow \quad x^2 - 2x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0$

$\Rightarrow \quad x = 1 \Rightarrow y = 2 - 10$ i.e. $y = -8$

The line intersects the curve at $t(1, -8)$

The curve intersects the x-axis at $y = 0$.

$y = x^2 - 9$

At $y = 0$: $\quad 0 = x^2 - 9$

$\Rightarrow x^2 = 9 \qquad \Rightarrow x = \pm 3$

$\therefore$ the coordinates of $b = (3, 0)$

The line $y = 2x - 10$ intersects the x-axis at $y = 0$

$\Rightarrow 0 = 2x - 10$

$\Rightarrow x = 5$

$\therefore$ the coordinates of c are $(5, 0)$

Shaded area = area Δact – area under the curve

$$\text{Area of } \Delta act = \frac{1}{2}|ac| \cdot |at|$$

$$= \frac{1}{2}(4)(8)$$

$$= 16$$

$$\text{Area under curve} = \int_1^3 y\,dx$$

$$= \int_1^3 \left(x^2 - 9\right)dx$$

$$= \left[\frac{x^3}{3} - 9x\right]_1^3$$

$$= (9 - 27) - \left(\frac{1}{3} - 9\right)$$

$$= -18 + 8\frac{2}{3}$$

$$= \left|-\frac{28}{3}\right| = \frac{28}{3}$$

$$\text{Shaded area} = 16 - \frac{28}{3}$$

$$= \frac{48 - 28}{3} = \frac{20}{3}$$

$$\text{Area of shaded region} = \underline{\underline{\frac{20}{3}}}$$

Leaving Certificate Mathematics
Higher Level 2007—Paper 2

1. (a) The following parametric equations define a circle:
$x = 5 + 7\cos\theta$, $y = 7\sin\theta$, where $\theta \in \mathbf{R}$.
What is the Cartesian equation of the circle?

$x = 5 + 7\cos\theta$ and $y = 7\sin\theta$

$\Rightarrow 7\cos\theta = x - 5$ $\Rightarrow 7\sin\theta = y$

$\Rightarrow 49\cos^2\theta = (x-5)^2$ $\Rightarrow 49\sin^2\theta = y^2$

$\Rightarrow 49\cos^2\theta + 49\sin^2\theta = (x-5)^2 + y^2$

$\Rightarrow 49(\cos^2\theta + \sin^2\theta) = (x-5)^2 + y^2$

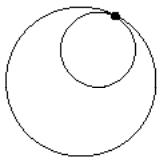
$\Rightarrow 49 = (x-5)^2 + y^2$

$\Rightarrow \underline{(x-5)^2 + y^2 = 49}$ is the equation of the circle

(b) $x^2 + y^2 - 4x - 6y + 5 = 0$ and $x^2 + y^2 - 6x - 8y + 23 = 0$ are two circles.

(i) Prove that the circles touch internally.

(ii) Find the coordinates of the point of contact of the two circles.



(i) $x^2 + y^2 - 4x - 6y + 5 = 0$

$\Rightarrow$ centre = (2, 3) and radius $= \sqrt{4+9-5}$

$= \sqrt{8} = 2\sqrt{2}$

$x^2 + y^2 - 6x - 8y + 23 = 0$

$\Rightarrow$ centre = (3, 4) and radius $= \sqrt{9+16-23}$

$= \sqrt{2}$

Let d = distance between the centres (2, 3) and (3, 4)

$\Rightarrow d = \sqrt{(3-2)^2 + (4-3)^2}$

$= \sqrt{1+1} = \sqrt{2}$

The circles touch internally if $d = r_1 - r_2$

$r_1 - r_2 = 2\sqrt{2} - \sqrt{2} = \sqrt{2}$

Since $d = r_1 - r_2 = \sqrt{2}$, the circles touch internally.

(ii) The point of contact is

$x^2 + y^2 - 4x - 6y + 5 - (x^2 + y^2 - 6x - 8y + 23) = 0$

$\Rightarrow x^2 + y^2 - 4x - 6y + 5 - x^2 - y^2 + 6x + 8y - 23 = 0$

$\Rightarrow 2x + 2y - 18 = 0$

$\Rightarrow x + y - 9 = 0$ ... i.e. the common tangent

$\Rightarrow y = -x + 9$

$x^2 + (-x + 9)^2 - 4x - 6(-x + 9) + 5 = 0$

$\Rightarrow x^2 + x^2 - 18x + 81 - 4x + 6x - 54 + 5 = 0$

$\Rightarrow 2x^2 - 16x + 32 = 0$

$\Rightarrow x^2 - 8x + 16 = 0$

$\Rightarrow (x - 4)(x - 4) = 0$

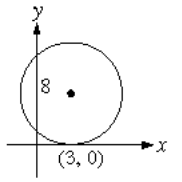
$\Rightarrow x = 4$

$x = 4 \Rightarrow y = -x + 9$

$= -4 + 9 = 5$

The point of contact is $\underline{\underline{(4, 5).}}$

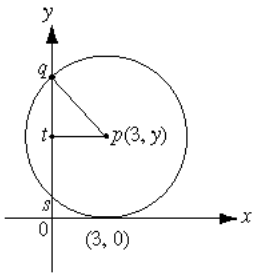
(c) A circle has its centre in the first quadrant.
The x-axis is a tangent to the circle at the point (3, 0).
The circle cuts the y-axis at points that are 8 units apart.
Find the equation of the circle.



From the diagram;

$[pt] \perp [sq]$ and t is the midpoint of $[sq]$.

$\Rightarrow |qt| = 4$ and $|pt| = 3$



By Pythagoras

$|pq|^2 = 4^2 + 3^2$

$= 25$

$\Rightarrow |pq| = 5 = \text{radius}$

$\Rightarrow p = (3, 5)$

$\therefore$ the circle has centre (3, 5) and radius = 5

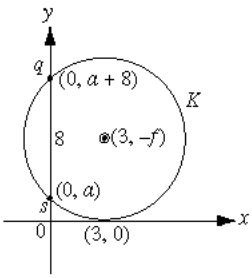
Equation is $(x - 3)^2 + (y - 5)^2 = 5^2$

or $\underline{x^2 + y^2 - 6x - 10y + 9 = 0}$

OR

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle.

Let $s = (0, a)$ and $q = (0, a + 8)$



$(0, a) \in K \Rightarrow a^2 + 2fa + c = 0$ ..... (A)

$(0, a+8) \in K \Rightarrow (a+8)^2 + 2f(a+8) + c = 0$

$\Rightarrow a^2 + 16a + 64 + 2fa + 16f + c = 0$

$\Rightarrow a^2 + a(16 + 2f) + 64 + 16f + c = 0$ ..... (B)

$(3, 0) \in K \Rightarrow 9 + 6g + c = 0$ ..... (C)

The centre of K is $(3, -f)$

$\Rightarrow (3, -f) = (-g, -f) \Rightarrow g = -3$

From (C): $9 + 6g + c = 0$

$\Rightarrow 9 + 6(-3) + c = 0$ ..... $(g = -3)$

$\Rightarrow 9 - 18 + c = 0$

$\Rightarrow c = 9$

(B) − (A): $a^2 + a(16 + 2f) + 64 + 16f + c - (a^2 + 2fa + c) = 0$

$\Rightarrow a^2 + 16a + 2af + 64 + 16f + 9 - a^2 - 2af - 9 = 0$

$\Rightarrow 16a + 64 + 16f = 0$

$\Rightarrow a + 4 + f = 0$

$\Rightarrow a = -f - 4$

Now substitute $(-f - 4)$ for a in equation (A)

$(-f - 4)^2 + 2f(-f - 4) + 9 = 0$

$\Rightarrow f^2 + 8f + 16 - 2f^2 - 8f + 9 = 0$

$\Rightarrow -f^2 + 25 = 0$

$\Rightarrow f^2 = 25 \Rightarrow f = \pm 5$

$\Rightarrow$ centre $(-g, -f) = (3, 5)$ and radius $= 5$

Equation is $(x - 3)^2 + (y - 5)^2 = 5^2$

i.e. $\underline{x^2 + y^2 - 6x - 10y + 9 = 0}$

2. (a) $\vec{x} = -2\vec{i} + 5\vec{j}$ and $\vec{xy} = -6\vec{i} - 8\vec{j}$. Express $\vec{y}$ in terms of $\vec{i}$ and $\vec{j}$.

$\vec{xy} = \vec{y} - \vec{x}$

$\Rightarrow -6\vec{i} - 8\vec{j} = \vec{y} - (-2\vec{i} + 5\vec{j})$

$\Rightarrow -6\vec{i} - 8\vec{j} = \vec{y} + 2\vec{i} - 5\vec{j}$

$\Rightarrow \vec{y} = -6\vec{i} - 8\vec{j} - 2\vec{i} + 5\vec{j}$

$\Rightarrow \underline{\underline{\vec{y} = -8\vec{i} - 3\vec{j}}}$

(b) $\vec{a} = 5\vec{i}$ and $\vec{b} = \sqrt{3}\vec{i} + 3\vec{j}$

(i) Show that $\vec{ab}$ is not perpendicular to $\vec{b}$.

(ii) Find the value of the real number k, given that $\vec{c} = k\vec{b}$ and $\vec{ac} \perp \vec{b}$.

(i) $\vec{ab} = \vec{b} - \vec{a}$

$= \sqrt{3}\vec{i} + 3\vec{j} - 5\vec{i}$

$= \left(\sqrt{3} - 5\right)\vec{i} + 3\vec{j}$

$\vec{ab}.\vec{b} = \left[\left(\sqrt{3} - 5\right)\vec{i} + 3\vec{j}\right].\left[\sqrt{3}\vec{i} + 3\vec{j}\right]$

$= \sqrt{3}\left[\sqrt{3} - 5\right] + 9$

$= 3 - 5\sqrt{3} + 9 \neq 0$, i.e., the dot product $\neq 0$

$\Rightarrow \underline{\underline{\vec{ab} \text{ is not perpendicular } \vec{b}}}$

(ii) $\vec{c} = k\vec{b} = k\left(\sqrt{3}\vec{i} + 3\vec{j}\right)$

$= k\sqrt{3}\vec{i} + 3k\vec{j}$

$\vec{ac} = \vec{c} - \vec{a}$

$= k\sqrt{3}\vec{i} + 3k\vec{j} - 5\vec{i}$

$= \left(k\sqrt{3} - 5\right)\vec{i} + 3k\vec{j}$

$\vec{ac} \perp \vec{b} \Rightarrow \left[\left(k\sqrt{3} - 5\right)\vec{i} + 3k\vec{j}\right].\left[\sqrt{3}\vec{i} + 3\vec{j}\right] = 0$

$\Rightarrow \left(k\sqrt{3}.\sqrt{3} - 5\sqrt{3}\right) + 9k = 0$

$\Rightarrow 3k - 5\sqrt{3} + 9k = 0$

$\Rightarrow 12k = 5\sqrt{3}$

$\Rightarrow \underline{\underline{k = \frac{5\sqrt{3}}{12}}}$

OR

$\overrightarrow{ac} \perp \overrightarrow{b}$

$\Rightarrow \overrightarrow{ac}.\overrightarrow{b} = 0$

$\Rightarrow \left(\overrightarrow{c} - \overrightarrow{a}\right).\overrightarrow{b} = 0$

$\Rightarrow \overrightarrow{c}.\overrightarrow{b} = \overrightarrow{a}.\overrightarrow{b}$

$\Rightarrow k\overrightarrow{b}.\overrightarrow{b} = \overrightarrow{a}.\overrightarrow{b}$

$$\Rightarrow k = \frac{\overrightarrow{a}.\overrightarrow{b}}{\overrightarrow{b}.\overrightarrow{b}} = \frac{\left(5\overrightarrow{i}\right).\left(\sqrt{3}\overrightarrow{i} + 3\overrightarrow{j}\right)}{\left(\sqrt{3}\overrightarrow{i} + 3\overrightarrow{j}\right).\left(\sqrt{3}\overrightarrow{i} + 3\overrightarrow{j}\right)}$$

$$= \frac{5\sqrt{3} + 0}{3 + 9} = \frac{5\sqrt{3}}{12}$$

$$\Rightarrow \underline{\underline{k = \frac{5\sqrt{3}}{12}}}$$

(c) $\overrightarrow{p} = 3\overrightarrow{i} + 4\overrightarrow{j}$ and $\overrightarrow{q} = 5\overrightarrow{i} + 12\overrightarrow{j}$.

$\overrightarrow{r} = \frac{65t}{16}\left(\frac{\overrightarrow{p}}{\left|\overrightarrow{p}\right|} + \frac{\overrightarrow{q}}{\left|\overrightarrow{q}\right|}\right)$, where $t > 0$.

(i) Express $\overrightarrow{r}$ in terms of $\overrightarrow{i}$ and $\overrightarrow{j}$.

(ii) Find $\overrightarrow{p}.\overrightarrow{r}$ and $\overrightarrow{q}.\overrightarrow{r}$.

(iii) Hence, show that r is on the bisector of $\angle poq$, where o is the origin.

(i) $\overrightarrow{p} = 3\overrightarrow{i} + 4\overrightarrow{j}$ and $\overrightarrow{q} = 5\overrightarrow{i} + 12\overrightarrow{j}$

$$\overrightarrow{r} = \frac{65t}{16}\left[\frac{\overrightarrow{p}}{\left|\overrightarrow{p}\right|} + \frac{\overrightarrow{q}}{\left|\overrightarrow{q}\right|}\right]$$

$$= \frac{65t}{16}\left[\frac{3\overrightarrow{i} + 4\overrightarrow{j}}{5} + \frac{5\overrightarrow{i} + 12\overrightarrow{j}}{13}\right]$$

$$= \frac{65t}{16}\left[\frac{39\vec{i}+52\vec{j}+25\vec{i}+60\vec{j}}{65}\right]$$

$$= \frac{t}{16}\left[64\vec{i}+112\vec{j}\right]$$

$$\underline{\underline{\vec{r} = t\left[4\vec{i}+7\vec{j}\right] = 4t\vec{i}+7t\vec{j}}}$$

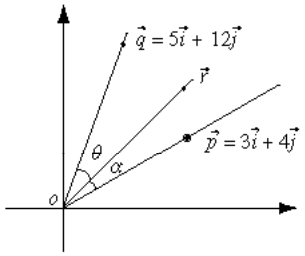
(ii) $\vec{p}.\vec{r} = \left(3\vec{i}+4\vec{j}\right).\left(4t\vec{i}+7t\vec{j}\right) = 12t+28t$

$$= \underline{\underline{40t}}$$

$$\vec{q}.\vec{r} = \left(5\vec{i}+12\vec{j}\right).\left(4t\vec{i}+7t\vec{j}\right)$$

$$= 20t+84t = \underline{\underline{104t}}$$

(iii) Let $\angle por = \alpha$



$$\cos\alpha = \frac{\vec{p}.\vec{r}}{\left|\vec{p}\right|\left|\vec{r}\right|}$$

$$= \frac{40t}{(5)\left(\sqrt{16t^2+49t^2}\right)}$$

$$= \frac{40t}{5\sqrt{65}.\sqrt{t^2}}$$

$$= \frac{40t}{5t\sqrt{65}} = \frac{8}{\sqrt{65}}$$

$$\therefore \cos\alpha = \frac{8}{\sqrt{65}}$$

Let $\angle qor = \theta$

$$\cos\theta = \frac{\vec{q}.\vec{r}}{\left|\vec{q}\right|\left|\vec{r}\right|}$$

$$= \frac{104t}{13\sqrt{65}t^2}$$

$$= \frac{104t}{13t\sqrt{65}} = \frac{8}{\sqrt{65}}$$

$$\therefore \cos\theta = \frac{8}{\sqrt{65}}$$

$$\therefore \cos\theta = \frac{8}{\sqrt{65}} = \cos\alpha$$

$$\Rightarrow \theta = \alpha$$

Therefore r is on the bisector of $\angle poq$.

OR

Equation of op .... points are (0, 0) and (3, 4)

$\Rightarrow op : 4x - 3y = 0 \quad \ldots \quad L_1$

Equation of oq .... points are (0, 0) and (5, 12)

$\Rightarrow oq : 12x - 5y = 0 \quad \ldots \quad L_2$

$(4t, 7t)$ is any point on or

Perpendicular distance from $(4t, 7t)$ to $L_1 : 4x - 3y = 0$ is

$$= \left|\frac{16t - 21t}{5}\right| = \left|\frac{-5t}{5}\right| = |-t| = t$$

Perpendicular distance from $(4t, 7t)$ to $L_2 : 12x - 5y = 0$

$$= \left|\frac{48t - 35t}{13}\right| = \left|\frac{13t}{13}\right| = t$$

Since $(4t, 7t)$ is equidistant from L_1 and L_2, r is on the bisector of the angle poq.

3. (a) Find the area of the triangle with vertices (1, 1), (8, –5) and (5, –2).

$(1, 1) \rightarrow (0, 0)$

$(8, -5) \rightarrow (7, -6) = (x_1, y_1)$

$(5, -2) \rightarrow (4, -3) = (x_2, y_2)$

$$\text{Area} = \frac{1}{2}|x_1 y_2 - x_2 y_1|$$

$$= \frac{1}{2}|7(-3) - (4)(-6)|$$

$$= \frac{1}{2}|-21 + 24| = \frac{1}{2}|3| = 1\frac{1}{2} \text{ sq. units}$$

(b) f is the transformation $(x, y) \rightarrow (x', y')$, where $x' - 4x + 2y$ and $y' - -3x - y$.
K is the line $x + y - 0$.

(i) Show that K is its own image under f.

(ii) $p(1, -1)$ and $q(3, -3)$ are two points.
Find the ratio $|pq| : |f(p)f(q)|$, giving your answer in its simplest form.

(i) We express x and y in terms of x' and y'.

$x' = 4x + 2y \quad \Rightarrow \quad x' = 4x + 2y$

$y' = -3x - y \quad \Rightarrow \quad 2y' = -6x - 2y$

$x' + 2y' = -2x$

$\Rightarrow \quad 2x = -x' - 2y'$

$\Rightarrow \quad x = \frac{1}{2}(-x' - 2y')$

$y' = -3x - y$

$\Rightarrow y = -3x - y'$

$= -\frac{3}{2}(-x' - 2y') - y'$

$= \frac{3x'}{2} + 3y' - y' = \frac{3x'}{2} + 2y' = \frac{1}{2}(3x' + 4y')$

$K : x + y = 0$

$\Rightarrow f(K): \frac{1}{2}(-x' - 2y') + \frac{1}{2}(3x' + 4y') = 0$

$\Rightarrow -x' - 2y' + 3x' + 4y' = 0$

$\Rightarrow 2x' + 2y' = 0$

$\Rightarrow \quad x' + y' = 0$

$\Rightarrow$ $f(K)$: $x + y = 0 \Rightarrow f(K) = K$

(ii) $(x, y) \to (x', y')$ where $x' = 4x + 2y$ and $y' = -3x - y$

$p(1, -1) \Rightarrow f(p) = (4 - 2, -3 + 1) = (2, -2)$

$q(3, -3) \Rightarrow f(q) = (12 - 6, -9 + 3) = (6, -6)$

$|pq| = \sqrt{(3-1)^2 + (-3+1)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$f(p) = (2,-2)$ and $f(q) = (6,-6)$

$|f(p)f(q)| = \sqrt{(6-2)^2 + (-6+2)^2} = \sqrt{16+16}$

$= \sqrt{32} = 4\sqrt{2}$

Thus $|pq| : |f(p)f(q)| = 2\sqrt{2} : 4\sqrt{2}$

$= 2 : 4$

$= 1 : 2$

(c) Consider the equation $k(3x - 5y + 6) + l(5x - 7y + 4) - 0$, where $k, l \in \mathbf{R}$.

(i) Show that for all k and l, the given equation represents a line passing through the point of intersection of $3x - 5y + 6 - 0$ and $5x - 7y + 4 - 0$.

(ii) Find the relationship between k and l for which the given equation represents a line of slope 2.

(iii) If $k = 1$, what line through the point of intersection cannot be represented by the given equation? Justify your answer.

(i) $k(3x - 5y + 6) + l(5x - 7y + 4) = 0$

$\Rightarrow 3kx - 5ky + 6k + 5lx - 7ly + 4l = 0$

$\Rightarrow x(3k + 5l) + y(-5k - 7l) + 6k + 4l = 0$

This equation represents a line as it is in the form $ax + by + c = 0$.

Let (x_1, y_1) be the point of intersection of the two lines.

Since $(x_1, y_1) \in 3x - 5y + 6 = 0 \Rightarrow 3x_1 - 5y_1 + 6 = 0$

$(x_1, y_1) \in 5x - 7y + 4 = 0 \Rightarrow 5x_1 - 7y_1 + 4 = 0$

$\Rightarrow \quad k(3x_1 - 5y_1 + 6) + l(5x_1 - 7y_1 + 4) = 0$

Thus $k(3x - 5y + 6) + l(5x - 7y + 4) = 0$ represents a line passing through the points of intersection of the two given lines.

(ii) The line $x(3k + 5l) + y(-5k - 7l) + 6k + 4l = 0$ has slope $m = \dfrac{3k+5l}{5k+7l}$.

$\Rightarrow \dfrac{3k+5l}{5k+7l} = 2$

$\Rightarrow 10k + 14l = 3k + 5l$

$\Rightarrow 7k = -9l$ i.e. $\underline{\underline{k = \dfrac{-9l}{7}}}$

If $k = 1$, the slope will be $\dfrac{3+5l}{5+7l}$

There is no value of l that can make this $= \dfrac{5}{7}$ (which is the slope of

$5x - 7y + 4 = 0$

i.e. $\dfrac{3+5l}{5+7l} = \dfrac{5}{7}$

i.e. $21 + 35l = 25 + 35l$

i.e. $21 = 25$, which is not true.

OR

(iii) The equation is $k(3x - 5y + 6) + l(5x - 7y + 4) = 0$

If $k = 1$ the equation cannot represent the line

$5x - 7y + 4 = 0$

Reason: There is no value of l that will get rid of

$3x - 5y + 6 = 0$

4. (a) Show that $(\cos A + \sin A)^2 - 1 + \sin 2A$.

$$(\cos A + \sin A)^2 = \cos^2 A + 2\sin A\cos A + \sin^2 A$$
$$= (\cos^2 A + \sin^2 A) + 2\sin A\cos A$$
$$= \underline{1 + \sin 2A}$$

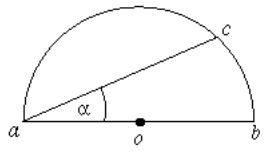
(b) Find all the solutions of the equation
$6\cos^2 x + \sin x - 5 - 0$, where $0° \le x \le 360°$.
Give the solutions correct to the nearest degree.

$$6\cos^2 x + \sin x - 5 = 0$$
$$\Rightarrow 6(1 - \sin^2 x) + \sin x - 5 = 0$$
$$\Rightarrow 6 - 6\sin^2 x + \sin x - 5 = 0$$
$$\Rightarrow 6\sin^2 x - \sin x - 1 = 0$$
$$\Rightarrow (2\sin x - 1)(3\sin x + 1) = 0$$
$$\Rightarrow 2\sin x = 1 \text{ or } 3\sin x = -1$$
$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \sin x = -\frac{1}{3}$$
$$\sin x = \frac{1}{2} \Rightarrow x = 30° \text{ or } 150°$$
$$\sin x = -\frac{1}{3} \Rightarrow \sin x = -0.3333$$
$$\Rightarrow x = 19.46° \text{ in the 3rd or 4th quadrants}$$
$$\Rightarrow x = 199° \text{ or } 341°$$
$$\therefore \underline{x = 30°, 150°, 199°, 341°}$$

(c) $[ab]$ is the diameter of a semicircle of centre o and radius-length r.
$[ac]$ is a chord such that $|\angle cab| = \alpha$, where α is in radian measure.

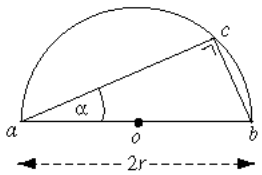
(i) Find $|ac|$ in terms of r and α.

(ii) $[ac]$ bisects the area of the semicircular region.
Show that $2\alpha + \sin 2\alpha = \frac{\pi}{2}$.

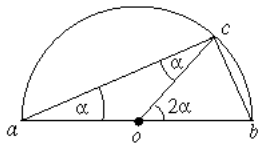


(i) $|\angle acb| = 90°$ … $[ab]$ is a diameter

$$\cos\alpha = \frac{|ac|}{|ab|} = \frac{|ac|}{2r}$$
$$\Rightarrow \underline{|ac| = 2r\cos\alpha}$$



(ii) Area of semicircle $= \frac{1}{2}\pi r^2$



Area of region $abc = \Delta aoc + \text{sector } obc$

Area of $\Delta aoc = \frac{1}{2}|ao||ac|\sin\alpha$

$= \frac{1}{2}(r)(2r\cos\alpha)(\sin\alpha)$

$= \frac{1}{2}r^2.2\sin\alpha\cos\alpha$

$= \frac{1}{2}r^2(\sin 2\alpha)$, i.e. $\frac{1}{2}r^2\sin 2\alpha \cdots ①$

Area of sector $obc = \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2(2\alpha) = r^2\alpha \cdots ②$

Area of region abc = ① + ②

$= \frac{1}{2}r^2\sin 2\alpha + r^2\alpha$

$\Rightarrow \frac{1}{2}r^2\sin 2\alpha + r^2\alpha = \text{area of } \frac{1}{2} \text{ semicircle} = \frac{1}{4} \text{ full circle}$

$\Rightarrow \frac{1}{2}r^2\sin 2\alpha + r^2\alpha = \frac{1}{4}\pi r^2$

$\Rightarrow \frac{1}{2}\sin 2\alpha + \alpha = \frac{\pi}{4} \ \ldots \ (\text{divide across by } r^2)$

$\Rightarrow \sin 2\alpha + 2\alpha = \frac{\pi}{2}$

$\Rightarrow \underline{\underline{2\alpha + \sin 2\alpha = \frac{\pi}{2}}}$

5. (a) Evaluate $\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x}$.

$\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2x}{3x}$

$= 1.1.\frac{2x}{3x} = \frac{2}{3}$

$\Rightarrow \underline{\underline{\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x} = \frac{2}{3}}}$

(b) Using the formula $\cos(A + B) - \cos A\cos B - \sin A\sin B$, derive a formula for $\cos(A - B)$ and hence prove that $\sin(A + B) - \sin A\cos B + \cos A\sin B$.

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

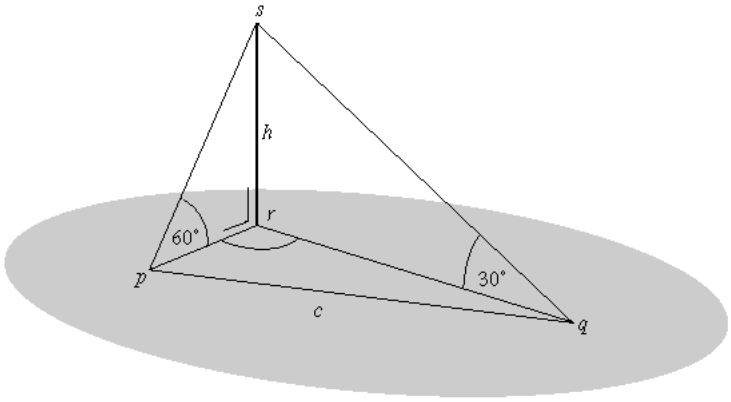
Putting in $(-B)$ for B, we get:

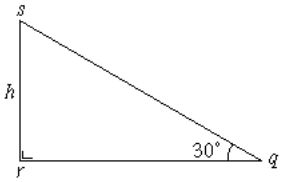
$$\cos(A - B) = \cos A\cos(-B) - \sin A\sin(-B)$$
$$= \underline{\cos A\cos B + \sin A\sin B}$$

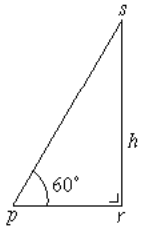
$$\sin\theta = \cos(90° - \theta)$$
$$\Rightarrow \sin(A+B) = \cos\left[90° - (A+B)\right]$$
$$= \cos\left[(90° - A) - B\right]$$
$$= \cos(90° - A)\cos B + \sin(90° - A)\sin B$$
$$\Rightarrow \underline{\underline{\sin(A+B) = \sin A\cos B + \cos A\sin B}}$$

(c) p, q and r are three points on horizontal ground.
$[sr]$ is a vertical pole of height h metres.
The angle of elevation of s from p is 60° and the angle of elevation of s from q is 30°.
$|pq| = c$ metres.

Given that $3c^2 = 13h^2$, find $|\angle prq|$.



① 

② 

From diagram ①: $\tan 30° = \dfrac{h}{|rq|}$

$$\Rightarrow |rq| = \frac{h}{\tan 30°} = \frac{h}{\frac{1}{\sqrt{3}}} = h\sqrt{3}$$

From diagram ②: $\tan 60° = \dfrac{h}{|pr|}$

$$\Rightarrow |pr| = \frac{h}{\tan 60°} = \frac{h}{\sqrt{3}}$$

$$\cos \angle prq = \frac{|pr|^2 + |rq|^2 - |pq|^2}{2|pr||rq|}$$

$$= \frac{\frac{h^2}{3} + 3h^2 - c^2}{2 \cdot \frac{h}{\sqrt{3}} \cdot h\sqrt{3}} = \frac{\frac{h^2}{3} + 3h^2 - c^2}{2h^2}$$

$$= \frac{h^2 + 9h^2 - 3c^2}{6h^2}$$

$$= \frac{10h^2 - 3c^2}{6h^2}$$

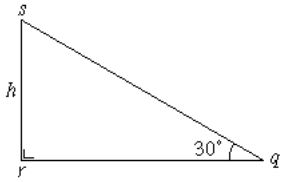
But $3c^2 = 13h^2$

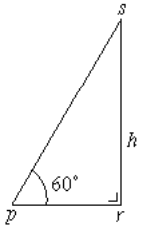
$$\Rightarrow \cos \angle prq = \frac{10h^2 - 13h^2}{6h^2} = \frac{-3h^2}{6h^2} = -\frac{1}{2}$$

$$\cos \angle prq = -\frac{1}{2}$$

$$\Rightarrow \underline{\underline{|\angle prq| = 120°}}$$

OR





$$\tan 30° = \frac{h}{|rq|} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{|rq|} \Rightarrow h = \frac{|rq|}{\sqrt{3}}$$

$$\tan 60° = \frac{h}{|pr|} \Rightarrow \sqrt{3} = \frac{h}{|pr|} \Rightarrow h = |pr|\sqrt{3}$$

$$h = h$$

$$\Rightarrow |pr|\sqrt{3} = \frac{|rq|}{\sqrt{3}}$$

$$\Rightarrow |pr| = \frac{|rq|}{3}$$

$$\cos\angle prq = \frac{|rq|^2 + |pr|^2 - |pq|^2}{2|rq||pr|}$$

$$= \frac{|rq|^2 + \frac{|rq|^2}{9} - c^2}{2|rq||pr|}$$

$$= \frac{|rq|^2 + \frac{|rq|^2}{9} - \frac{13|rq|^2}{9}}{2|rq||pr|} \quad \cdots \left[c^2 = \frac{13|rq|^2}{9}\right]$$

$$= \frac{9|rq|^2 + |rq|^2 - 13|rq|^2}{18|rq||pr|}$$

$$= \frac{-3|rq|^2}{18|rq||pr|} = \frac{-3|rq|}{18|pr|} = \frac{-3|rq|}{18 \cdot \frac{|rq|}{3}}$$

$$= \frac{-3}{6} = -\frac{1}{2}$$

$$\Rightarrow \cos\angle prq = -\frac{1}{2}$$

$$\Rightarrow \underline{\underline{|\angle prq| = 120^\circ}}$$

$$h = \frac{|rq|}{\sqrt{3}}$$

$$\Rightarrow \quad h^2 = \frac{|rq|^2}{3}$$

$$\Rightarrow \quad 3h^2 = |rq|^2$$

But $3c^2 = 13h^2$

$$\Rightarrow \quad h^2 = \frac{3c^2}{13}$$

$$\Rightarrow \quad 3h^2 = \frac{9c^2}{13}$$

$$\Rightarrow \quad \frac{9c^2}{13} = |rq|^2$$

$$\Rightarrow \quad c^2 = \frac{13|rq|^2}{9}$$

6. (a) Six people, including Mary and John, sit in a row.

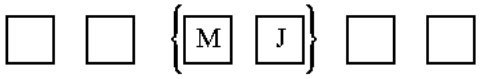
(i) How many different arrangements of the six people are possible?

(ii) In how many of these arrangements are Mary and John next to each other?

(i) No. of arrangements = 6!

= 6 × 5 × 4 × 3 × 2 × 1

= <u>720</u>

(ii) 

Regard Mary and John as 1 'unit'

They can be arranged when together in 2! ways

No. of arrangements = 5! × 2!

= 5 × 4 × 3 × 2 × 1 × 2

= 240

(b) α and β are the roots of the quadratic equation $px^2 + qx + r - 0$.

$u_n - l\alpha^n + m\beta^n$, for all $n \in$ N.

Show that $pu_{n+2} + qu_{n+1} + ru_n - 0$, for all $n \in$ N.

Since α and β are roots of the equation $px^2 + qx + r = 0$,
$\Rightarrow p\alpha^2 + q\alpha + r = 0$ and $p\beta^2 + q\beta + r = 0$ ... (1)

Also since $u_n = l\alpha^n + m\beta^n$
$\Rightarrow u_{n+1} = l\alpha^{n+1} + m\beta^{n+1} = l\alpha^n . \alpha + m\beta^n . \beta$
$\Rightarrow u_{n+2} = l\alpha^{n+2} + m\beta^{n+2} = l\alpha^n . \alpha^2 + m\beta^n . \beta^2$

Thus $pu_{n+2} + qu_{n+1} + ru_n$
$= pl\alpha^n . \alpha^2 + pm\beta^n . \beta^2 + ql\alpha^n . \alpha + qm\beta^n . \beta + rl\alpha^n + rm\beta^n$
$= l\alpha^n (p\alpha^2 + q\alpha + r) + m\beta^n (p\beta^2 + q\beta + r)$
$= l\alpha^n (0) + m\beta^n (0)$ ... from (1) above
$= 0$
$\Rightarrow pu_{n+2} + qu_{n+1} + ru_n = 0$, for all $n \in$ N

(c) w white discs and r red discs are placed in a box. Two of the discs are drawn at random from the box. The probability that both discs are red is p.

(i) Find p in terms of w and r.

(ii) When $w - 1$, find the value of r for which $p = \frac{1}{2}$.

(iii) There are other values of w and r that also give $p = \frac{1}{2}$.

The next smallest such value of w is even.
By investigating the even numbers in turn, find this value of w and the corresponding value of r.

(i) w = white discs ; r = red discs
Total number of discs = $r + w$
$P(2 \text{ red}) = P(1\text{st red}) . P(2\text{nd red})$

$$= \frac{r}{r+w} \times \frac{r-1}{r+w-1}$$

$$\Rightarrow p = \frac{r(r-1)}{(r+w)(r+w-1)}$$

OR

Number of ways of picking 2 discs from $(r + w)$ disc is

$$\binom{r+w}{2} = \frac{(r+w)(r+w-1)}{1.2} = \text{total possible}$$

Number of ways of picking 2 red discs from r red discs is

$$\binom{r}{2} = \frac{r(r-1)}{1.2} = \text{total favourable}$$

$$P(2\text{ red discs}) = \frac{\text{Favourable}}{\text{Possible}} = \frac{\dfrac{r(r-1)}{1.2}}{\dfrac{(r+w)(r+w-1)}{1.2}}$$

$$\Rightarrow p = \frac{r(r-1)}{(r+w)(r+w-1)}$$

(ii) $w = 1\ ;\ p = \dfrac{1}{2}$

$$\Rightarrow \frac{1}{2} = \frac{r(r-1)}{(r+1)(r+1-1)}$$

$$= \frac{r(r-1)}{r(r+1)} = \frac{r-1}{r+1}$$

$$\frac{1}{2} = \frac{r-1}{r+1}$$

$$\Rightarrow 2r - 2 = r + 1$$

$$\Rightarrow \underline{r = 3}$$

(iii) $\dfrac{r(r-1)}{(r+w)(r+w-1)} = \dfrac{1}{2}$

Take $w = 2$: $\dfrac{r(r-1)}{(r+2)(r+2-1)} = \dfrac{1}{2}$

$$\Rightarrow \quad 2r\,(r-1) = (r+2)\,(r+1)$$

$$\Rightarrow \quad 2r^2 - 2r = r^2 + 3r + 2$$

$$\Rightarrow \quad r^2 - 5r - 2 = 0$$

There is no solution to this equation for $r \in$ N.

Take $w = 4$: $\dfrac{r(r-1)}{(r+4)(r+3)} = \dfrac{1}{2}$

$$\Rightarrow \quad 2r^2 - 2r = r^2 + 7r + 12$$

$$\Rightarrow \quad r^2 - 9r - 12 = 0$$

There is no solution for $r \in$ N.

Take $w = 6$: $\dfrac{r(r-1)}{(r+6)(r+5)} = \dfrac{1}{2}$

$$\Rightarrow \quad 2r^2 - 2r = r^2 + 11r + 30$$

$$\Rightarrow \quad r^2 - 13r - 30 = 0$$

$$\Rightarrow \quad (r - 15)\,(r + 2) = 0$$

$$\Rightarrow \quad r = 15 \text{ as } r \neq -2$$

$\therefore$ $\underline{w = 6 \text{ and } r = 15}$

7. (a) (i) How many different selections of four letters can be made from the letters of the word FLORIDA?
(ii) How many of these selections contain at least one vowel?

(i) $\binom{7}{4} = \frac{7\times6\times5\times4}{1\times2\times3\times4} = \underline{\underline{35 \text{ selections}}}$

(ii) There are 3 vowels in the word.
No. of selections of 4 letters with no vowel is $\binom{4}{4} = 1$.
No. of selections with at least one vowel is

$$\binom{7}{4} - \binom{4}{4} = 35 - 1$$
$$= \underline{\underline{34 \text{ selections}}}$$

(b) Two dice are thrown.
(i) What is the probability of getting two identical numbers or a total of five?
(ii) What is the probability that the product of the two numbers thrown is at least twice their sum?

(i) Possible outcomes = 36
Favourable outcomes = (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (2, 3), (3, 2)

Probability (2 identical or total of 5) $= \frac{10}{36} = \underline{\underline{\frac{5}{18}}}$

(ii)

Outcomes of interest	Product	Sum
(6, 6)	36	12
(6, 5) or (5, 6)	30	11
(6, 4) or (4, 6)	24	10
(6, 3) or (3, 6)	18	9
(5, 5)	25	10
(5, 4) or (4, 5)	20	9
(4, 4)	16	8

Probability $= \underline{\underline{\frac{11}{36}}}$

OR

(b) (i)

	1	2	3	4	5	6
1	•			□		
2		•	□			
3		□	•			
4	□			•		
5					•	
6						•

Favourable outcomes = 10

Total outcomes = 36

$$P(E) = \frac{10}{36} = \underline{\underline{\frac{5}{18}}}$$

(b) (ii)

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	***18***
4	4	8	12	***16***	***20***	***24***
5	5	10	15	***20***	***25***	***30***
6	6	12	***18***	***24***	***30***	***36***

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Total outcomes = 36

Total favourable = 11, (i.e. the numbers printed in bold)

$$P(E) = \underline{\underline{\frac{11}{36}}}$$

(c) (i) Find, in terms of a and d, the mean of the first seven terms of an arithmetic sequence with first term a and common difference d.
(ii) Show that the standard deviation of these seven numbers is $2d$.

(i) Sum of the first n terms is $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$$S_7 = \frac{7}{2}[2a + 6d]$$
$$= 7a + 21d$$

$$\text{Mean} = \frac{7a + 21d}{7} = \underline{\underline{a + 3d}}$$

(ii)

Number	Mean	Deviation	(Deviation)^2
a	$a + 3d$	$3d$	$9d^2$
$a + d$	$a + 3d$	$2d$	$4d^2$
$a + 2d$	$a + 3d$	d	d^2
$a + 3d$	$a + 3d$	0	0
$a + 4d$	$a + 3d$	d	d^2
$a + 5d$	$a + 3d$	$2d$	$4d^2$
$a + 6d$	$a + 3d$	$3d$	$9d^2$

$$\text{Standard deviation} = \sqrt{\frac{9d^2 + 4d^2 + d^2 + d^2 + 4d^2 + 9d^2}{7}}$$
$$= \sqrt{\frac{28d^2}{7}} = \sqrt{4d^2} = 2d$$

$\therefore$ Standard deviation, $\sigma = 2d$

8. (a) p and q are real numbers such that $p + q - 1$.
Find the value of p that maximizes the product pq.

$p + q = 1 \Rightarrow q = 1 - p$
$pq = p\,(1 - p)$
Let $A = pq = p\,(1 - p)$
$\Rightarrow A = p - p^2$

$\frac{dA}{dp} = 1 - 2p$ $\qquad$ $\frac{d^2A}{dp^2} = -2$

$1 - 2p = 0 \quad \Rightarrow p = \frac{1}{2}$ $\qquad$ $\Rightarrow$ maximum value

$\therefore$ Maximum product occurs when $\underline{p = 2}$

(b) (i) Derive the Maclaurin series for $f(x) - (1 + x)^m$ up to and including the term containing x^3.
(ii) Given that the general term of the series $f(x)$ is

$$\frac{m(m-1)(m-2)\cdots\cdots(m-r+1)}{r!}x^r,$$

show that the series converges for $-1 < x < 1$.

(i) $f(x) = (1 + x)^m$ $\qquad$ $\Rightarrow f(0) = 1$
$f'(x) = m(1 + x)^{m-1}$ $\qquad$ $\Rightarrow f'(0) = m$
$f''(x) = m(m - 1)(1 + x)^{m-2}$ $\qquad$ $\Rightarrow f''(0) = m(m - 1)$
$f'''(x) = m(m - 1)(m - 2)(1 + x)^{m-3}$ $\qquad$ $\Rightarrow f'''(0) = m(m - 1)(m - 2)$

Since $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$$\Rightarrow (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

(ii) $u_{r+1} = \frac{m(m-1)(m-2)\ldots(m-r+1)}{r!}x^r$

$$\Rightarrow u_r = \frac{m(m-1)(m-2)\cdots(m-r+2)}{(r-1)!}x^{r-1}$$

$$\lim_{r\to\infty}\frac{u_{r+1}}{u_r} = \lim_{r\to\infty}\left|\frac{m(m-1)(m-2)\cdots(m-r+1)x^r}{r!}\cdot\frac{(r-1)!}{m(m-1)\cdots(m-r+2)x^{r-1}}\right|$$

$$= \lim_{r\to\infty}\left|\frac{m-r+1}{r}\cdot x\right|$$

$$= \lim_{r\to\infty}\left|\left(\frac{m}{r} - 1 + \frac{1}{r}\right)x\right|$$

$$= |(0 - 1 + 0)x|$$

$$= |-x| = |x|$$

The series is convergent for $|x| < 1$

i.e. for $\underline{-1 < x < 1}$

(c) Evaluate $\int_0^1 \tan^{-1} x\,dx$.

$$\int_0^1 \tan^{-1} x\,dx: \qquad \int u\,dv = uv - \int v\,du$$

Let $u = \tan^{-1} x$ and $dv = dx$

$$\Rightarrow du = \frac{1}{1+x^2}\,dx \text{ and } v = x$$

$$\Rightarrow \int_0^1 \tan^{-1} x = (\tan^{-1} x)x - \int x\left(\frac{1}{1+x^2}\right)dx$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

We now find $\int \frac{x}{1+x^2}\,dx$

Thus $\int \frac{x}{1+x^2}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du$ | Let $u = 1+x^2$

$= \frac{1}{2}\int \frac{1}{u}\,du$ | $\frac{du}{dx} = 2x$

$= \frac{1}{2}\ln u$ | $\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\ln\left(1+x^2\right)$

Thus $\int_0^1 \tan^{-1} x = \left[x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$

$$= \left[\tan^{-1} 1 - \frac{1}{2}\ln(2)\right] - \left[0 - \frac{1}{2}\ln 1\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 - [0-0]$$

$$= \underline{\underline{\frac{\pi}{4} - \frac{1}{2}\ln 2}}$$

9. (a) Two events E_1 and E_2 are independent. $P(E_1) = \frac{1}{5}$ and $P(E_2) = \frac{1}{7}$. Find

(i) $P(E_1 \cap E_2)$

(ii) $P(E_1 \cup E_2)$.

(i) $P(E_1) = \frac{1}{5}$ and $P(E_2) = \frac{1}{7}$

$$P(E_1 \cap E_2) = P(E_1).P(E_2) = \frac{1}{5}\cdot\frac{1}{7} = \underline{\underline{\frac{1}{35}}}$$

(ii) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{1}{5} + \frac{1}{7} - \frac{1}{35} = \underline{\underline{\frac{11}{35}}}$$

(b) Five unbiased coins are tossed.

(i) Find the probability of getting three heads and two tails.

(ii) The five coins are tossed eight times. Find the probability of getting three heads and two tails exactly four times.
Give your answer correct to three decimal places.

(i) $p = \frac{1}{2}; \quad q = \frac{1}{2}$

$$P(3 \text{ heads, } 2 \text{ tails}) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$$

$$= \frac{5.4.3}{1.2.3} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{10}{32} = \underline{\underline{\frac{5}{16}}}$$

(ii) $p = \frac{5}{16}$ … from (i); $q = \frac{11}{16}$

$$\text{Probability} = \binom{8}{4}\left(\frac{5}{16}\right)^4\left(\frac{11}{16}\right)^4$$

$$= 70\left(\frac{625}{65536}\right)\left(\frac{14641}{65536}\right)$$

$$= \underline{\underline{0.149}}$$

(c) The amounts due on monthly mobile phone bills are normally distributed with mean €53 and standard deviation €15.

(i) If a bill is chosen at random, find the probability that the amount due is between €47 and €74.

(ii) A random sample of 900 bills is taken. Find the probability that the mean amount due on the bills in the sample is greater than €53.30.

(i) $\bar{x} = 53\,; \ \sigma = 15$

$P(47 < x < 74) = P(z_1 < z < z_2)$

$$z_1 = \frac{x - \bar{x}}{\sigma} = \frac{47 - 53}{15} = \frac{-6}{15} = -0.4$$

$$z_2 = \frac{x - \bar{x}}{\sigma} = \frac{74 - 53}{15} = \frac{21}{15} = 1.4$$

$$
\begin{aligned}
&P(-0.4 < x < 1.4)\\
&= P(z \le 1.4) - P(z > 0.4)\\
&= 0.9192 - [1 - P(z \le 0.4)]\\
&= 0.9192 - [1 - 0.6554]\\
&= 0.9192 - [0.3446]\\
&= \underline{\underline{0.5746}}
\end{aligned}
$$

(ii) $n = 900,\ \bar{x} = 53,\ \sigma = 15$

$$
\Rightarrow \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{900}} = \frac{15}{30} = \frac{1}{2} = 0.5
$$

$$
\begin{aligned}
P(x > 53.3) &= P\left(z > \frac{53.3 - 53}{0.5}\right)\\
&= P(z > 0.6)\\
&= 1 - P(z \le 0.6)\\
&= 1 - 0.7527\\
&= \underline{\underline{0.2743}}
\end{aligned}
$$

10. (a) For each of the following, give a reason why it is not a group.

(i) The set of natural numbers under subtraction.

(ii) The set of real numbers under multiplication.

(i) Not closed under subtraction as $6 - 14 = -8 \notin \mathrm{N}$

$\therefore$ not a group

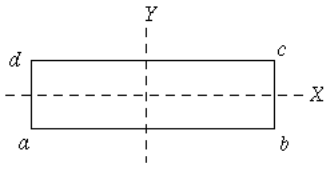
(ii) $0 \in R$ but 0 does not have an inverse

$\therefore$ not a group

(b) $G = \{I_\pi, R_{180°}, S_X, S_Y\}$ is the set of symmetries of the rectangle $abcd$.

(i) Show that G is a group under composition. You may assume that composition of symmetries is associative.

(ii) Find $Z(G)$, the centre of the group.



(i) From the Cayley Table

	I_π	$R_{180°}$	S_X	S_Y
I_π	I_π	$R_{180°}$	S_X	S_Y
$R_{180°}$	$R_{180°}$	I_π	S_Y	S_X
S_X	S_X	S_Y	I_π	$R_{180°}$
S_Y	S_Y	S_X	$R_{180°}$	I_π

(i) The set is closed

(ii) Assosiative: Given

(iii) I_π is the identity element

(iv) Each element has an inverse,

i.e. $I_\pi^{-1} = I_\pi;\ R_{180°}^{-1} = R_{180°};$

$S_X^{-1} = S_X;\ S_Y^{-1} = S_Y$

Thus the set is a group.

(ii) The centre of a group, G, is the set of all elements of G which commute with every element of G.
From the table $x \circ y = y \circ x$ for all $x, y \in G$
i.e., each element commutes with every other element
$\therefore\ Z(G) = G = \{I_\pi, R_{180°}, S_X, S_Y\}$

OR

In the table the elements are symmetrical about the main diagonal.
$\Rightarrow G$ is a commutative group
i.e. $Z(G) = G$

(c) Use Lagrange's theorem to prove that

(i) any group of prime order is cyclic.

(ii) the order of any element of a finite group G divides the order of G.

(i) Let $(G, *)$ be a group of order k, where k is prime.
Let $a \in G$ and $a \neq e$.
$\therefore$ $\langle a \rangle$ is a subgroup of G
But the order of $\langle a \rangle$ is a factor of k ... (Lagrange)
As k is prime $\Rightarrow \langle a \rangle = k$ as $\langle a \rangle \neq 1$.
$\therefore$ $\langle a \rangle = G \Rightarrow G$ is cyclic

(ii) Let $(G, *)$ be a group of order n.
Let $a \in G$ and let the order of a be m.
But, by Lagrange, the order of a subgroup divides the order of the group, G
$\therefore$ m is a factor of n
Therefore the order of an element divides the order of G.

11. (a) Find the eccentricity of the ellipse with equation $\frac{x^2}{64} + \frac{y^2}{48} = 1$.

$$\frac{x^2}{64} + \frac{y^2}{48} = 1$$

$a^2 = 64,\ b^2 = 48$ and $b^2 = a^2(1 - e^2)$

$$\Rightarrow 48 = 64(1 - e^2)$$

$$\Rightarrow 1 - e^2 = \frac{48}{64} = \frac{3}{4}$$

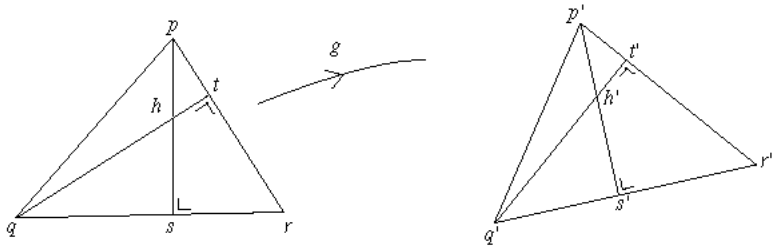
$$\Rightarrow 4 - 4e^2 = 3$$

$$\Rightarrow 4e^2 = 1 \ \Rightarrow e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$$

$\therefore$ the eccentricity $= \frac{1}{2}$

(b) Prove that a similarity transformation maps the orthocentre of a triangle onto the orthocentre of the image of the triangle.

Let pqr be a triangle with orthocentre h and let g be a similarity transformation.
Let $p'q'r'$ be the image of pqr under g.
h' is the orthocentre of the triangle $p'q'r'$
We are required to prove that h is mapped onto h' by g.

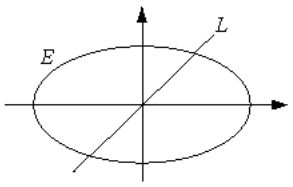


g is a similarity transformation.
h is the orthocentre of Δpqr.
By g: Δpqr maps to $\Delta p'q'r'$.
To prove: $g(h)$ is the orthocentre of $\Delta p'q'r'$.

By g, $[ps]$ maps to $[p's']$ and $[qr]$ maps to $[q'r']$.
But $[ps] \perp [qr] \Rightarrow [p's'] \perp [q'r']$ as perpendicularity is invariant under g.
Similarly $[q't'] \perp [p'r']$
$\Rightarrow h'$ is the orthocentre of $\Delta p'q'r'$
$h = [ps] \cap [qt]$ maps to $g(h) = [p's'] \cap [q't']$
$\therefore\ g(h) = h'$
$\therefore$ $g(h)$ is the orthocentre of $\Delta p'q'r'$

(c) E is the ellipse $\dfrac{x^2}{4} + y^2 = 1$ and L is the line $y = x$.

Using a transformation that maps E to the unit circle, or otherwise, find the equation of the diameter that is conjugate to L in E.



Let f be the transformation $(x, y) \to (x', y')$, where $x = 2x'$ and $y = y'$.

$$f(E) = \frac{(2x')^2}{4} + (y')^2 = 1 \Rightarrow (x')^2 + (y')^2 = 1 \ \cdots \text{ a circle}$$

L is the line $x - y = 0$
$f(L) = 2x' - y' = 0$

But $f(E)$ is a circle $\Rightarrow$ conjugate diameter (slope $= -1$)
$\Rightarrow$ conjugate diameter (slope $= -1$) of $f(L)$ is $f(K)$:

$$x' + 2y' = 0$$

By $f^{-1}(K)$, $f(K)$ maps to K, the conjugate diameter of L.
Mapping back to the ellipse,

$$K: \frac{x}{2} + 2y = 0$$

$\Rightarrow L$ conjugate diameter is $\dfrac{x}{2} + 2y = 0$

i.e. $x + 4y = 0$